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New approximations of the gamma function in terms of the digamma function

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ABSTRACT

The goal of this paper is to prove the following asymptotic formula

$$\Gamma(x) \approx \sqrt{2\pi} e^{-b} (x+b)^x \exp\left(-x - \frac{1}{2}\psi(x+c)\right) \quad \text{as } x \in \mathbb{N}, x \to \infty,$$

where Γ is the Euler Gamma function and ψ is the digamma function, namely, the logarithmic derivative of Γ. Moreover, optimal values of parameters b, c are calculated in such a way that this asymptotic convergence is the best possible.

1. Introduction

In the recent paper [1], H. Alzer and N. Batir considered the function

$$G_c(x) = \log\Gamma(x) - x\log x + x - \frac{1}{2}\log 2\pi + \frac{1}{2}\psi(x+c) \quad (x > 0; c \geq 0),$$

where Γ is the Euler Gamma function and $\psi = \Gamma'/\Gamma$ is the digamma function. They proved that G_a is completely monotonic on $(0, \infty)$ if and only if $a \geq 1/3$ and $-G_b$ is completely monotonic on $(0, \infty)$ if and only if $b = 0$.

As an application, they provided sharp bounds for $\Gamma(x)$, *i.e.*, they determined the smallest number $\alpha = 1/3$ and the largest number $\beta = 0$ such that for $x > 0$, the following inequalities are valid

$$\sqrt{2\pi} x^x \exp\left(-x - \frac{1}{2}\psi(x+\alpha)\right) < \Gamma(x) < \sqrt{2\pi} x^x \exp\left(-x - \frac{1}{2}\psi(x+\beta)\right). \tag{1.1}$$

We extend this class of approximations by introducing a new parameter b to obtain the asymptotic formula

$$\Gamma(x) \approx \sqrt{2\pi} e^{-b} (x+b)^x \exp\left(-x - \frac{1}{2}\psi(x+c)\right) \equiv G_{b,c}(x). \tag{1.2}$$

We find the best approximations G_{b_*,c_*} and $G_{b_\#,c_\#}$, where $b_* > b_\#$ are the real roots of the polynomial $18b^4 + 24b^3 + 6b^2 - 1$ and $c_* = \frac{1}{3} - b_*^2$, $c_\# = \frac{1}{3} - b_\#^2$. Computer softwares such as MAPLE provide us the exact values of

$$b_* = \frac{v+2}{6} + \frac{1}{6}\sqrt{\frac{4}{v} - v^2 + 6}, \qquad b_\# = \frac{v+2}{6} - \frac{1}{6}\sqrt{\frac{4}{v} - v^2 + 6}$$

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and

$$c_* = \frac{v-2-2v^2}{18v} + \frac{v+2}{18}\sqrt{\frac{4}{v} - v^2 + 6}, \qquad c_\# = \frac{v-2-2v^2}{18v} - \frac{v+2}{18}\sqrt{\frac{4}{v} - v^2 + 6}$$

where $v = \sqrt{9u + 2 - \frac{5}{9u}}$, with $u = \sqrt[3]{\frac{1}{243}\sqrt{46} - \frac{17}{729}}$. The numerical values are: $b_* = 0.269448666249\ldots$, $b_\# = -1.0665985070008\ldots$, $c_* = 0.260730749589\ldots$, $c_\# = -0.804299041803\ldots$.

At the end of this work, we discuss the superiority of our approximations G_{b_*,c_*} and $G_{b_\#,c_\#}$ over the approximations $G_{0,1/3}$ and $G_{0,0}$ which appear in (1.1).

With every approximation (1.2), we associate the sequence $(\omega_n)_{n\geq 1}$ defined by

$$\Gamma(n) = \sqrt{2\pi}\,\mathrm{e}^{-b}(n+b)^n \exp\left(-n - \frac{1}{2}\psi(n+c)\right) \cdot \exp\omega_n. \tag{1.3}$$

It is easy to see that $(\omega_n)_{n\geq 1}$ converges to zero, since

$$\psi(x) \approx \log x - \frac{1}{2x} - \frac{1}{12x^2} + \cdots$$

(see [2, pp. 259]), but our study is based on the elementary idea that the approximation (1.2) is as better as $(\omega_n)_{n\geq 1}$ faster converges to zero.

In order to calculate the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$, we use the following:

Lemma. *If* $(\omega_n)_{n\geq 1}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = l \in \mathbb{R}, \tag{1.4}$$

with $k > 1$, *then there exists the limit:*

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{l}{k-1}.$$

Proof. For $\varepsilon > 0$, assume that $l - \varepsilon \leq n^k(\omega_n - \omega_{n+1}) \leq l + \varepsilon$, for every integer n greater than or equal to the rank n_0. By adding the inequalities of the form

$$(l-\varepsilon)\cdot\frac{1}{n^k} \leq \omega_n - \omega_{n+1} \leq (l+\varepsilon)\cdot\frac{1}{n^k},$$

we get

$$(l-\varepsilon)\sum_{i=n}^{n+p-1}\frac{1}{i^k} \leq \omega_n - \omega_{n+p} \leq (l+\varepsilon)\sum_{i=n}^{n+p-1}\frac{1}{i^k},$$

for every $n \geq n_0$ and $p \geq 2$. By taking the limit as $p \to \infty$, then multiplying by n^{k-1}, we obtain

$$(l-\varepsilon)\cdot n^{k-1}\left(\zeta(k) - \sum_{i=1}^{n-1}\frac{1}{i^k}\right) \leq n^{k-1}\omega_n \leq (l+\varepsilon)\cdot n^{k-1}\left(\zeta(k) - \sum_{i=1}^{n-1}\frac{1}{i^k}\right), \tag{1.5}$$

where $\zeta(k)$ is the Riemann-zeta function.

Now, by using the rate of convergence of the generalized harmonic series

$$\lim_{n\to\infty} n^{k-1}\left(\zeta(k) - \sum_{i=1}^{n-1}\frac{1}{i^k}\right) = \frac{1}{k-1},$$

(see [2,3]) we can take the limit as $n \to \infty$ in (1.5), to complete the proof of the Lemma. □

We can see from the Lemma that the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ increases together with the value k satisfying (1.4).

This Lemma is also an important tool for constructing asymptotic expansions, or to accelerate some convergences. See, *e.g.*, [4–6].

2. Main results

From (1.3), we deduce that

$$\omega_n = \ln\Gamma(n) - \ln\sqrt{2\pi} + b - n\ln(n+b) + n + \frac{1}{2}\psi(n+c),$$

thus

$$\omega_n - \omega_{n+1} = (n+1)\ln(n+b+1) - n\ln(n+b) - \ln n - \frac{1}{2(n+c)} - 1,$$

where we have used the recurrence formula $\psi(x+1) = \psi(x) + 1/x$.

As we are interested to compute a limit of the form (1.4), we prefer to write

$$\omega_n - \omega_{n+1} = \left(\frac{1}{2}b^2 + \frac{1}{2}c - \frac{1}{6}\right)\frac{1}{n^2} - \left(\frac{2}{3}b^3 + \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{12}\right)\frac{1}{n^3} + \left(\frac{3}{4}b^4 + b^3 + \frac{1}{2}b^2 + \frac{1}{2}c^3 - \frac{1}{20}\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right).$$

Using this expression for $\omega_n - \omega_{n+1}$ written as the power sum of n^{-1} and Lemma, we can state the main result of this work.

Theorem. (i) *If $c \neq \frac{1}{3} - b^2$, then the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ is n^{-1} since*

$$\lim_{n\to\infty} n\omega_n = \frac{1}{2}b^2 + \frac{1}{2}c - \frac{1}{6} \neq 0.$$

(ii) *If $c = \frac{1}{3} - b^2$ and $\frac{2}{3}b^3 + \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{12} \neq 0$, then the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ is n^{-2} since*

$$\lim_{n\to\infty} n^2\omega_n = -\frac{1}{2}\left(\frac{2}{3}b^3 + \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{12}\right) \neq 0.$$

(iii) *If $c = \frac{1}{3} - b^2$ and $\frac{2}{3}b^3 + \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{12} = 0$ (equivalent with $(b, c) = (b_*, c_*)$, or $(b, c) = (b_\#, c_\#)$), then the speed of convergence of $(\omega_n)_{n\geq 1}$ is n^{-3}.*

If $(b, c) = (b_*, c_*)$, then

$$\lim_{n\to\infty} n^3\omega_n = \frac{1}{3}\left(\frac{3}{4}b_*^4 + b_*^3 + \frac{1}{2}b_*^2 + \frac{1}{2}c_*^3 - \frac{1}{20}\right) \neq 0$$

and if $(b, c) = (b_\#, c_\#)$, then

$$\lim_{n\to\infty} n^3\omega_n = \frac{1}{3}\left(\frac{3}{4}b_\#^4 + b_\#^3 + \frac{1}{2}b_\#^2 + \frac{1}{2}c_\#^3 - \frac{1}{20}\right) \neq 0.$$

By replacing $c = \frac{1}{3} - b^2$, in $\frac{2}{3}b^3 + \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{12} = 0$, we deduce that b_* and $b_\#$ are the real solutions of the equation $6b^2 + 24b^3 + 18b^4 - 1 = 0$, that is

$$b_* = 0.269448666249\ldots, \qquad b_\# = -1.0665985070008\ldots.$$

The approximations from [1] are obtained in cases (i) and (ii) of the Theorem, for $b = 0, c = 0$, respectively $b = 0$ and $c = \frac{1}{3}$. Indeed, let us introduce the sequences $(u_n)_{n\geq 1}$, $(v_n)_{n\geq 1}$ associated with the approximations (1.1) from [1], by

$$\Gamma(n) = G_{0,0}(n)\exp u_n, \qquad \Gamma(n) = G_{0,1/3}(n)\exp v_n.$$

According to the Theorem, we have

$$\lim_{n\to\infty} nu_n = -\frac{1}{6} \quad\text{and}\quad \lim_{n\to\infty} n^2 v_n = \frac{1}{36},$$

while

$$\lim_{n\to\infty} n^3\omega_n = \frac{1}{3}\left(\frac{3}{4}b^4 + b^3 + \frac{1}{2}b^2 + \frac{1}{2}c^3 - \frac{1}{20}\right) \neq 0,$$

where $b \in \{b_*, b_\#\}$. In other words, the remainder $(\omega_n)_{n\geq 1}$ becomes much smaller as n increases, since

$$\omega_n = O\left(\frac{u_n}{n^2}\right) \quad\text{and}\quad \omega_n = O\left(\frac{v_n}{n}\right).$$

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