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# On semi-planar Steiner quasigroups

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**Abstract**

A Steiner triple system (briefly **ST**) is in 1–1 correspondence with a Steiner quasigroup or squag (briefly **SQ**) [B. Ganter, H. Werner, Co-ordinatizing Steiner systems, Ann. Discrete Math. 7 (1980) 3–24; C.C. Lindner, A. Rosa, Steiner quadruple systems: A survey, Discrete Math. 21 (1979) 147–181]. It is well known that for each $n \equiv 1$ or 3 (mod 6) there is a planar squag of cardinality $n$ [J. Doyen, Sur la structure de certains systems triples de Steiner, Math. Z. 111 (1969) 289–300]. Quackenbush expected that there should also be semi-planar squags [R.W. Quackenbush, Varieties of Steiner loops and Steiner quasigroups, Canad. J. Math. 28 (1976) 1187–1198]. A simple squag is semi-planar if every triangle either generates the whole squag or the 9-element squag. The first author has constructed a semi-planar squag of cardinality $3n$ for all $n > 3$ and $n \equiv 1$ or 3 (mod 6) [M.H. Armanious, Semi-planar Steiner quasigroups of cardinality $3n$, Australas. J. Combin. 27 (2003) 13–27]. In fact, this construction supplies us with semi-planar squags having only nontrivial subsquags of cardinality 9. Our aim in this article is to give a recursive construction as $n \to 3n$ for semi-planar squags. This construction permits us to construct semi-planar squags having nontrivial subsquags of cardinality $>9$. Consequently, we may say that there are semi-planar $\mathbf{SQ}(3^m n)$s (or semi-planar $\mathbf{ST}(3^m n)$s) for each positive integer $m$ and each $n \equiv 1$ or 3 (mod 6) with $n > 3$ having only medial subsquags at most of cardinality $3^\nu$ (sub-$\mathbf{ST}(3)^\nu$) for each $\nu \in \{1, 2, \dots, m+1\}$.
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## 1. Introduction

A *Steiner quasigroup* (or a *squag*) is a groupoid $\boldsymbol{Q} = (Q;\ .\ )$ satisfying the identities:

$$x \cdot x = x, \qquad x \cdot y = y \cdot x, \qquad x \cdot (x \cdot y) = y.$$

A squag is called *medial*, if it satisfies the medial law:

$$(x \cdot y) \cdot (z \cdot w) = (x \cdot z) \cdot (y \cdot w).$$

A *Steiner triple system* (briefly *triple system*) $\boldsymbol{P}$ is a pair $(P; B)$, where $P$ is a set of points and $B$ is a set of 3-element subsets of $P$ called blocks such that for distinct points $p_1, p_2 \in P$, there is a unique block $b \in B$ such that $\{p_1, p_2\} \subseteq b$. Triple systems are in 1–1 correspondence with the squags [6,12].

The associated squag $\boldsymbol{Q} = (P;\ .\ )$ with the triple system $\boldsymbol{P} = (P; B)$ is defined by:

$$x \cdot x = x \quad \text{for all } x \in P \text{ and for each pair } \{x, y\} \subseteq P,\ x \cdot y = z \text{ if and only if } \{x, y, z\} \in B\ [6,11].$$

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If the cardinality of $P$ is equal to $n$, then $(P; B)$ and $(P;\ .\ )$ are called of *order n* (or of *cardinality n*), and briefly written $\mathbf{ST}(n)$ and $\mathbf{SQ}(n)$, respectively.

It is well known that the necessary and sufficient condition for an $\mathbf{ST}(n)$ to exist is that $n \equiv 1$ or 3 (mod 6) [6,11]. In fact, there is a 1–1 correspondence between the subsquags (or sub-**SQ**s) of the co-ordinatizing squag $\boldsymbol{Q} = (P;\ .\ )$ and the subspaces (or sub-**ST**s) of the underlying triple system $(P; B)$ [6].

A subsquag $\boldsymbol{N} = (N; .)$ of a squag $\boldsymbol{Q} = (Q; .)$ is called *normal* if and only if $\boldsymbol{N}$ is a congruence class of $\boldsymbol{Q}$ [6,12]. In the following theorem, Quackenbush [12] has given a necessary and sufficient condition for a large subsquag $\boldsymbol{N_1}$ of a finite squag $\boldsymbol{Q}$ to be normal.

**Theorem 1** *([12]). If $\boldsymbol{N_1} = (N_1;\ .\ )$ and $\boldsymbol{N_2} = (N_2;\ .\ )$ are two subsquags of a finite squag $\boldsymbol{Q} = (Q; .)$ such that $N_1 \cap N_2 = \varnothing$ and $|Q| = 3|N_1| = 3|N_2|$, then $\mathbf{N_i}$; for $i = 1, 2$ and 3 are normal subsquags, where $\boldsymbol{N_3} = (N_3; .)$ and $N_3 = Q - (N_1 \cup N_2)$.*

The author [3] has shown that there is a subsquag $\boldsymbol{N_1} = (N_1;\ .\ )$ of a finite squag $\boldsymbol{Q} = (Q;\ .\ )$ with $|Q| = 3|N_1|$ and $\boldsymbol{N_1}$ is not normal. This means that a subsquag $\boldsymbol{N_1} = (N_1;\ .\ )$ of a finite squag $\boldsymbol{Q} = (Q;\ .\ )$ with $|Q| = 3|N_1|$ is normal if and only if the set $Q - N_1$ can be divided into two subsquags of $\boldsymbol{Q}$ of cardinality $|N_1|$.

Quackenbush [12] also proved that squags have permutable, regular, and Lagrangian congruences. Basic concepts of universal algebra and properties of squags can be found in [4,6,7].

A squag is called *simple* if it has only the trivial congruences. Guelzow [8] and the author [2] have constructed examples of non-simple squags (and not medial, of course).

An **ST** is *planer* if it is generated by every triangle and contains a triangle. A planer $\mathbf{ST}(n)$ exists for each $n \geq 7$ and $n \equiv 1$ or 3 (mod 6) [5]. Quackenbush has also shown in the next theorem that the only non-simple finite planar squag has 9 elements.

**Theorem 2** *([12]). Let $(Q; B)$ be a planar $\mathbf{ST}(n)$ and let $\boldsymbol{Q} = (Q;\ .\ )$ be the corresponding squag. Then either $\boldsymbol{Q}$ is simple or $n = 9$.*

Quackenbush [12] has expected that there should be *semi-planar squags* that are simple squags and each of whose triangles either generates the whole squag or the 9-element subsquag. We observe that any planar squag (except of cardinality 9) is semi-planar and the inverse is not true.

The triple system $\mathbf{ST}(n)$ associated with a semi-planar squag $\mathbf{SQ}(n)$ will also be called semi-planar (for much more precision it may be called semi-9-planar). In other words, one may say that a triple system $\mathbf{ST}(n)$ is *semi-planar* if the $\mathbf{ST}(n)$ has no proper $a$-normal subsystems (see [13]) (equivalently, the corresponding $\mathbf{SQ}(n)$ is simple) and each triangle generates a sub-$\mathbf{ST}(9)$ or the whole triple system $\mathbf{ST}(n)$.

Indeed, for $n = 7, 9, 13, 15$ there are only planar squags. In [1] the first author has constructed semi-planar squags of cardinality $n$ for all $n > 9$ and $n \equiv 3$ or 9 (mod 18) having only nontrivial subsquags of cardinality 9.

In this article, we give a recursive construction as $n \to 3n$ for semi-planar squags. This construction permits us to construct semi-planar squags having nontrivial subsquags of cardinality >9. In fact, we may construct a semi-planar squag having only medial subsquags of cardinality $3^m$ for each finite positive integer $m$.

## 2. Construction of semi-planar squags of cardinality 3*n*

In this section we describe the construction of semi-planar squags given in [1]. Let $\boldsymbol{T_i} = (S_i; B_i)$ be a triple system with $\boldsymbol{S_i} = (S_i;\ .\ )$ the corresponding squag for $i = 1, 2$. The direct product $\boldsymbol{T_1} \times \boldsymbol{T_2}$ of the two triple systems can be obtained from the underlying triple system of the direct product squag $\boldsymbol{S_1} \times \boldsymbol{S_2}$ [6].

Let $\boldsymbol{T_0} = (Q_0; B_0)$ be a triple system of cardinality $n$, and let $Q_0 = \{a_1, a_2, \ldots, a_n\}$. We consider the direct product $\boldsymbol{T_0} \times \boldsymbol{C_3}$, where $\boldsymbol{C_3}$ is the $\mathbf{ST}(3)$ on the set $\{1, 2, 3\}$ and $\boldsymbol{I_3}$ is its corresponding squag. The direct product $\boldsymbol{T_0} \times \boldsymbol{C_3} = (Q_1; B_1)$ is formed by the usual tripling of $(Q_0; B_0)$. Namely, $(Q_1; B_1)$ is an $\mathbf{ST}(3n)$, where $Q_1 = Q_0 \times \{1, 2, 3\}$ and the set of triples $B_1$ is obtained by:

$$\begin{aligned} B_1 &= \{\{(a_i, 1), (a_j, 2), (a_k, 3)\} \mid \{a_i, a_j, a_k\} \in B_0 \text{ or } a_i = a_j = a_k\} \\ &\quad \cup \{\{(a_i, i), (a_j, i), (a_k, i)\} \mid \{a_i, a_j, a_k\} \in B_0 \ \&\ i \in \{1, 2, 3\}\}. \end{aligned}$$

We denote the squag $(Q_0;\ ._0)$ associated with $\boldsymbol{T_0}$ by $\boldsymbol{Q_0}$ and the squag $\mathbf{3} \times \boldsymbol{Q_0} = (Q_1; \times) = \boldsymbol{Q_0} \times \boldsymbol{I_3}$ associated with $\boldsymbol{T_1} \times \boldsymbol{C_3} := (Q_1; B_1)$.

Without loss of generality, we may assume that $A_0 = \{a_1, a_2, a_3\}$ is a block of $B_0$, then the triple system $(Q_0; B_0)$ contains the subsystem $(A; R)$, where $A = A_0 \times C_3$ and the set of blocks $R$ obtained by:

$$\begin{aligned} R = &\{\{(a_1, i), (a_2, i), (a_3, i)\} : i \in \{1, 2, 3\}\} \\ &\cup \{\{(x, 1), (x, 2), (x, 3)\} : x \in \{a_1, a_2, a_3\}\} \\ &\cup \{\{(x, i), (y, j), (z, k)\} : \{x, y, z\} = \{a_1, a_2, a_3\} \;\&\; \{i, j, k\} = \{1, 2, 3\}\}. \end{aligned}$$

Define on the subset $A$ the set of triples $H$ as follows:

$$\begin{aligned} H = \{&\{(a_3, 1), (a_3, 2), (a_1, 3)\}, \{(a_2, 1), (a_2, 2), (a_2, 3)\}, \{(a_1, 1), (a_1, 2), (a_3, 3)\}, \\ &\{(a_3, 1), (a_2, 2), (a_1, 1)\}, \{(a_3, 2), (a_2, 3), (a_1, 2)\}, \{(a_1, 3), (a_2, 1), (a_3, 3)\}, \\ &\{(a_3, 1), (a_2, 3), (a_3, 3)\}, \{(a_2, 2), (a_1, 2), (a_1, 3)\}, \{(a_1, 1), (a_2, 1), (a_3, 2)\}, \\ &\{(a_1, 3), (a_2, 3), (a_1, 1)\}, \{(a_2, 2), (a_3, 2), (a_3, 3)\}, \{(a_1, 2), (a_2, 1), (a_3, 1)\}\}. \end{aligned}$$

Each of $(A; R)$ and $(A; H)$ are isomorphic to the affine plane over $\mathbf{GF}(3)$. Note that the block $\{(a_2, 1), (a_2, 2), (a_2, 3)\}$ is the only block lying in the intersection of $R$ and $H$.

Using the replacement property by interchanging the two sets of blocks $R$ and $H$ in $(Q_1; B_1)$, then we get again an $\mathbf{ST}(3n) = (Q_1; \underline{B}_1)$, where $\underline{B}_1 := B_1 - R \cup H$ [6,11]. In fact, the sub-**ST** formed by the direct product of $\{a_1, a_2, a_3\}$ and $\{1, 2, 3\}$ is replaced with an isomorphic copy on the same set of points. We denote the squag associated with the $\mathbf{ST}(3n) = (Q_1; \underline{B}_1)$ by $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0} = (Q_1; \;\boldsymbol{.}\;)$. Observe that the difference between the binary operations '$\times$' and '$\boldsymbol{.}$' depends only on the elements of $A$.

**Theorem 3** (*[1]*). *If $\boldsymbol{Q_0}$ is a planar squag of cardinality n, then the constructed squag $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$ is semi-planar of cardinality* $3\,n$ *for all* $n \neq 3$ *or* 9 *and* $n \equiv 1$ *or* 3 (mod 6).

Moreover, in [1] an example of a semi-planar squag of cardinality 27 was given. According to Theorems 2 and 3, we may say that there is always a semi-planar $\mathbf{SQ}(3\,n)$ for all $n > 3$ and $n \equiv 1$, or 3 (mod 6).

Also, according to the proof of Theorem 3 given in [1] we may directly deduce the following result.

**Corollary 4.** *Any subsquag $\mathbf{S}$ of the constructed semi-planar squag $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$ satisfies that:*

1. *If $|S \cap A| = 3$, then $S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\}$.*
2. *If $S \supset A$, then $\mathbf{S} = \boldsymbol{Q_1}$.*
3. *The only nontrivial subsquags of $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$ are of cardinality* 9.

In the next section, we will discuss the following problem:

Is there a semi-planar squag having nontrivial subsquags of cardinality $>9$?

## 3. Recursive construction of semi-planar squags

According to Theorem 3 and Corollary 4, we may always assume that there is a semi-planar squag of cardinality $n$ having only nontrivial subsquags of cardinality 9, for all $n > 9$ and $n \equiv 3$ or 9 (mod 18). In other words, the subsquags of the constructed semi-planar squag $\boldsymbol{Q_1}$ are exactly of cardinality 1, 3, 9 and $n$. In the next theorem we generalize the results of Theorem 3 and Corollary 4 to construct semi-planar squags of cardinality $3^m\, n$ for $n > 3$ and $n \equiv 1$ or 3 (mod 6) having only medial subsquags at most of cardinality $3^\nu$ for each $\nu = 1, 2, \ldots$ or $m + 1$ and for each positive integer $m$. If a squag $\boldsymbol{Q}$ has only medial subsquags of cardinality $3^{\nu'}$ for each $\nu' \leq \nu$ (i.e.; all subsquags are medial with the maximum cardinality $3^\nu$), we will say that $\boldsymbol{Q}$ is a squag with sub-$\mathbf{SQ}(3)^\nu$s.

We note that if $\boldsymbol{Q_0}$ is planar, then each sub-$\mathbf{SQ}(3)$ of $\boldsymbol{Q_0}$ is only covered by the whole squag $\boldsymbol{Q_0}$, this means that each sub-$\mathbf{SQ}(3)$ of a planar squag $\boldsymbol{Q_0}$ is a maximal subsquag in $\boldsymbol{Q_0}$. We attempt in the next lemma to show that the constructed semi-planar $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$ has a sub-$\mathbf{SQ}(3) = \boldsymbol{A_1}$ satisfying that the only subsquag covering $\boldsymbol{A_1}$ is $\boldsymbol{Q_1}$.

**Lemma 5.** *Let $\boldsymbol{Q_0}$ be a planar squag of cardinality $n \neq 9$, then the constructed semi-planar squag $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$ has a maximal sub-*$\mathbf{SQ}(3)$*; i.e. $\boldsymbol{Q_1}$ has a sub-*$\mathbf{SQ}(3)$ *covered only by $\boldsymbol{Q_1}$.*

**Proof.** Let $A_1 = \{(a_1, 1), (b, 1), (c, 1)\}$ be a subsquag in $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$ with $b, c \notin A_0 = \{a_1, a_2, a_3\}$. We want to prove that the only subsquag containing $A_1$ is $\boldsymbol{Q_1}$.

Assume that there is a subsquag $\boldsymbol{S} \supset A_1$. Then we have two cases:
**(i)** $S \cap A$ has more than one element or **(ii)** $S \cap A = \{(a_1, 1)\}$, where $\boldsymbol{A_0}$ is a subsquag in $\boldsymbol{Q_0}$ and $\boldsymbol{A} = \boldsymbol{A_0} \times \boldsymbol{I_3}$ is given as in the construction $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$.

**For the case (i)**: We may say that $|S \cap A| = 3$ or 9. But we have $(a_1, 1) \in S \cap A$, hence if $|S \cap A| = 3$, then $S \cap A$ is a sub-$\mathbf{SQ}(3) \neq \{(a_2, 1), (a_2, 2), (a_2, 3)\}$ contradicting Corollary 4 that the only possible case with the condition $|S \cap A| = 3$ is $S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\}$.

If $|S \cap A| = 9$, then $S \supseteq A$, moreover $\boldsymbol{S} \supset A_1$, hence $|S| > 9$. Again, according to Corollary 4, the only subsquag containing $A$ with cardinality $>9$ is the whole squag, hence $\boldsymbol{S} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$.

**For the second case (ii)** $S \cap A = \{(a_1, 1)\}$: Since $\boldsymbol{S} \supset A_1$, there is an element $(x, i) \in S - A_1$. Hence we have 3 possible cases:

$$\textbf{(1)}\ (x, i) = (x, 1), \qquad \textbf{(2)}\ (x, i) = (x, 2) \quad \text{or} \quad \textbf{(3)}\ (x, i) = (x, 3).$$

**For the case (1)**: Since $(x, 1) \notin A_1$, it follows that $x \notin \{a_1, b, c\}$. Hence $S$ contains the 4-element subset $\{(x, 1), (a_1, 1), (b, 1), (c, 1)\}$, this means that $|S| = 9$ or $\boldsymbol{S} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$. But $|S| = 9$ means that the number of elements of the first components of $\boldsymbol{S}$ is greater than 3, which contradicts the fact that $\boldsymbol{Q_0}$ is planar. Then $\boldsymbol{S} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$.

**Case (2):** For $(x, i) = (x, 2)$, we have two cases $x \notin \{a_1, b, c\}$ or $x \in \{a_1, b, c\}$. If $x \notin \{a_1, b, c\}$, then $S$ contains the 4 distinct elements $\{(x, 2), (a_1, 1), (b, 1), (c, 1)\}$. Hence the set of the first components $\mathbf{P_1}(\boldsymbol{S})$ of $\boldsymbol{S}$ forms a subsquag of cardinality $>4$ of $\boldsymbol{Q_0}$. Since $\boldsymbol{Q_0}$ is planar, it follows that $\mathbf{P}_1(S) = \boldsymbol{Q_0}$. Hence $\boldsymbol{S} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$ for the same reason given in the preceding case.

If $x \in \{a_1, b, c\}$, then $x = b$ or $c$ because of $(x, i) \in S - A_1$ and $S \cap A = \{(a_1, 1)\}$. But $x = b$ or $c$ tends to $(b, 2) \cdot (c, 1) = (a_1, 3)$ or $(c, 2) \cdot (b, 1) = (a_1, 3) \in \boldsymbol{S}$ which contradicts the assumption that $S \cap A = \{(a_1, 1)\}$.

The same discussion holds for the **case (3):** $(x, i) = (x, 3)$. Then the only possible case for a subsquag $\boldsymbol{S}$ containing the block $\{(a_1, 1), (b, 1), (c, 1)\}$ is the whole squag $\mathbf{3} \otimes_A \boldsymbol{Q_0}$. This completes the proof. $\square$

Note that we may prove the same result if we choose $A_1 = \{(a_1, 1), (b, 2), (c, 3)\}$ with $b, c \notin A_0 = \{a_1, a_2, a_3\}$.

For $n = 9$, the author [1] has constructed an example of a semi-planar $\mathbf{SQ}(27)$. It is also easy to find a sub-$\mathbf{SQ}(3) = \boldsymbol{A_1}$ covered by the whole squag $\mathbf{SQ}(27)$.

In the next theorem we assume that $\boldsymbol{Q_m}$ is a semi-planar squag and $\mathbf{3} \times \boldsymbol{Q_m} := (Q_{m+1}; \times)$ is the direct product squag $\boldsymbol{Q_m} \times \boldsymbol{I_3}$. For any sub-$\mathbf{SQ}(3) = \boldsymbol{A_m}$ of $\boldsymbol{Q_m}$ the set $A = A_m \times I_3$ forms a sub-$\mathbf{SQ}(9)$ in $\mathbf{3} \times \boldsymbol{Q_m} = (\boldsymbol{Q_{m+1}}; \times)$. Consider the same subsystems $(A; R)$ and $(A; H)$ exactly as in the construction $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$. So the construction $\boldsymbol{Q_{m+1}} = \mathbf{3} \otimes_A \boldsymbol{Q_m}$ is still a squag (but not necessarily be semi-planar). Indeed, if we choose a sub-$\mathbf{SQ}(3) = \boldsymbol{A_m} = \{a_1, a_2, a_3\}$ of $\boldsymbol{Q_m}$ satisfying that $\boldsymbol{A_m}$ is only covered by the whole squag $\boldsymbol{Q_m}$, then we will show in the next two theorems that the constructed squag $\boldsymbol{Q_{m+1}} = \mathbf{3} \otimes_A \boldsymbol{Q_m}$ is semi-planar and preserves the first two properties as in Corollary 4.

**Lemma 6.** *Assume that $\boldsymbol{Q_m}$ is a semi-planar squag having a sub-$\mathbf{SQ}(3)$ $\boldsymbol{A_m} = \{a_1, a_2, a_3\}$ covered only by $\boldsymbol{Q_m}$. Let $\boldsymbol{S}$ be a subsquag of the constructed squag $\boldsymbol{Q_{m+1}} = \mathbf{3} \otimes_A \boldsymbol{Q_m}$, It follows that if $|S \cap A| = 3$, then $S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\}$.*

**Proof.** Let $\boldsymbol{S} \cap \boldsymbol{A}$ be a 3-element subsquag $\neq \{(a_2, 1), (a_2, 2), (a_2, 3)\}$. We will show that any other choice of $S \cap A$ leads to a contradiction. The intersection $S \cap A$ belongs to one of two essential cases.

$$\begin{aligned} \text{First, let } S \cap A = {} & \{(a_1, 1), (a_2, 2), (a_3, 1)\}, \{(a_1, 3), (a_2, 1), (a_3, 3)\}, \\ & \{(a_1, 2), (a_2, 3), (a_3, 2)\}, \{(a_1, 1), (a_2, 1), (a_3, 2)\} \text{ or} \\ & \{(a_1, 2), (a_2, 1), (a_3, 1)\}. \end{aligned}$$

In this case, the set of the second components of the elements of $S \cap A$ is a 2-element subset of $\{1, 2, 3\}$. But one can easily see that the set of second components of the elements of $S - A$ consists of $\{1, 2, 3\}$ for any choice of $S \cap A$.

Let $\{i, j, k\} = \{1, 2, 3\}$ and let the maximum number of distinct elements of $S$, having second components $i$, be equal to $r$. Let the values of second components of the sub-$\mathbf{SQ}(3)$ of $S \cap A$ be $i$, $i$ and $j$. If $(y, k) \in S - A$, then the product of any element $(x, i)$ of $S$ by $(y, k)$ gives an element of $S$ having a second component $j$; i.e. $(x, i) \cdot (y, k) = (z, j)$. This means that $S$ contains exactly $r$ elements having second components equal to $j$. Also, let $(y, j) \in S - A$, then the product $(x, i) \cdot (y, j) = (z, k)$, which means that $S$ contains also $r$ distinct elements having second components equal to $k$. Accordingly, we may deduce that $S$ consists exactly of an $r$-element subset of pairs with second components $i$, an $r$-element subset of pairs with second components $j$ and an $r$-element subset of pairs with second components $k$.

Since the second components of $S \cap A$ are $i$, $i$ and $j$, each of the $r$-element subsets of $S$ with second components $j$ or $k$ forms a subsquag of $S$. According to Theorem 1, the third $r$-element subset of $S$ with second component equal to $i$ must be a subsquag of $S$, contradicting the choice that $S \cap A = \{(a_1, i), (a_2, i), (a_3, j)\}$. Therefore, this case is ruled out.

$$\text{Next, let } S \cap A = \{(a_1, 1), (a_1, 2), (a_3, 3)\}, \{(a_3, 1), (a_2, 3), (a_3, 3)\},$$
$$\{(a_2, 2), (a_1, 2), (a_1, 3)\}, \{(a_1, 1), (a_2, 3), (a_1, 3)\},$$
$$\{(a_3, 1), (a_3, 2), (a_1, 3)\} \text{ or } \{(a_2, 2), (a_3, 2), (a_3, 3)\}.$$

For any choice of $S \cap A$, the set of second components of the elements of $S \cap A$ is the set $\{1, 2, 3\}$ or a 2-element subset of $\{1, 2, 3\}$. But one can directly see that the set of second components of the elements of $S - A$ consists of all elements of $\{1, 2, 3\}$. Also, the index set of the first components of any choice of $S \cap A$ is a 2-element subset of $\{1, 2, 3\}$.

Let $\{i, j, k\} = \{1, 2, 3\}$ and let $S \cap A = \{(a_i, l), (a_i, n), (a_j, m)\}$ with $l \neq n$ and $l, n, m \in \{1, 2, 3\}$.

Let $(b, l) \in S - A$ and $(b, l) \times (a_i, l) = (b, l) \,.\, (a_i, l) = (c, l)$, then $(b, l) \times (a_i, n) = (b, l) \,.\, (a_i, n) = (c, k)$ with $\{l, n, k\} = \{1, 2, 3\}$. Hence $(c, l) \times (c, k) = (c, l) \,.\, (c, k) = (c, n)$ and accordingly $(c, n) \times (b, l) = (c, n) \,.\, (b, l) = (a_i, k) \in S$. On the other side, we have $(a_i, k) \in A$ contradicting the choice that $S \cap A = \{(a_i, l), (a_i, n), (a_j, m)\}$ and $\{l, n, k\} = \{1, 2, 3\}$. We will get the same contradiction, if we choose $(b, n)$ or $(b, k) \in S - A$ instated of $(b, l)$. Therefore, the second possible case of $S \cap A$ is also ruled out.

This means that the only possible case for a subsquag $\boldsymbol{S}$ of $\boldsymbol{Q_{m+1}}$ with $|S \cap A| = 3$ is $S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\}$. This completes the proof of the lemma. □

The next theorem permits us to construct semi-planar squags of cardinality $3^m n$ for $n \neq 3$ or 9 and each $n \equiv 1, 3 \pmod 6$ with subsquags $\cong \mathbf{SQ}(3)^\nu$ for all positive integers $m$ and $\nu$ with $\nu \leq m + 1$.

**Theorem 7.** *Let $\boldsymbol{Q_m}$ be a semi-planar squag of cardinality $3^m\, n$ with sub-$\mathbf{SQ}(3)^\nu$s for some $\nu = 1, 2, \ldots$ or $m + 1$. Also, let $\boldsymbol{Q_m}$ have a sub-$\mathbf{SQ}(3) = \boldsymbol{A_m}$ covered by $\boldsymbol{Q_m}$. Then the constructed squag $\boldsymbol{Q_{m+1}} = \mathbf{3} \otimes_{\boldsymbol{A}} \boldsymbol{Q_m}$ is semi-planar of cardinality $3^{m+1}\, n$ with sub-$\mathbf{SQ}(3)^{\nu+1}$s for all possible values $n \neq 3$ or 9 and $n \equiv 1, 3 \pmod 6$.*

**Proof.** Let $\boldsymbol{S}$ be a subsquag of $\boldsymbol{Q_{m+1}}$ and $S$ has more than one element. First we have to prove that $\boldsymbol{S}$ is a sub-$\mathbf{SQ}(3)^{\nu+1}$ or $\boldsymbol{S} = \boldsymbol{Q_{m+1}}$. Let $A_m = \{a_1, a_2, a_3\}$ and $A = A_m \times \boldsymbol{C_3} = \{a_1, a_2, a_3\} \times \{1, 2, 3\}$. And choose the two sets of triples $R$ and $H$ that are defined exactly as in the construction $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$.

In general, there are only four essential cases for the relation between the subsquags $\boldsymbol{S}$ and $\boldsymbol{A}$:

**(i)** $S \subseteq A$, **(ii)** $|S \cap A| = 0, 1$ or $S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\}$

**(iii)** $|S \cap A| = 3$, **(iv)** $A \subset S$.

**(i)** If $S \subseteq A$, then $\boldsymbol{S}$ is a sub-$\mathbf{SQ}(3)$ or a sub-$\mathbf{SQ}(9)$.

**(ii)** If $|S \cap A| = 0, 1$ or $S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\}$, then $\boldsymbol{S}$ is a subsquag in $\boldsymbol{Q_{m+1}}$ so also in $\mathbf{3} \times \boldsymbol{Q_m} = \boldsymbol{Q_m} \times \{1, 2, 3\}$, hence the set of first components of $\boldsymbol{S}$ is a subsquag of $\boldsymbol{Q_m}$. But in these three cases given in **(ii)**, the set of first components of $S \neq \boldsymbol{Q_m}$ because the number of elements of the first component of $S \cap A < 3$. Hence the set of the first components of $\boldsymbol{S}$ forms a medial subsquag at most of cardinality $3^\nu$ of $\boldsymbol{Q_m}$ for some positive integer $\nu \leq m+1$. Also, the second projection $\mathbf{P_2}(\boldsymbol{S})$ consists of one element or three elements, then we may say that $\boldsymbol{S} \cong \mathbf{SQ}(3)^\nu$ or $\mathbf{SQ}(3)^\nu \times \{1, 2, 3\} = \mathbf{SQ}(3)^{\nu+1}$ of $\mathbf{3} \times \boldsymbol{Q_m}$ and so also of $\boldsymbol{Q_{m+1}}$.

**(iii)** According to the above lemma, we may say directly that $S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\}$. Then we get the same result as in the preceding case.

**(iv)** Note that the sub-$\mathbf{SQ}(3)$ $\boldsymbol{A_m} = \{a_1, a_2, a_3\}$ is covered by $\boldsymbol{Q_m}$. So $\boldsymbol{A} = \boldsymbol{A_m} \times \boldsymbol{I_3}$ is a sub-$\mathbf{SQ}(9)$ of $\mathbf{3} \times \boldsymbol{Q_m}$ satisfying that the only subsquag containing $A$ is $\mathbf{3} \times \boldsymbol{Q_m}$.

Let $\boldsymbol{S} = (S; \,.\,)$ be a subsquag satisfying $S \supset A$ and with $(S; B_S)$ the corresponding $\mathbf{ST}$, then $(S; B_S - H \cup R)$ is a sub-$\mathbf{ST}$ of $(Q; B)$ ($(Q; B)$ is the $\mathbf{ST}$ associated with $\mathbf{3} \times \boldsymbol{Q_m}$). As a consequence, the squag $\boldsymbol{S'} = (S, \times)$ associated with $(S; B_S - H \cup R)$ is a subsquag of $\mathbf{3} \times \boldsymbol{Q_m}$ containing $A$. Since $S \supset A$, it follows that $\boldsymbol{S'}$ is equal to $\mathbf{3} \times \boldsymbol{Q_m}$. But the subsquag $\boldsymbol{S}$ in $\boldsymbol{Q_{m+1}}$ has the same set of points as the subsquag $\boldsymbol{S'}$ in $\mathbf{3} \times \boldsymbol{Q_m}$. Because $\boldsymbol{S}$ differs only from

$\boldsymbol{S'}$ in the binary operations ' $\times$' and ' **.** '. This implies that $\boldsymbol{S}$ is equal to $\boldsymbol{Q_{m+1}}$. So we may say that the only subsquag containing $A$ is the whole squag $\boldsymbol{Q_{m+1}}$.

Finally we may say that the semi-planar squag $\boldsymbol{Q_{m+1}}$ has only medial proper subsquags at most of cardinality $3^{\nu+1}$ for $\nu \le m+1$. This completes the proof of the first part of the theorem.

Now, we need only show that $\boldsymbol{Q_{m+1}}$ is a simple squag. Assume that $\boldsymbol{Q_{m+1}}$ has a proper congruence $\theta$, since $[(x,i)]\theta$ is a subsquag of $\boldsymbol{Q_{m+1}}$, then $[(x,i)]\theta$ must be a medial subsquag $\cong \mathbf{SQ}(3)^{\nu'}$ with $\nu'-1 \le \nu \le m+1$. On the other hand, for any 3-element subsquag $X$ of $\boldsymbol{Q_{m+1}}$ the set $[X]\theta$ forms a proper subsquag of $\boldsymbol{Q_{m+1}}$ and so forms a medial subsquag. Hence, if $\nu'-1=\nu=m+1$, then $[(x,i)]\theta = \mathbf{SQ}(3)^{m+2}$. But we may choose a 3-element subsquag $X$ satisfying that the subsquag $[X]\theta$ is of cardinality $\mathbf{SQ}(3)^{m+3}$. This is a contradiction. Therefore, in light of the first part of the proof, we may deduce that $[(x,i)]\theta = \mathbf{SQ}(3)^{\nu'}$ with $\nu' \le \nu$ for some positive integer $\nu \le m+1$.

If $[(x,i)]\theta = \mathbf{SQ}(3)^{\nu'}$ with $2 \le \nu' \le \nu$, so for $(a,1) \in A$, we have three cases: **(a)** $|[(a,1)]\theta \cap A| = 1$, **(b)** $|[(a,1)]\theta \cap A| = 3$ or **(c)** $A \subseteq [(a,1)]\theta$.

**For case (a):** $[A]\theta$ is a proper subsquag of $\boldsymbol{Q_{m+1}}$, so $[A]\theta \cong \mathbf{SQ}(3)^{\nu'+2}$. According to the first part of the theorem if $\nu'=\nu=m$ or $m+1$, it contradicts the preceding fact that the maximum cardinality of subsquags is $3^{m+1}$. In general, for $2 \le \nu' \le \nu$, the subset $S_1 = \{(a_1,1),(a_1,2),(a_3,3)\}$ forms a sub-$\mathbf{SQ}(3) \subseteq A$, then $[\boldsymbol{S_1}]\theta$ is a sub-$\mathbf{SQ}$ intersects $A$ in $\boldsymbol{S_1}$ which contradicts the result of Lemma 6.

**For case (b):** According to Lemma 6, we have $[(a_2,1)]\theta \cap A = \{(a_2,1),(a_2,2),(a_2,3)\}$. Again the set $B = (a_1,1)\ .\ [(a_2,1)]\theta$ is a medial subsquag and $\cong \mathbf{SQ}(3)^{\nu'}$. But $(a_1,1)\ .\ \{(a_2,1),(a_2,2),(a_2,3)\} = \{(a_3,2),(a_3,1),(a_1,3)\} \subseteq B$ contradicts the result of Lemma 6 that $B \cap A$ must be equal to $\{(a_2,1),(a_2,2),(a_2,3)\}$.

**For case (c):** $A \subseteq [(a,1)]\theta$, but according to case **(iv)** of the first part we may say that $[(a,1)]\theta$ is equal to $\boldsymbol{A}$ or $\boldsymbol{Q_{m+1}}$. If $[(a,1)]\theta$ is equal to $\boldsymbol{A}$ we choose $(b,2) \notin A$, then $(b,2)\ .\ [(a,1)]\theta$ is a sub-$\mathbf{SQ}(3)^2$ i.e.; $(b,2)\ .\ [(a,1)]\theta$ is a medial subsquag and so it is distributive. But $(b,2)\ .\ ((a_1,1)\ .\ (a_2,1)) = (b,2)\ .\ (a_3,2) = (c,2)$ and $((b,2)\ .\ (a_1,1))\ .\ ((b,2)\ .\ (a_2,1)) = (c_1,3)\ .\ (c_2,3) = (c_3,3)$, which contradicts the fact that $(b,2)\ .\ [(a,1)]\theta$ must be distributive. Consequently, these three cases **(a)**, **(b)** and **(c)** are ruled out.

**If** $|[(x,i)]\theta| = 3$, so for $(a,1) \in A$, we have two essential cases:

**(I)** $|[(a,1)]\theta \cap A| = 1$, or **(II)** $|[(a,1)]\theta \cap A| = 3$.

**For case (I)**: Choose a block $X \in H - \{(a_2,1),(a_2,2),(a_2,3)\}$, the subsquag $[X]\theta$ is an $\mathbf{SQ}(9)$ and $[X]\theta \cap A$ is a block $\in H - \{(a_2,1),(a_2,2),(a_2,3)\}$, which contradicts the result of Lemma 6.

**Case (II)** tends to $[(a,1)]\theta \cap A = [(a,1)]\theta$, then $\theta_A$ is a nontrivial congruence on the sub-$\mathbf{SQ}(9) = \boldsymbol{A}$. Then we may choose $(a_i,j) \in A - \{(a_2,1),(a_2,2),(a_2,3)\}$, so $[(a_i,j)]\theta \in H - \{(a_2,1),(a_2,2),(a_2,3)\}$. By choosing a block $X$ satisfying $X \cap A = (a_i,j)$, hence $[X]\theta$ is an $\mathbf{SQ}(9)$ containing a block in the set $H - \{(a_2,1),(a_2,2),(a_2,3)\}$. That contradicts again the result of Lemma 6.

Therefore, the constructed squag $\boldsymbol{Q_{m+1}} = \mathbf{3} \otimes_A \boldsymbol{Q_m}$ is simple. This completes the proof of the theorem. □

Now, we may say that the planar squag $\boldsymbol{Q_0}$ is semi-planar with sub-$\mathbf{SQ}(3)$s and has a sub-$\mathbf{SQ}(3) = \boldsymbol{A_0}$ covered by $\boldsymbol{Q_0}$ (indeed each sub-$\mathbf{SQ}(3)$ covered by $\boldsymbol{Q_0}$). According to Theorem 3, Corollary 4 and Lemma 5, the constructed squag $\boldsymbol{Q_1} = \mathbf{3} \otimes_A \boldsymbol{Q_0}$ is semi-planar with sub-$\mathbf{SQ}(3)^2$s and has a sub-$\mathbf{SQ}(3) = \boldsymbol{A_1}$ covered by $\boldsymbol{Q_1}$. To complete the requirements of the mathematical induction we have to prove that if the squag $\boldsymbol{Q_m}$ is semi-planar with sub-$\mathbf{SQ}(3)^{\nu}$s for some positive integer $\nu \le m+1$ and has a sub-$\mathbf{SQ}(3) = \boldsymbol{A_m}$ covered by $\boldsymbol{Q_m}$, then the constructed $\boldsymbol{Q_{m+1}} = \mathbf{3} \otimes_A \boldsymbol{Q_m}$ is semi-planar with sub-$\mathbf{SQ}(3)^{\nu+1}$s and has also a sub-$\mathbf{SQ}(3) = \boldsymbol{A_{m+1}}$ covered by $\boldsymbol{Q_{m+1}}$. The first part is already established by Theorem 7 and the second part will be proven in the next theorem.

**Theorem 8.** *Let $\boldsymbol{Q_m}$ be a semi-planar squag with sub-$\mathbf{SQ}(3)^{\nu}$s for some $\nu = 1,2,\ldots,m+1$ having a sub-$\mathbf{SQ}(3) = \boldsymbol{A_m}$ covered by $\boldsymbol{Q_m}$. Then the constructed semi-planar squag $\boldsymbol{Q_{m+1}} = \mathbf{3} \otimes_A \boldsymbol{Q_m}$ has also a sub-$\mathbf{SQ}(3) = \boldsymbol{A_{m+1}}$ covered by $\boldsymbol{Q_{m+1}}$.*

**Proof.** Let $A_m = \{a_1,a_2,a_3\}$ be a sub-$\mathbf{SQ}(3)$ of $\boldsymbol{Q_m}$ covered only by $\boldsymbol{Q_m}$. Also, let $A$ be the set $A_m \times \{1,2,3\}$ of $\boldsymbol{Q_{m+1}}$. According to Theorem 7, the construction $\boldsymbol{Q_{m+1}} = \mathbf{3} \otimes_A \boldsymbol{Q_m}$ is a semi-planar squag having sub-$\mathbf{SQ}(3)^{\nu+1}$s. Consider the subsquag $A_{m+1} = \{(a_1,1),(b,2),(c,3)\}$ of $\boldsymbol{Q_{m+1}}$ with $b$ and $c \notin A_m$. We will show that the only subsquag containing $A_{m+1}$ is $\boldsymbol{Q_{m+1}}$.

Suppose $\boldsymbol{S}$ be a subsquag of $\boldsymbol{Q_{m+1}}$ containing $A_{m+1}$. We have three cases:

**(i)** $S \cap A = A$, **(ii)** $|S \cap A| = 3$ or **(iii)** $S \cap A = \{(a_1,1)\}$.

**Case (i)** $S \cap A = A$ means that $\boldsymbol{S} \supset A$, according to the result of case **(iv)** of the proof of Theorem 7 that $\boldsymbol{Q_{m+1}}$ is the only subsquag containing $A$, hence $\boldsymbol{S} = \boldsymbol{Q_{m+1}}$.

**Case (ii)** $|\boldsymbol{S} \cap A| = 3$, according to Lemma 6 we have $S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\}$, which contradicts that $(a_1, 1) \in \boldsymbol{S} \cap A$. Hence, this case is ruled out.

**For the case (iii):** If $\boldsymbol{S} \cap A = \{(a_1, 1)\}$, then $\boldsymbol{S}$ is a subsquag of $\mathbf{3} \times \boldsymbol{Q_m}$ and also of $\boldsymbol{Q_{m+1}}$. Assume that $|S| > 3$. According to Theorem 7, $\boldsymbol{S} \cong \mathbf{SQ}(3)^{\nu'+1}$ for $1 \leq \nu' \leq \nu \leq m+1$. Let $\mathbf{P_1}(S)$ and $\mathbf{P_2}(S)$ be the projections of $S$ on the first and second components, respectively. Then $\mathbf{P_1}(\boldsymbol{S}) \cong \mathbf{SQ}(3)^{\nu'}$ for $\nu' \leq \nu$ and $\mathbf{P_2}(S) = \{1, 2, 3\}$. Note that $\boldsymbol{S}$ contains more than one element with second component $i = 1, 2$ or 3. Similar to the proof of case **(ii)** of the preceding theorem, we may say that $\boldsymbol{S} \cong \mathbf{SQ}(3)^{\nu'+1} = \mathbf{SQ}(3)^{\nu'} \times \{1, 2, 3\}$, where $\mathbf{SQ}(3)^{\nu'+1}$ is a subsquag of both $\mathbf{3} \times \boldsymbol{Q_m}$ and $\boldsymbol{Q_{m+1}}$. Also, we have $\mathbf{P_1}(S) \cap \mathbf{P_1}(A) = \{a_1\}$, so $S \cap A = \{(a_1, 1), (a_1, 2), (a_1, 3)\}$, which is impossible. Because of $S \cap A = \{(a_1, 1)\}$ and also the result of Lemma 6 says that the set $\{(a_1, 1), (a_1, 2), (a_1, 3)\}$ does not form a subsquag of $\boldsymbol{Q_{m+1}}$. Therefore, this case is also ruled out.

This means that the only possible case for $\boldsymbol{S} \supseteq A_{m+1}$ is $\boldsymbol{S} \supseteq A$. So we go back to the result of case **(i)** that $\boldsymbol{S}$ must be equal to $\boldsymbol{Q_{m+1}}$. This completes the proof of the lemma. □

According to the 1–1 correspondence between squags and triple systems, we may say that there are semi-planar $\mathbf{TS}(3^m\, n)$s having only subsystems $\cong \mathbf{ST}(3)^\nu$s for each positive integer $\nu \leq m+1$ and for each possible number $n$ ($n \neq 9$ and $n \equiv 1$ or 3 (mod 6)). These triple systems satisfy that each triangle generates a sub-$\mathbf{ST}(9)$ or the whole triple system and whose corresponding squag is simple.

For $n = 9$, the author [1] has constructed an example of semi-planar squag of cardinality 27. Also, it easy to find a sub-$\mathbf{SQ}(3) = A_1$ covered by the whole squag, so we may apply Lemma 6, Theorems 7 and 8 to get a semi-planar squag of cardinality 81 having only medial subsquags of cardinality $3^3$ at most. Equivalently, there are three semi-planar $\mathbf{ST}(81)$s with subsystems $\cong \mathbf{ST}(3)^\nu$s for $\nu = 1, 2$ and 3.

Finally, one may say that there is a semi-planar $\mathbf{SQ}(3^m\, n) := \boldsymbol{Q_{m,\nu}}$ for all $n > 3$ and $n \equiv 1$ or 3 (mod 6) and each positive integer $m$ with medial subsquags of maximum cardinality $3^\nu$ for each positive integer $\nu \leq m+1$.

Quackenbush [12] proved that the variety $\mathbf{V}(\boldsymbol{Q})$ generated by a simple planar squag $\boldsymbol{Q}$ has only two subdirectly irreducible squags $\boldsymbol{Q}$ and the 3-element squag $\mathbf{SQ}(3)$ and then $\mathbf{V}(\boldsymbol{Q})$ covers the smallest nontrivial subvariety (the class of all medial squags).

Similarly, if $\boldsymbol{Q_{m,\nu}} = \mathbf{SQ}(3^m\, n)$ is a semi-planar squag having only medial subsquags of cardinality $3^\nu$ at most, then one can prove that the variety $\mathbf{V}(\boldsymbol{Q_{m,\nu}})$ generated by $\boldsymbol{Q_{m,\nu}}$ has only two subdirectly irreducible squags $\boldsymbol{Q_{m,\nu}}$ and the 3-element squag $\mathbf{SQ}(3)$. And hence we deduce the same result that each semi-planar squag $\boldsymbol{Q_{m,\nu}}$ with sub-$\mathbf{SQ}(3)^\nu$s for each positive integer $\nu \leq m+1$ generates a variety $\mathbf{V}(\boldsymbol{Q_{m,\nu}})$ which covers also the smallest nontrivial subvariety (the class of all medial squags).

Hall [9] constructed a Steiner triple system in which each triangle generates a sub-$\mathbf{ST}(9)$, such a class is called Hall triple systems. The corresponding squags of such class is the class of distributive squags. Klossik [10] gave a construction of distributive squags as a vector space over $\mathbf{GF}(3)$ of dimension $\geq 4$. Using the interchange property to inject a distributive subsquag $\mathbf{SQ}(3^\nu)$ instead of a medial sub-SQ$(3)^\nu$, we get a construction of a squag having distributive subsquags but not medial. Consequently, we are faced with the question:

Is there a semi-planar squag having distributive (not medial) subsquags?

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