On semi-planar Steiner quasigroups

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Abstract

A Steiner triple system (briefly \textbf{ST}) is in 1–1 correspondence with a Steiner quasigroup or squag (briefly \textbf{SQ}) [B. Ganter, H. Werner, Co-ordinatizing Steiner systems, Ann. Discrete Math. 7 (1980) 3–24; C.C. Lindner, A. Rosa, Steiner quadruple systems: A survey, Discrete Math. 21 (1979) 147–181]. It is well known that for each \(n \equiv 1\) or \(3\) (mod 6) there is a planar squag of cardinality \(n\) [J. Doyen, Sur la structure de certains systems triples de Steiner, Math. Z. 111 (1969) 289–300]. Quackenbush expected that there should also be semi-planar squags [R.W. Quackenbush, Varieties of Steiner loops and Steiner quasigroups, Canad. J. Math. 28 (1976) 1187–1198]. A simple squag is semi-planar if every triangle either generates the whole squag or the 9-element squag. The first author has constructed a semi-planar squag of cardinality 3\(n\) for all \(n > 3\) and \(n \equiv 1\) or \(3\) (mod 6) [M.H. Armanious, Semi-planar Steiner quasigroups of cardinality 3\(n\), Australas. J. Combin. 27 (2003) 13–27]. In fact, this construction supplies us with semi-planar squags having only nontrivial subsquags of cardinality 9. Our aim in this article is to give a recursive construction as \(n \to 3n\) for semi-planar squags. This construction permits us to construct semi-planar squags having nontrivial subsquags of cardinality \(> 9\). Consequently, we may say that there are semi-planar SQ\((3^v n)\)s (or semi-planar ST\((3^v n)\)s) for each positive integer \(m\) and each \(n \equiv 1\) or \(3\) (mod 6) with \(n > 3\) having only medial subsquags at most of cardinality 3\(v\) (sub-ST\((3^v)\)) for each \(v \in \{1, 2, \ldots, m + 1\}\).

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1. Introduction

A \textit{Steiner quasigroup} (or a \textit{squag}) is a groupoid \(Q = (Q; .)\) satisfying the identities:

\[ x \cdot x = x, \quad x \cdot y = y \cdot x, \quad x \cdot (x \cdot y) = y. \]

A squag is called \textit{medial}, if it satisfies the medial law:

\[ (x \cdot y) \cdot (z \cdot w) = (x \cdot z) \cdot (y \cdot w). \]

A \textit{Steiner triple system} (briefly \textit{triple system}) \(P\) is a pair \((P; B)\), where \(P\) is a set of points and \(B\) is a set of 3-element subsets of \(P\) called blocks such that for distinct points \(p_1, p_2 \in P\), there is a unique block \(b \in B\) such that \(\{p_1, p_2\} \subseteq b\). Triple systems are in 1–1 correspondence with the squags [6,12].

The associated squag \(Q = (P; .)\) with the triple system \(P = (P; B)\) is defined by:

\[ x \cdot x = x \quad \text{for all } x \in P \text{ and for each pair } \{x, y\} \subseteq P, x \cdot y = z \text{ if and only if } \{x, y, z\} \in B \text{ [6,11].} \]
If the cardinality of $P$ is equal to $n$, then $(P; B)$ and $(P; r)$ are called of order $n$ (or of cardinality $n$), and briefly written $ST(n)$ and $SQ(n)$, respectively.

It is well known that the necessary and sufficient condition for an $ST(n)$ to exist is that $n \equiv 1$ or $3 \mod 6$ [6,11]. In fact, there is a 1–1 correspondence between the subsquags (or sub-$SQ$s) of the co-ordinatizing squag $Q = (P; r)$ and the subspaces (or sub-$ST$s) of the underlying triple system $(P; B)$ [6].

A subsquag $N = (N; \cdot)$ of a squag $Q = (Q; \cdot)$ is called normal if and only if $N$ is a congruence class of $Q$ [6,12]. In the following theorem, Quackenbush [12] has given a necessary and sufficient condition for a large subsquag $N_1$ of a finite squag $Q$ to be normal.

**Theorem 1** ([12]). If $N_1 = (N_1; \cdot)$ and $N_2 = (N_2; \cdot)$ are two subsquags of a finite squag $Q = (Q; \cdot)$ such that $N_1 \cap N_2 = \emptyset$ and $|Q| = 3|N_1| = 3|N_2|$, then $N_i$; for $i = 1, 2$ and $3$ are normal subsquags, where $N_3 = (N_3; \cdot)$ and $N_3 = Q - (N_1 \cup N_2)$.

The author [3] has shown that there is a subsquag $N_1 = (N_1; \cdot)$ of a finite squag $Q = (Q; \cdot)$ with $|Q| = 3|N_1|$ and $N_1$ is not normal. This means that a subsquag $N_1 = (N_1; \cdot)$ of a finite squag $Q = (Q; \cdot)$ with $|Q| = 3|N_1|$ is normal if and only if the set $Q - N_1$ can be divided into two subsquags of $Q$ of cardinality $|N_1|$.

Quackenbush [12] also proved that squags have permutable, regular, and Lagrangian congruences. Basic concepts of universal algebra and properties of squags can be found in [4,6,7].

A squag is called simple if it has only the trivial congruences. Guelzow [8] and the author [2] have constructed examples of non-simple squags (and not medial, of course).

An $ST$ is planar if it is generated by every triangle and contains a triangle. A planar $ST(n)$ exists for each $n \geq 7$ and $n \equiv 1$ or $3 \mod 6$ [5]. Quackenbush has also shown in the next theorem that the only non-simple finite planar squag has 9 elements.

**Theorem 2** ([12]). Let $(Q; B)$ be a planar $ST(n)$ and let $Q = (Q; \cdot)$ be the corresponding squag. Then either $Q$ is simple or $n = 9$.

Quackenbush [12] has expected that there should be semi-planar squags that are simple squags and each of whose triangles either generates the whole squag or the 9-element subsquag. We observe that any planar squag (except of cardinality 9) is semi-planar and the inverse is not true.

The triple system $ST(n)$ associated with a semi-planar squag $SQ(n)$ will also be called semi-planar (for much more precision it may be called semi-9-planar). In other words, one may say that a triple system $ST(n)$ is semi-planar if the $ST(n)$ has no proper $a$-normal subsystems (see [13]) (equivalently, the corresponding $SQ(n)$ is simple) and each triangle generates a sub-$ST(9)$ or the whole triple system $ST(n)$.

Indeed, for $n = 7, 9, 13, 15$ there are only planar squags. In [1] the first author has constructed semi-planar squags of cardinality $n$ for all $n > 9$ and $n \equiv 3$ or $9 \mod 18$ having only nontrivial subsquags of cardinality 9.

In this article, we give a recursive construction as $n \rightarrow 3n$ for semi-planar squags. This construction permits us to construct semi-planar squags having nontrivial subsquags of cardinality $>9$. In fact, we may construct a semi-planar squag having only medial subsquags of cardinality $3^m$ for each finite positive integer $m$.

2. Construction of semi-planar squags of cardinality $3n$

In this section we describe the construction of semi-planar squags given in [1]. Let $T_i = (S_i; B_i)$ be a triple system with $S_i = (S_i; \cdot)$ the corresponding squag for $i = 1, 2$. The direct product $T_1 \times T_2$ of the two triple systems can be obtained from the underlying triple system of the direct product squag $S_1 \times S_2$ [6].

Let $T_0 = (Q_0; B_0)$ be a triple system of cardinality $n$, and let $Q_0 = \{a_1, a_2, \ldots, a_n\}$. We consider the direct product $T_0 \times C_3$, where $C_3$ is the $ST(3)$ on the set $\{1, 2, 3\}$ and $I_3$ is its corresponding squag. The direct product $T_0 \times C_3 = (Q_1; B_1)$ is formed by the usual tripling of $(Q_0; B_0)$. Namely, $(Q_1; B_1)$ is an $ST(3n)$, where $Q_1 = Q_0 \times \{1, 2, 3\}$ and the set of triples $B_1$ is obtained by:

$B_1 = \{\{(a_i, 1), (a_j, 2), (a_k, 3)\} \mid a_i, a_j, a_k \in B_0 \text{ or } a_i = a_j = a_k\}$

$\cup \{\{(a_i, i), (a_j, i), (a_k, i)\} \mid a_i, a_j, a_k \in B_0 \text{ and } i \in \{1, 2, 3\}\}$.

We denote the squag $(Q_0; \cdot_0)$ associated with $T_0$ by $Q_0$ and the squag $3 \times Q_0 = (Q_1; \times) = Q_0 \times I_3$ associated with $T_1 \times C_3 := (Q_1; B_1)$. 
Without loss of generality, we may assume that \(A_0 = \{a_1, a_2, a_3\}\) is a block of \(B_0\), then the triple system \((Q_0; B_0)\) contains the subsystem \((A; R)\), where \(A = A_0 \times \mathbb{C}_3\) and the set of blocks \(R\) obtained by:

\[
R = \{( (a_1, 1), (a_2, i), (a_3, i) ) : i \in \{1, 2, 3\}\} \\
\cap \{ (x, 1), (x, 2), (x, 3) ) : x \in \{a_1, a_2, a_3\}\} \\
\cap \{ (x, i), (y, j), (z, k) ) : \{x, y, z\} = \{a_1, a_2, a_3\} \& \{i, j, k\} = \{1, 2, 3\}\}.
\]

Define on the subset \(A\) the set of triples \(H\) as follows:

\[
H = \{(a_3, 1), (a_3, 2), (a_1, 3)\}, \{(a_2, 1), (a_2, 2), (a_2, 3)\}, \{(a_1, 1), (a_1, 2), (a_3, 3)\}, \\
\{(a_3, 1), (a_2, 2), (a_1, 1)\}, \{(a_3, 2), (a_2, 3), (a_1, 2)\}, \{(a_1, 3), (a_2, 1), (a_3, 3)\}, \\
\{(a_3, 1), (a_2, 3), (a_1, 3)\}, \{(a_2, 2), (a_1, 2), (a_1, 1)\}, \{(a_1, 1), (a_2, 1), (a_3, 2)\}, \\
\{(a_1, 3), (a_2, 3), (a_1, 1)\}, \{(a_2, 2), (a_3, 2), (a_3, 3)\}, \{(a_1, 2), (a_2, 1), (a_3, 1)\}.
\]

Each of \((A; R)\) and \((A; H)\) are isomorphic to the affine plane over \(\mathbb{GF}(3)\). Note that the block \({(a_2, 1), (a_2, 2), (a_2, 3)}\) is the only block lying in the intersection of \(R\) and \(H\).

Using the replacement property by interchanging the two sets of blocks \(R\) and \(H\) in \((Q_1; B_1)\), then we get again an \(\text{ST}(3n) = (Q_1; B_1)\), where \(B_1 := B_1 - R \cup H\). In fact, the sub-\(\text{ST}\) formed by the direct product of \(\{a_1, a_2, a_3\}\) and \(\{1, 2, 3\}\) is replaced with an isomorphic copy on the same set of points. We denote the squag associated with the \(\text{ST}(3n) = (Q_1; B_1)\) by \(Q_1 = 3 \otimes_A Q_0 = (Q_1; \cdot)\). Observe that the difference between the binary operations ‘\(\times\)’ and ‘\(\cdot\)’ depends only on the elements of \(A\).

**Theorem 3 (f1).** If \(Q_0\) is a planar squag of cardinality \(n\), then the constructed squag \(Q_1 = 3 \otimes_A Q_0\) is semi-planar of cardinality \(3n\) for all \(n \equiv 3\) or \(9\) and \(n \equiv 1\) or \(3\) (mod \(6\)).

Moreover, in [1] an example of a semi-planar squag of cardinality 27 was given. According to Theorems 2 and 3, we may say that there is always a semi-planar \(\text{SQ}(3n)\) for all \(n > 3\) and \(n \equiv 1\) or \(3\) (mod \(6\)).

Also, according to the proof of Theorem 3 given in [1] we may directly deduce the following result.

**Corollary 4.** Any subsquag \(S\) of the constructed semi-planar squag \(Q_1 = 3 \otimes_A Q_0\) satisfies that:

1. If \(|S \cap A| = 3\), then \(S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\}\).
2. If \(S \supseteq A\), then \(S = Q_1\).
3. The only nontrivial subsquags of \(Q_1 = 3 \otimes_A Q_0\) are of cardinality 9.

In the next section, we will discuss the following problem:

Is there a semi-planar squag having nontrivial subsquags of cardinality >9?

### 3. Recursive construction of semi-planar squags

According to Theorem 3 and Corollary 4, we may always assume that there is a semi-planar squag of cardinality \(n\) having only nontrivial subsquags of cardinality 9, for all \(n > 9\) and \(n \equiv 3\) or \(9\) (mod \(18\)). In other words, the subsquags of the constructed semi-planar squag \(Q_1\) are exactly of cardinality 1, 3, 9 and \(n\). In the next theorem we generalize the results of Theorem 3 and Corollary 4 to construct semi-planar squags of cardinality \(3^m n\) for \(n > 3\) and \(n \equiv 1\) or \(3\) (mod \(6\)) having only medial subsquags at most of cardinality \(3^m v\) for each \(v = 1, 2, \ldots\) or \(m + 1\) and for each positive integer \(m\). If a squag \(Q\) has only medial subsquags of cardinality \(3^m v\) for each \(v \leq v\) (i.e.; all subsquags are medial with the maximum cardinality \(3^m\)), we will say that \(Q\) is a squag with sub-\(\text{SQ}(3^m)v\)’s.

We note that if \(Q_0\) is planar, then each sub-\(\text{SQ}(3)\) of \(Q_0\) is only covered by the whole squag \(Q_0\), this means that each sub-\(\text{SQ}(3)\) of a planar squag \(Q_0\) is a maximal subsquag in \(Q_0\). We attempt in the next lemma to show that the constructed semi-planar \(Q_1 = 3 \otimes_A Q_0\) has a sub-\(\text{SQ}(3)\) = \(A_1\) satisfying that the only subsquag covering \(A_1\) is \(Q_1\).

**Lemma 5.** Let \(Q_0\) be a planar squag of cardinality \(n \neq 9\), then the constructed semi-planar squag \(Q_1 = 3 \otimes_A Q_0\) has a maximal sub-\(\text{SQ}(3)\); i.e. \(Q_1\) has a sub-\(\text{SQ}(3)\) covered only by \(Q_1\).

**Proof.** Let \(A_1 = \{(a_1, 1), (b, 1), (c, 1)\}\) be a subsquag in \(Q_1 = 3 \otimes_A Q_0\) with \(b, c \notin A_0 = \{a_1, a_2, a_3\}\). We want to prove that the only subsquag containing \(A_1\) is \(Q_1\).
Assume that there is a subsquag \( S \supseteq A_1 \). Then we have two cases:

(i) \( S \cap A \) has more than one element or (ii) \( S \cap A = \{(a_1, 1)\} \), where \( A_0 \) is a subsquag in \( Q_0 \) and \( A = A_0 \times I_3 \) is given as in the construction \( Q_1 = 3 \otimes_A Q_0 \).

**For the case (i):** We may say that \( |S \cap A| = 3 \) or 9. But we have \( (a_1, 1) \in S \cap A \), hence if \( |S \cap A| = 3 \), then \( S \cap A \) is a sub-SQ(3) \( \neq \{(a_2, 1), (a_2, 2), (a_2, 3)\} \) contradicting Corollary 4 that the only possible case with the condition \( |S \cap A| = 3 \) is \( \{a_2, 1, (a_2, 2), (a_2, 3)\} \).

If \( |S \cap A| = 9 \), then \( S \supseteq A_1 \) moreover \( S \supseteq A_1 \), hence \( |S| > 9 \). Again, according to Corollary 4, the only subsquag containing \( A \) with cardinality \( >9 \) is the whole squag, hence \( S = 3 \otimes_A Q_0 \).

**For the second case (ii) \( S \cap A = \{(a_1, 1)\} \):** Since \( S \supseteq A_1 \), there is an element \((x, i) \in S - A_1 \). Hence we have 3 possible cases:

- (1) \((x, i) = (x, 1)\)
- (2) \((x, i) = (x, 2)\)
- (3) \((x, i) = (x, 3)\)

**For the case (1):** Since \((x, 1) \not\in A_1 \), it follows that \( x \not\in \{a_1, b, c\} \). Hence \( S \) contains the 4-element subset \( \{(x, 1), (a_0, 1), (b, 1), (c, 1)\} \), this means that \( |S| = 9 \) or \( S = 3 \otimes_A Q_0 \). But \( |S| = 9 \) means that the number of elements of the first components of \( S \) is greater than \( 3 \), which contradicts the fact that \( Q_0 \) is planar. Then \( S = 3 \otimes_A Q_0 \).

**Case (2):** For \((x, i) = (x, 2)\), we have two cases \( x \not\in \{a_1, b, c\} \) or \( x \in \{a_1, b, c\} \). If \( x \not\in \{a_1, b, c\} \), then \( S \) contains the 4 distinct elements \( \{(x, 2), (a_1, 1), (b, 1), (c, 1)\} \). Hence the set of the first components \( P_1(S) \) of \( S \) forms a sub-squag of cardinality \( >4 \) of \( Q_0 \). Since \( Q_0 \) is planar, it follows that \( P_1(S) = Q_0 \). Hence \( S = 3 \otimes_A Q_0 \) for the same reason given in the preceding case.

If \( x \in \{a_1, b, c\} \), then \( x = b \) or \( c \) because of \((x, i) \in S - A_1 \) and \( S \cap A = \{(a_1, 1)\} \). But \( x = b \) or \( c \) tends to \((b, 2)\) \( \cdot (c, 1) \) \( = (a_1, 3) \) or \((c, 2) \cdot (b, 1) = (a_1, 3) \in S \) which contradicts the assumption that \( S \cap A = \{(a_1, 1)\} \).

The same discussion holds for the case (3): \((x, i) = (x, 3)\). Then the only possible case for a subsquag \( S \) containing the block \( \{(a_1, 1), (b, 1), (c, 1)\} \) is the whole squag \( 3 \otimes_A Q_0 \). This completes the proof.

Note that we may prove the same result if we choose \( A_1 = \{(a_1, 1), (b, 2), (c, 3)\} \) with \( b, c \not\in A_0 = \{a_1, a_2, a_3\} \).

For \( n = 9 \), the author [1] has constructed an example of a semi-planar \( SQ(27) \). It is also easy to find a sub-SQ(3) \( = A_1 \) covered by the whole squag \( SQ(27) \).

In the next theorem we assume that \( Q_m \) is a semi-planar squag and \( 3 \times Q_m := (Q_{m+1}; x) \) is the direct product squag \( Q_m \times I_3 \). For any sub-SQ(3) \( = A_m \) of \( Q_m \) the set \( A = A_m \times I_3 \) forms a sub-SQ of \( 3 \times Q_m \) \( = (Q_{m+1}; x) \).

Consider the same subsystems \( (A; R) \) and \( (A; H) \) exactly as in the construction \( Q_1 = 3 \otimes_A Q_0 \). So the construction \( Q_{m+1} = 3 \otimes_A Q_m \) is still a squag (but not necessarily be semi-planar). Indeed, if we choose a sub-SQ(3) \( = A_m = \{a_1, a_2, a_3\} \) of \( Q_m \) satisfying that \( A_m \) is only covered by the whole squag \( Q_m \), then we will show in the next two theorems that the constructed squag \( Q_{m+1} = 3 \otimes_A Q_m \) is semi-planar and preserves the first two properties as in Corollary 4.

**Lemma 6.** Assume that \( Q_m \) is a semi-planar squag having a sub-SQ(3) \( = A_m = \{a_1, a_2, a_3\} \) covered only by \( Q_m \). Let \( S \) be a subsquag of the constructed squag \( Q_{m+1} = 3 \otimes_A Q_m \). It follows that if \( |S \cap A| = 3 \), then \( S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\} \).

**Proof.** Let \( S \cap A \) be a 3-element subsquag \( \neq \{(a_2, 1), (a_2, 2), (a_2, 3)\} \). We will show that any other choice of \( S \cap A \) leads to a contradiction. The intersection \( S \cap A \) belongs to one of two essential cases.

First, let \( S \cap A = \{(a_1, 1), (a_2, 2), (a_3, 1)\}, \{(a_1, 3), (a_2, 1), (a_3, 3)\} \), \( \{(a_1, 2), (a_2, 3), (a_3, 2)\}, \{(a_1, 1), (a_2, 1), (a_3, 2)\} \) or \( \{(a_1, 2), (a_2, 1), (a_3, 3)\} \).

In this case, the set of the second components of the elements of \( S \cap A \) is a 2-element subset of \( \{1, 2, 3\} \). But one can easily see that the set of second components of the elements of \( S \cap A \) consists of \( \{1, 2, 3\} \) for any choice of \( S \cap A \).

Let \( \{i, k, j\} = \{1, 2, 3\} \) and let the maximum number of distinct elements of \( S \), having second components \( i \), be equal to \( r \). Let the values of second components of the sub-SQ(3) \( = \{(a_2, 1), (a_2, 2), (a_2, 3)\} \) of \( S \cap A \) be \( i, j \) and \( k \). If \( (y, k) \in S - A \), then the product of any element \((x, i) \cdot (y, k) \) gives an element of \( S \) having a second component \( j \).

This means that \( S \) contains exactly \( r \) elements having second components equal to \( j \). Also, let \( (y, j) \in S - A \), then the product \((x, i) \cdot (y, j) \) \( = (z, k) \), which means that \( S \) contains also \( r \) distinct elements having second components equal to \( k \). Accordingly, we may deduce that \( S \) consists exactly of an \( r \)-element subset of
pairs with second components \( i \), an \( r \)-element subset of pairs with second components \( j \) and an \( r \)-element subset of pairs with second components \( k \).

Since the second components of \( S \cap A \) are \( i \), \( i \) and \( j \), each of the \( r \)-element subsets of \( S \) with second components \( j \) or \( k \) forms a subsquag of \( S \). According to Theorem 1, the third \( r \)-element subset of \( S \) with second component equal to \( i \) must be a subsquag of \( S \), contradicting the choice that \( S \cap A = \{(a_1, i), (a_2, i), (a_3, j)\} \). Therefore, this case is ruled out.

Next, let \( S \cap A = \{(a_1, 1), (a_1, 2), (a_3, 3)\}, \{(a_3, 1), (a_2, 3), (a_3, 3)\}, \{(a_2, 2), (a_1, 2), (a_1, 3)\}, \{(a_1, 1), (a_2, 3), (a_1, 3)\}, \{(a_3, 1), (a_3, 2), (a_1, 3)\} \) or \( \{(a_2, 2), (a_3, 2), (a_3, 3)\} \).

For any choice of \( S \cap A \), the set of second components of the elements of \( S \cap A \) is the set \( \{1, 2, 3\} \) or a 2-element subset of \( \{1, 2, 3\} \). But one can directly see that the set of second components of the elements of \( S - A \) consists of all elements of \( \{1, 2, 3\} \). Also, the index set of the first components of any choice of \( S \cap A \) is a 2-element subset of \( \{1, 2, 3\} \).

Let \( \{i, j, k\} = \{1, 2, 3\} \) and let \( S \cap A = \{(a_1, l), (a_i, n), (a_j, m)\} \) with \( l \neq n \) and \( l, n, m \in \{1, 2, 3\} \).

Let \( (b, l) \in S - A \) and \( (b, l) \times (a_1, l) \neq (b, l) \times (a_i, n) \neq (b, l) \times (a_j, m) \neq (b, l) \times (a_k, k) \) with \( l, n, k \in \{1, 2, 3\} \). Hence \( (a_1, l) \times (a_i, n) \neq (a_1, l) \times (a_j, m) \), and accordingly, \( (a_i, n) \times (a_j, m) \neq (a_i, n) \times (a_k, k) \) and \( (a_j, m) \times (a_k, k) \neq (a_j, m) \times (a_i, n) \), \( (a_k, k) \times (a_i, n) \neq (a_k, k) \times (a_j, m) \). On the other side, we have \( (a_1, k) \in A \) contradicting the choice that \( S \cap A = \{(a_1, l), (a_i, n), (a_j, m)\} \) and \( \{l, n, k\} = \{1, 2, 3\} \). We will get the same contradiction, if we choose \( (b, n) \) or \( (b, k) \in S - A \) instated of \( (b, l) \). Therefore, the second possible case of \( S \cap A \) is also ruled out.

This means that the only possible case for a subsquag \( S \) of \( Q_{m+1} \) with \( |S \cap A| = 3 \) is \( S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\} \). This completes the proof of the lemma.

The next theorem permits us to construct semi-planar squags of cardinality \( 3^n \) for all positive integers \( m \) and \( n \) with \( n \neq 3 \) or \( 9 \).

**Theorem 7.** Let \( Q_m \) be a semi-planar squag of cardinality \( 3^n \) with sub-SQ(3)\( ^v \)'s for some \( v = 1, 2, \ldots \) or \( m + 1 \). Also, let \( Q_m \) have a sub-SQ(3) = \( A_m \) covered by \( Q_m \). Then the constructed squag \( Q_{m+1} = 3 \times A \) \( Q_m \) is semi-planar of cardinality \( 3^{m+1} \) with sub-SQ(3)\( ^{v+1} \)'s for all possible values \( n \neq 3 \) or \( 9 \) and \( n \equiv 1, 3 \) (mod 6).

**Proof.** Let \( S \) be a subsquag of \( Q_{m+1} \) and \( S \) has more than one element. First we have to prove that \( S \) is a sub-SQ(3)\( ^{v+1} \) or \( S = Q_{m+1} \). Let \( A_m = \{a_1, a_2, a_3\} \) and \( A = A_m \times C_3 = \{a_1, a_2, a_3\} \times \{1, 2, 3\} \). And choose the two sets of triples \( R \) and \( H \) that are defined exactly as in the construction \( Q_1 = 3 \times A \) \( Q_0 \).

In general, there are only four essential cases for the relation between the subsquags \( S \) and \( A \):

(i) \( S \subseteq A \),
(ii) \( |S \cap A| = 0 \) or \( S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\} \),
(iii) \( |S \cap A| = 3 \),
(iv) \( A \subseteq S \).

(i) If \( S \subseteq A \), then \( S \) is a sub-SQ(3) of \( \) or sub-SQ(9).

(ii) If \( |S \cap A| = 0 \) or \( S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\} \), then \( S \) is a subsquag in \( Q_{m+1} \) so also in \( 3 \times Q_m = Q_m \times \{1, 2, 3\} \), hence the set of first components of \( S \) is a subsquag of \( Q_m \). But in these three cases given in (ii), the set of first components of \( S \neq Q_m \) because the number of elements of the first component of \( S \cap A \neq 3 \). Hence the set of the first components of \( S \) forms a medial subsquag at most of cardinality \( 3^n \) of \( Q_m \) for some positive integer \( v \leq m + 1 \). Also, the second projection \( P_2(S) \) consists of one element or three elements, then we may say that \( S \subseteq SQ(3)^{v} \) or \( SQ(3)^{v+1} \times \{1, 2, 3\} \).

(iii) According to the above lemma, we may say directly that \( S \cap A = \{(a_2, 1), (a_2, 2), (a_2, 3)\} \). Then we get the same result as in the preceding case.

(iv) Note that the sub-SQ(3) \( A_m = \{a_1, a_2, a_3\} \) is covered by \( Q_m \). So \( A = A_m \times I_3 \) is a sub-SQ(9) of \( 3 \times Q_m \) satisfying the only subsquag containing \( A \).
S′ in the binary operations ‘×’ and ‘.’. This implies that S is equal to \( Q_{m+1} \). So we may say that the only sub-squag containing A is the whole squag \( Q_{m+1} \).

Finally we may say that the semi-planar squag \( Q_{m+1} \) has only medial proper sub-squags at most of cardinality \( 3^{v+1} \) for \( v \leq m + 1 \). This completes the proof of the first part of the theorem.

Now, we need only show that \( Q_{m+1} \) is a simple squag. Assume that \( Q_{m+1} \) has a proper congruence \( \theta \), since \( [(x, i)]\theta \) is a sub-squag of \( Q_{m+1} \), then \( [(x, i)]\theta \) must be a medial sub-squag \( \cong SQ(3)^{v′} \) with \( v′ - 1 \leq v \leq m + 1 \). On the other hand, for any 3-element sub-squag X of \( Q_{m+1} \) the set \( [X]\theta \) forms a proper sub-squag of \( Q_{m+1} \) and so forms a medial sub-squag. Hence, if \( v′ - 1 = v = m + 1 \), then \( [(x, i)]\theta = SQ(3)^{m+2} \). But we may choose a 3-element sub-squag X satisfying that the sub-squag \( [X]\theta \) is of cardinality \( SQ(3)^{m+3} \). This is a contradiction. Therefore, in light of the first part of the proof, we may deduce that \( [(x, i)]\theta = SQ(3)^{v′} \) with \( v′ \leq v \) for some positive integer \( v \leq m + 1 \).

If \( [(x, i)]\theta = SQ(3)^{v′} \) with \( 2 \leq v′ \leq v \), so for \( (a, 1) \in A \), we have three cases: (a) \( [(a, 1)]\theta \cap A = 1 \), (b) \( [(a, 1)]\theta \cap A = 3 \) or (c) A \( \subseteq [(a, 1)]\theta \).

For case (a): \( [A]\theta \) is a proper sub-squag of \( Q_{m+1} \), so \( [A]\theta \cong SQ(3)^{v′+2} \). According to the first part of the theorem for \( v′ = v = m + 1 \), it contradicts the preceding fact that the maximum cardinality of sub-squags is \( 3^{m+1} \). In general, for \( 2 \leq v′ \leq v \), the subset \( S_1 = \{(a, 1), (a, 2), (a, 3)\} \) forms a sub-SQ(3) \( \subseteq A \), then \( S_1 \cap A \) is a sub-SQ intersects A in \( S_1 \) which contradicts the result of Lemma 6.

For case (b): According to Lemma 6, we have \( [(a, 2)]\theta \cap A = \{(a, 2), (a, 2), (a, 3)\} \). Again the set \( B = \{(a, 1), [(a, 2)]\theta \} \) is a medial sub-squag and \( \cong SQ(3)^{v′} \). But \( (a, 1) \in [(a, 2), (a, 2), (a, 3)] = \{(a, 3), (a, 1), (a, 3)\} \subseteq B \) contradicts the result of Lemma 6 that \( B \cap A \) must be equal to \( \{(a, 1), (a, 2), (a, 3)\} \).

For case (c): A \( \subseteq [(a, 1)]\theta \), but according to case (iv) of the first part we may say that \( [(a, 1)]\theta \) is equal to A or \( Q_{m+1} \). If \( [(a, 1)]\theta \) is equal to A we choose \( (b, 2) \notin A \), then \( (b, 2) \in [(a, 1)]\theta \) is a sub-SQ(3) \( \times c \); \( (b, 2), [(a, 1)]\theta \) is a medial sub-squag and so it is distributive. But \( (b, 2), [(a, 1), (a, 1)] \) \( \in (b, 2), (a, 2), (a, 2) \) \( \subseteq (c, 2) \) \( \subseteq (b, 2), (a, 1) \) \( \subseteq (b, 2), (a, 2) \) \( \subseteq (c, 2), (c, 3) \), which contradicts the fact that \( (b, 2), [(a, 1)]\theta \) must be distributive. Consequently, these three cases (a), (b) and (c) are ruled out.

If \( [(x, i)]\theta = 3 \), so for \( (a, 1) \in A \), we have two essential cases:

(I) \( [(a, 1)]\theta \cap A = 1 \), or (II) \( [(a, 1)]\theta \cap A = 3 \).

For case (I): Choose a block \( X \in H - \{(a, 2), (a, 2), (a, 3)\} \), the sub-squag \( [X]\theta \) is an SQ(9) and \( [X]\theta \cap A \) is a block in \( H - \{(a, 1), (a, 2), (a, 2), (a, 3)\} \), which contradicts the result of Lemma 6.

Case (II) tends to \( [(a, 1)]\theta \cap A = \{(a, 1)]\theta \), then \( \theta_A \) is a nontrivial congruence on the sub-SQ(9) = A. Then we may choose \( (a, j) \in A - \{(a, 2), (a, 2), (a, 3)\} \), so \( [(a, j)]\theta \in H - \{(a, 2), (a, 2), (a, 3)\} \). By choosing a block X satisfying \( X \cap A = \{(a, j) \} \), hence \( [X]\theta \) is an SQ(9) containing a block in the set \( H - \{(a, 1), (a, 2), (a, 2), (a, 3)\} \). That contradicts again the result of Lemma 6.

Therefore, the constructed squag \( Q_{m+1} = 3 \otimes_A Q_m \) is simple. This completes the proof of the theorem. \( \square \)

Now, we may say that the planar squag \( Q_0 \) is semi-planar with sub-SQ(3) and has a sub-SQ(3) = \( A_0 \) covered by \( Q_0 \) (indeed each sub-SQ(3) covered by \( Q_0 \)). According to Theorem 3, Corollary 4 and Lemma 5, the constructed squag \( Q_1 = 3 \otimes_A Q_0 \) is semi-planar with sub-SQ(3) squared and has a sub-SQ(3) = \( A_1 \) covered by \( Q_1 \). To complete the requirements of the mathematical induction we have to prove that if the squag \( Q_m \) is semi-planar with sub-SQ(3) for some positive integer \( v \leq m + 1 \) and has a sub-SQ(3) = \( A_m \) covered by \( Q_m \), then the constructed \( Q_{m+1} = 3 \otimes_A Q_m \) is semi-planar with sub-SQ(3) squared and has also a sub-SQ(3) = \( A_{m+1} \) covered by \( Q_{m+1} \). The first part is already established by Theorem 7 and the second part will be proven in the next theorem.

**Theorem 8.** Let \( Q_m \) be a semi-planar squag with sub-SQ(3) for some \( v = 1, 2, \ldots, m + 1 \) having a sub-SQ(3) = \( A_m \) covered by \( Q_m \). Then the constructed semi-planar squag \( Q_{m+1} = 3 \otimes_A Q_m \) has also a sub-SQ(3) = \( A_{m+1} \) covered by \( Q_{m+1} \).

**Proof.** Let \( A_m = \{a_1, a_2, a_3\} \) be a sub-SQ(3) of \( Q_m \) covered only by \( Q_m \). Also, let \( A \) be the set \( A_m \times \{1, 2, 3\} \) of \( Q_{m+1} \). According to Theorem 7, the construction \( Q_{m+1} = 3 \otimes_A Q_m \) is a semi-planar squag having sub-SQ(3) squared. Consider the sub-squag \( A_{m+1} = \{(a_1, 1), (b, 2), (c, 3)\} \) of \( Q_{m+1} \) with \( b \) and \( c \notin A_m \). We will show that the only sub-squag containing \( A_{m+1} \) is \( Q_{m+1} \).

Suppose \( S \) be a sub-squag of \( Q_{m+1} \) containing \( A_{m+1} \). We have three cases:

(i) \( S \cap A = A \), (ii) \( |S \cap A| = 3 \) or (iii) \( S \cap A = \{(a_1, 1)\} \).
Theorem 7

that we have Lemma 6 says that the set to get a semi-


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For the case (iii): If $S \cap A = \{(a_1, 1)\}$, then $S$ is a subquag of $3 \times Q_m$ and also of $Q_{m+1}$. Assume that $|S| > 3$.

According to Theorem 7, $S \cong SQ(3)^{v'+1}$ for $1 \leq v' \leq v \leq m + 1$. Let $P_1(S)$ and $P_2(S)$ be the projections of $S$ on the first and second components, respectively. Then $P_1(S) \cong SQ(3)^{v'}$ for $v' \leq v$ and $P_2(S) = \{(1, 2, 3)\}$. Note that $S$ contains more than one element with second component $i = 1, 2$ or $3$. Similar to the proof of case (ii) of the preceding theorem, we may say that $S \cong SQ(3)^{v'+1} = SQ(3)^{v'} \times \{(1, 2, 3)\}$, where $SQ(3)^{v'+1}$ is a subquag of both $3 \times Q_m$ and $Q_{m+1}$. Also, we have $P_1(S) \cap P_1(A) = \{(a_1, 1)\}$, so $S \cap A = \{(a_1, 1), (a_1, 2), (a_1, 3)\}$, which is impossible. Because of $S \cap A = \{(a_1, 1)\}$ and also the result of Lemma 6 says that the set of $\{(a_1, 1), (a_1, 2), (a_1, 3)\}$ does not form a subquag of $Q_{m+1}$. Therefore, this case is also ruled out.

This means that the only possible case for $S \supset A_{m+1}$ is $S \supset A$. So we go back to the result of case (i) that $S$ must be equal to $Q_{m+1}$. This completes the proof of the lemma. □

According to the 1–1 correspondence between squags and triple systems, we may say that there are semi-planar $TS(3^n,n)$ having only subsystems $\cong ST(3)\, s$ for each positive integer $v \leq m + 1$ and for each possible number $n$ ($n \neq 9$ and $n \equiv 1$ or $3 \mod 6$). These triple systems satisfy that each triangle generates a sub- $ST(9)$ or the whole triple system and whose corresponding squag is simple.

For $n = 9$, the author [1] has constructed an example of semi-planar squag of cardinality 27. Also, it easy to find a sub-$SQ(3) = A_1$ covered by the whole squag, so we may apply Lemma 6, Theorems 7 and 8 to get a semi-planar squag of cardinality 81 having only medial subquags of cardinality $3^v$ at most. Equivalently, there are three semi-planar $ST(81)\, s$ with subsystems $\cong ST(3)\, s$ for $v = 1, 2$ and $3$.

Finally, one may say that there is a semi-planar $SQ(3^n,n) := Q_{m,v}$ for all $n > 3$ and $n \equiv 1$ or $3 \mod 6$ and each positive integer $m$ with medial subquags of maximum cardinality $3^v$ for each positive integer $v \leq m + 1$.

Quackenbush [12] proved that the variety $V(Q)$ generated by a simple planar squag $Q$ has only two subdirectly irreducible squags $Q$ and the 3-element squag $SQ(3)$ and then $V(Q)$ covers the smallest nontrivial subvariety (the class of all medial squags).

Similarly, if $Q_{m,v} = SQ(3^n,n)$ is a semi-planar squag having only medial subquags of cardinality $3^v$ at most, then one can prove that the variety $V(Q_{m,v})$ generated by $Q_{m,v}$ has only two subdirectly irreducible squags $Q_{m,v}$ and the 3-element squag $SQ(3,v)$.

And hence we deduce the same result that each semi-planar squag $Q_{m,v}$, with sub-$SQ(3)\, s$ for each positive integer $v \leq m + 1$ generates a variety $V(Q_{m,v})$ which covers also the smallest nontrivial subvariety (the class of all medial squags).

Hall [9] constructed a Steiner triple system in which each triangle generates a sub-$ST(9)$, such a class is called Hall triple systems. The corresponding squags of such class is the class of distributive squags. Klossek [10] gave a construction of distributive squags as a vector space over $GF(3)$ of dimension $\geq 4$. Using the interchange property to inject a distributive subquag $SQ(3^v)$ instead of a medial sub-$SQ(3)\, s$, we get a construction of a squag having distributive subquags but not medial. Consequently, we are faced with the question:

Is there a semi-planar squag having distributive (not medial) subquags?

References


