Note

2-Colored Triangles in Edge-Colored Complete Graphs

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If each edge of complete graph $K_n$ is colored with one of $k$ colors then it contains a triangle having two colors if $k < 1 + n^{1/2}$. The result is best possible when $n$ is the square of a prime.

Let each of the edges of the finite complete graph $K_n$ be colored with one of $k$ colors. We call a subgraph $I_n$ of $K_n$ a triangle and say it is monochromatic, bichromatic, polychromatic according to whether it has 1, 2, 3 colors. Sufficient conditions for the existence of monochromatic and polychromatic triangles appear in [1] and [2, 3], respectively. Here we prove Theorem 1 and give an example to show that the theorem is best possible in certain cases.

**Theorem 1.** If $2 \leq k < 1 + n^{1/2}$ then $K_n$ has a bichromatic triangle.

**Proof.** Suppose $K_n$ has no bichromatic triangle. Let $x$ be a vertex with the maximum possible number $t$ of edges incident with it of the same color, which is color 1 say. By counting edges at $x$ we have $kt \geq n - 1$. Let $Y$ be the set of vertices adjacent to $x$ with edges of color 1. Then every edge of the complete subgraph $K_{t+1}$ with vertex set $\{x\} \cup Y$ has color 1. Since $k \geq 2$ we have $R = K_n \setminus K_{t+1}$ nonempty. Let $r$ be a fixed vertex of $R$. Then edges of the form $\{r, z\}$ where $z \in \{x\} \cup Y$ do not have color 1, for otherwise we contradict the definition of $t$. Furthermore all such edges have different colors, so $k \geq t + 2$. From the two inequalities we get

$$(k - (1 + n^{1/2}))(k - (1 - n^{1/2})) \geq 0$$

and the theorem follows.
EXAMPLE. Let $n = p^2$ and $k = p + 1$ where $p$ is a prime. Let the vertices of $K_n$ be $x_{ij}$ for $1 \leq i, j \leq p$. Make edge $\{x_{ij}, x_{rs}\}$ color $p + 1$ if $i = r$ but color $c$ if $i \neq r$, $1 \leq c \leq p$, and $c(i - r) = j - s \pmod{p}$. When the vertices are set out in a matrix $(x_{ij})$ it can be seen that this graph has no bichromatic triangle.

The following result is due to D. E. Daykin and is an easy application of Turán's theorem.

**Theorem 2.** If each of the edges of $K_{n+1}^{p+1}$ is colored with one of $t + 1$ colors then there is a subgraph $K_{t+1}$ with $\leq t$ colors.

**References**