Solvability of a Nonlinear Two Point Boundary Value Problem at Resonance

Chung-Cheng Kuo*

Department of Mathematics, Fu Jen University, Taipei, Taiwan

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1. INTRODUCTION

Let $k \in \mathbb{N}$ be fixed. We consider the boundary value problem

$$
\begin{align*}
  u'' + k^2 u + g(x, u) &= h & \text{in } (0, \pi), \\
  u(0) &= u(\pi) = 0,
\end{align*}
$$

where $h \in L^1(0, \pi)$ is given and $g: (0, \pi) \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function. That is, $g(x, u)$ is measurable in $x \in (0, \pi)$ for each $u \in \mathbb{R}$, continuous in $u \in \mathbb{R}$ for a.e. $x \in (0, \pi)$ and satisfies for each $r > 0$, there exists $a_r \in L^1(0, \pi)$ such that

$$
|g(x, u)| \leq a_r(x),
$$

for a.e. $x \in (0, \pi)$ and $|u| \leq r$.

Concerning the growth condition of the nonlinear term $g$, we assume that:

(H) There exist a constant $r_0 > 0$ and nonnegative functions $p$, $b \in L^1(0, \pi)$ such that

$$
\|p\|_{L^1} < 2k(k+1) \tan \frac{\pi}{2(k+1)},
$$

and for a.e. $x \in (0, \pi)$ and $|u| \geq r_0$

$$
|g(x, u)| \leq p(x) |u| + b(x).
$$

The solvability of the problem (1) has been extensively studied if $p$ is assumed to be bounded, existence theorems for a solution to (1) are proved

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if \( p(x) \leq 2k + 1 \) for a.e. \( x \in (0, \pi) \) with strict inequality on a positive measurable subset of \((0, \pi)\) (see [2, 5]). Recently, Dancer and Gupta [4] has to give a solvability theorem for (1) under the growth condition (H) when \( k = 1 \) and satisfies

\[
\int_0^\pi h(x) \sin x \, dx = 0. \tag{5}
\]

The purpose of this paper is to extend the main result of Dancer and Gupta [4] when \( k = 1 \) and (5) is excluded, and improve the main theorem in Ha and Kuo [3] where it assumed that \( \|p\|_{L^1} \leq 2k \) and satisfies a Landesman–Lazer condition,

\[
\int_0^\pi h(x) v(x) \, dx < \int_{\pi > 0} g_+ (x) v(x) \, dx + \int_{\pi < 0} g_- (x) v(x) \, dx, \tag{6}
\]

where \( g_+ (x) = \lim_{x \to -\infty} \inf g(x, u), \quad g_- (x) = \lim_{x \to -\infty} \sup g(x, u) \) and \( v(x) = \pi \sin kx \) for \( \pi \in \mathbb{R} \setminus \{0\} \). To prove our results using a Lyapunov type inequality shown in Lemma 2 and the well-known Leray–Schauder continuation methods (see [1]).

In what follows we shall make use of the real Banach spaces \( L^p(0, \pi) \) and \( C^k[0, \pi] \) with the norms denoted by \( \|u\|_{L^p} \) and \( \|u\|_{C^k} \), respectively, and the Sobolev spaces \( W^{2, 1}(0, \pi) \) and \( H^1_0(0, \pi) \). By a solution of (1) we mean a function \( u \in W^{2, 1}(0, \pi) \cap H^1_0(0, \pi) \) satisfies the differential equation (1) a.e. on \((0, \pi)\). Finally, we note that for each \( u \in W^{2, 1}(0, \pi) \cap H^1_0(0, \pi) \),

\[
\int_0^\pi u(x) \sin kx = 0
\]

for all \( x \in (0, \pi) \), where

\[
G(x, \xi) = \sin kx (\xi \cos k\xi) / k - \left\{ \begin{array}{ll}
\cos kx (\sin k\xi) / k & \text{if } 0 \leq \xi \leq x \\
\sin kx (\cos k\xi) / k & \text{if } x \leq \xi \leq \pi.
\end{array} \right.
\]

2. EXISTENCE THEOREMS

First, we state the following lemma, which is a modification of [4, Lemma 3]. Its proof can be obtained in analogy to [4, Lemma 3], and so is omitted.
Lemma 1. Let $a, b \in \mathbb{R}$, $a < b$, $b - a \leq \pi/k$ and let $p$ be a nonnegative function in $L^1(a, b)$. Assume that $u \in W^{2, 1}(a, b) \cap H^1_0(a, b)$ is a nontrivial solution of the boundary value problem

$$u''(x) + k^2 u(x) + p(x) u(x) = 0 \quad \text{in} \ (a, b), \quad u(a) = u(b) = 0,$$

then $\int_a^b p(x) \, dx \geq 2k \cot(k/2)(b - a)$.

In order, we prove in the following lemma a Lyapunov type inequality which is an extension of [4, Theorem 2] to the case $k \geq 2$ and is also an improvement of [3, Lemma 1] from $\|p\|_{L^1} \leq 2k$ to $\|p\|_{L^1} < 2k(k + 1) \tan(\pi/2(k + 1))$.

Lemma 2. Let $k \in \mathbb{N}$ and let $p$ be a nonnegative function in $L^1(0, \pi)$ such that $\|p\|_{L^1} < 2k(k + 1) \tan(\pi/2(k + 1))$. Assume that $u \in W^{2, 1}(0, \pi) \cap H^1_0(0, \pi)$ is a nontrivial solution of the problem

$$u''(x) + k^2 u(x) + p(x) u(x) = 0 \quad \text{in} \ (0, \pi), \quad u(0) = u(\pi) = 0.$$

Then $p(x) = 0$ a.e. $x \in (0, \pi)$, or equivalently $u = \beta \sin kx$ for some $\beta \in \mathbb{R}$, $\beta \neq 0$.

Proof. Step I: $u$ has at most finite zeros in $(0, \pi)$. If not, then there exist $2(k + 1)$ zeros $a_i, b_i \in (0, \pi)$ such that $a_i < b_i < a_{i+1} < b_{i+1}$ for all $i = 1, 2, 3, ..., k$ and $\sum_{i=1}^{k+1} (b_i - a_i) \leq \pi/k$; it follows from Lemma 1 that

$$\|p\|_{L^1} = \int_0^\pi p(x) \, dx$$

$$\geq \sum_{i=1}^{k+1} \int_{a_i}^{b_i} p(x) \, dx \geq \sum_{i=1}^{k+1} 2k \cot\left(\frac{k}{2}(b_i - a_i)\right)$$

$$\geq 2k(k + 1) \cot\left(\frac{k}{2} \frac{\sum_{i=1}^{k+1} (b_i - a_i)}{k + 1}\right)$$

$$\geq 2k(k + 1) \cot\left(\frac{k\pi}{2(k + 1)}\right)$$

$$= 2k(k + 1) \tan\left(\frac{\pi}{2(k + 1)}\right)$$

because $\cot x$ is a convex decreasing function on $(0, \pi/2)$. We obtain a contradiction.
Step II: \( b - a \leq \pi/k \) for any two continuous zeros \( a, b \) of \( u \) in \([0, \pi]\). If not, by taking the inner product of (8) with \( \sin(\pi/(b-a))(x-a) \) in \( L^2(a,b) \) we have

\[
\int_a^b \left[ k^2 - \left( \frac{\pi}{b-a} \right)^2 \right] u(x) \sin \frac{\pi}{b-a} (x-a) \, dx
+ \int_a^b p(x) u(x) \sin \frac{\pi}{b-a} (x-a) \, dx = 0,
\]

so that \( u(x) \sin(\pi/(b-a))(x-a) = 0 \) because of \( p(x) \geq 0, k^2 - (\pi/(b-a))^2 > 0 \) and \( u(x) \sin(\pi/(b-a))(x-a) \) has a fixed sign on \((a, b)\); hence \( u(x) = 0 \) on \((a, b)\) which contradicts the fact that \( u \) has only finite zeros in \((0, \pi)\).

Step III: \( u \) has at most \( k - 1 \) zeros in \((0, \pi)\). In particular, \( u \) has no zeros in \((0, \pi)\) when \( k = 1 \). If not, we choose continuous zeros \( x_0, x_1, x_2, \ldots, x_k, x_{k+1} \) of \( u \) in \([0, \pi]\), \( x_0 < x_1 < \cdots < x_{k+1} \), then \( \Delta x_i = x_i - x_{i-1} \leq \pi/k, i = 1, 2, 3, \ldots, k + 1 \), so that

\[
\int_0^\pi p(x) \, dx \geq \sum_{i=1}^{k+1} \int_{x_{i-1}}^{x_i} p(x) \, dx
\geq 2k \sum_{i=1}^{k+1} \cot \frac{k}{2} \Delta x_i
\geq 2(k + 1) \cot \frac{k}{2(k+1)} (\Delta x_1 + \Delta x_2 + \cdots + \Delta x_{k+1})
\geq 2(k + 1) \cot \frac{k\pi}{2(k+1)} = 2(k + 1) \tan \frac{\pi}{2(k+1)}
\]

which contradicts the assumption that \( \|p\|_1 < 2(k + 1) \tan(\pi/2(k + 1)) \).

Step IV: \( u \) has exactly \((k - 1)\) zeros in \((0, \pi)\). If not, then there exists \( s \in \mathbb{N}, s < k \) such that \( x_0 = 0, x_1, x_2, \ldots, x_{s-1} \), \( x_s = \pi \) are all zeros of \( u \) in \([0, \pi]\); it follows from Step II that

\[
\pi = \sum_{i=1}^s x_i - x_{i-1} \leq \sum_{i=1}^s \frac{\pi}{k} = \frac{s\pi}{k} \leq \frac{k\pi}{k} = \pi.
\]

We obtain a contradiction.
Step V: $x_i = i\pi/k, i = 0, 1, 2, 3, ..., k$ are all zeros of $u$ in $[0, \pi]$. Indeed, if $x_0 = 0, x_1, x_2, ..., x_k = \pi$ are all zeros of $u$ in $[0, \pi]$. Using Step II, $Ax_i = x_i - x_{i-1} \leq \pi/k, i = 1, 2, ..., k$, we have

$$\pi = \sum_{i=1}^{k} Ax_i$$

if and only if $Ax_i = \frac{\pi}{k}, i = 1, 2, ..., k$.

or equivalently, $x_i = i\pi/k, i = 0, 1, 2, ..., k$.

Step VI: $p(x) = 0$ a.e. on $(0, \pi)$ or equivalently,

$$v = \beta \sin kx$$

for some $\beta \in \mathbb{R}, \beta \neq 0$.

By taking the inner product in $L^2(x_{i-1}, x_i)$ of (9) with $\sin kx$ we have

$$0 = \int_{x_{i-1}}^{x_i} [u^\prime(x) + k^2u(x)] \sin kx \, dx + \int_{x_{i-1}}^{x_i} p(x) u(x) \sin kx \, dx$$

$$= \int_{x_{i-1}}^{x_i} p(x) u(x) \sin kx \, dx,$$

for all $i = 1, 2, 3, ..., k$. Consequently, $p(x) = 0$ a.e. on $(x_{i-1}, x_i)$ for each $i = 1, 2, 3, ..., k$ because of $p$ is nonnegative on $(0, \pi)$ and $u(x) \sin kx$ has no change of the sign on $(x_{i-1}, x_i)$ for all $i = 1, 2, 3, ..., k$.

Remark. $k\pi < 2k(k+1) \tan(\pi/2(k+1)) \leq 4k$ for all $k \in \mathbb{N}$.

By slightly modifying the proof of [2, Lemma 2.2], the following lemma can be obtained. Before stating the lemma, we introduce some notations. For $v \in W^{2,1}(0, \pi) \cap H^1_0(0, \pi)$, $v$ into a sine series $v = \sum_{n=1}^{\infty} b_n \sin nx$, we write $v = v^+ + v^0 + v^-$ and $v^+ = v^0 + v^- \in W^{2,1}(0, \pi) \cap H^1_0(0, \pi)$ are defined by

$$v^+ = \sum_{n=1}^{\infty} b_n \sin nx, \quad v^0 = b_k \sin kx$$

and

$$v^- = \begin{cases} \sum_{n=1}^{k-1} b_n \sin nx & \text{if } k \geq 2 \\ 0 & \text{if } k = 1. \end{cases}$$

(10)
3. Let \( \{ p_n \} \) be a sequence in \( L^1(0, \pi) \) such that \( p_n(x) \geq 0 \) for a.e. \( x \in (0, \pi) \) and for all \( n \in \mathbb{N} \), and \( p_n \to 0 \) weakly in \( L^1(0, \pi) \). Then there exists a constant \( \delta > 0 \) such that for all \( v \in W^{2,1}(0, \pi) \cap H_0^1(0, \pi) \)

\[
\int_0^\pi (v^+ + (k^2 + p_n(x)) v^-) (v^- + v^0 - v^+) \, dx \geq \delta \| v^+ \|_{H^1}^2
\]  

(11)

for \( n \) large enough.

**Theorem 4.** Let \( g: (0, \pi) \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function satisfying (H). If there exist \( c, d \in L^1(0, \pi) \) such that for a.e. \( x \in (0, \pi) \) and all \( u \geq r_0 \)

\[
c(x) \leq g(x, u),
\]

(12)

and for a.e. \( x \in (0, \pi) \) and all \( u \leq -r_0 \)

\[
g(x, u) \leq d(x),
\]

(13)

then the problem (1) is solvable for any \( h \in L^1(0, \pi) \) provided that (6) holds.

**Proof.** Let \( \alpha \in \mathbb{R} \) be fixed, \( 0 < \alpha < k \). We consider the boundary value problems

\[
u'' + k^2 u + (1 - t) \alpha u + \alpha g(x, u) = \alpha h \quad \text{in} \quad (0, \pi), \quad u(0) = u(\pi) = 0
\]

(14)

for \( 0 \leq t \leq 1 \). Then the problem (14) has only a trivial solution when \( t = 0 \), and becomes the original problem (1) when \( t = 1 \). To apply the Leray–Schauder degree theory, it suffices to show that there exists \( R_0 > 0 \) such that \( |u|_{C^0} < R_0 \) for all possible solutions \( u \) of (14) and \( 0 < t < 1 \). We first note that there exist \( e \in L^1(0, \pi) \) and Caratheodory functions \( g_1, g_2: (0, \pi) \times \mathbb{R} \to \mathbb{R} \) such that for a.e. \( x \in (0, \pi) \) and all \( u \in \mathbb{R} \)

\[
g = g_1 + g_2, \quad 0 \leq u g_1(x, u), \quad |g_1(x, u)| \leq p(x) |u| \quad \text{and} \quad |g_2(x, u)| \leq e(x).
\]

(15)

This may be done by defining

\[
g_1(x, u) = \begin{cases} 
\min \{ g(x, u) + e(x), p(x) u \} \, \theta(u) & \text{if} \quad u \geq 0 \\
\max \{ g(x, u) - e(x), p(x) u \} \, \theta(u) & \text{if} \quad u \leq 0,
\end{cases}
\]

\[
g_2 = g - g_1 \quad \text{and} \quad e(x) = \max \{ a_1(x), b(x), |c(x)|, |d(x)| \},
\]

where \( \theta: \mathbb{R} \to \mathbb{R} \) is a continuous function such that for \( u \in \mathbb{R}, \ 0 \leq \theta \leq 1, \ \theta(u) = 0 \) for \( |u| \leq r_0 \)\) and \( \theta(u) = 1 \) for \( |u| \geq 2r_0 \). To show that solutions of (14) for \( 0 < t < 1 \) have an a priori bound in \( C[0, \pi] \), we argue by contradiction and suppose that
there exist a sequence \( \{ u_n \} \) in \( W^{2,1}(0, \pi) \cap H^1_0(0, \pi) \) and a corresponding sequence \( \{ t_n \} \) in \((0, 1)\) such that \( u_n \) is a solution of (14) when \( t = t_n \) and

\[ \| u_n \|_C \geq n \text{ for all } n \in \mathbb{N}. \]

Let \( v_n = u_n/\| u_n \|_C \). Then \( \| v_n \|_C = 1 \) and

\[ v_n^*(x) + k^2 v_n(x) + (1 - t_n) \pi v_n(x) + t_n m_n(x) v_n(x) = h_n(x) \quad \text{in} \ (0, \pi), \]

\[ v_n(0) = v_n(\pi) = 0, \tag{16} \]

where

\[ m_n(x) = \begin{cases} g_1(x, u_n(x))/u_n(x) & \text{if } u_n(x) \neq 0 \\ 0 & \text{if } u_n(x) = 0, \end{cases} \]

\[ 0 \leq m_n(x) \leq p(x) \quad \text{for a.e. } x \in (0, \pi) \tag{17} \]

and \( h_n(x) = t_n\{h(x) - g_2(x, u_n(x))\}/\| u_n \|_C \). By (15), (17) and the Dunford–Pettis theorem (see [1]), the sequence \( \{ m_n \} \) has a subsequence convergent weakly in \( L^1(0, \pi) \) and \( h_n \to 0 \) in \( L^1(0, \pi) \) as \( n \to \infty \), and then using the boundedness of \( \{ p, v_n \} \) in \( L^1(0, \pi) \) and the compactness of (7) we have that \( \{ v_n^1 \} \) has a subsequence convergent in \( C[0, \pi] \), where

\[ p_n(x) = (1 - t_n) \pi + t_n m_n(x) \text{ for all } n \in \mathbb{N} \text{ and a.e. } x \in (0, \pi). \]

Since \( \{ v_n^0 \} \) is bounded in a one-dimensional subspace of \( C[0, \pi] \), we may assume that \( \{ v_n \} \) is convergent in \( C[0, \pi] \). From (16) it follows that \( \{ v_n^0 \} \) is dominated by a function in \( L^1(0, \pi) \). Since \( v_n^0 \) vanishes somewhere in \((0, \pi)\), the sequence \( \{ v_n^0 \} \) is equicontinuous and uniformly bounded on \([0, \pi]\) and so by the Ascoli theorem that \( \{ v_n^0 \} \) has a subsequence convergent in \( C[0, \pi] \). We may assume without loss of generality that \( m_n \to m \) weakly in \( L^1(0, \pi) \), \( t_n \to t \), and \( v_n \to v \) in \( C^1[0, \pi] \). It follows from the Mazur theorem that \( 0 \leq m(x) \leq p(x) \) for a.e. \( x \in (0, \pi) \). By (7) and (16) we have

\[ v^*(x) + k^2 v(x) + [(1 - t_0) \pi + t_0 m(x)] v(x) = 0 \quad \text{in} \ (0, \pi), \]

\[ u(0) = u(\pi) = 0. \tag{18} \]

Clearly, \( \| v \|_C = 1 \). Since \( 0 < \pi < k < (1/\pi) \) \( 2k(k + 1) \tan \pi/2(k + 1) \) and \( \| m \|_L^1 \leq \| p \|_L^1 < 2k(k + 1) \tan \pi/2(k + 1) \), it follows from Lemma 2 that \((1 - t_0) \pi + t_0 m(x) = 0 \) a.e. \( x \in (0, \pi) \), and consequently \( t_0 = 1, m(x) = 0 \) a.e. \( x \in (0, \pi) \) and \( v = \beta \sin kx \) for some \( \beta \neq 0 \).

Obviously, \( \{ v_n^0 \} \) also converges to \( v = \beta \sin kx \) in \( C^1[0, \pi] \). Taking the inner product of (14) with \( v_n^0 \) in \( L^2(0, \pi) \) when \( u = u_n \) and \( t = t_n \), we have

\[ t_n \int_0^\pi g(x, u_n(x)) v_n^0(x) \, dx \]

\[ \leq (1 - t_0) \int_0^\pi u_n(x) v_n^0(x) \, dx + t_n \int_0^\pi g(x, u_n(x)) v_n^0(x) \, dx \]

\[ = t_n \int_0^\pi h(x) v_n^0(x) \, dx \tag{19} \]
for $n$ large enough. Moreover, using $v_n \to v$ in $C[0, \pi]$ and $|u_n|_{C^0} \geq n$ we have $u_n(x) \to \infty$ if $v(x) > 0$, and $u_n(x) \to -\infty$ if $v(x) < 0$. We assume for the moment that \{\{ g(x, u_n(x)) v_n^0(x) \}\} is bounded from below by a function in $L^1(0, \pi)$ for a.e. $x \in (0, \pi)$ and $n$ large enough. Applying the Fatou lemma to the inequality

$$
\int_{\{v_n^0 \geq 0\}} g(x, u_n(x)) v_n^0(x) \, dx + \int_{\{v_n^0 < 0\}} g(x, u_n(x)) v_n^0(x) \, dx
$$

we have

$$
= \int_{0}^{\pi} g(x, u_n(x)) v_n^0(x) \, dx < \int_{0}^{\pi} h(x) v_n^0(x) \, dx
$$

we have

$$
= \int_{\{v_n^0 \geq 0\}} g(x, u_n(x)) v_n^0(x) \, dx + \int_{\{v_n^0 < 0\}} g(x, u_n(x)) v_n^0(x) \, dx
$$

$$
\leq \liminf_{n \to \infty} \int_{\{v_n^0 \geq 0\}} g(x, u_n(x)) v_n^0(x) \, dx
$$

$$
+ \liminf_{n \to \infty} \int_{\{v_n^0 < 0\}} g(x, u_n(x)) v_n^0(x) \, dx
$$

$$
= \liminf_{n \to \infty} \int_{\{v_n^0 \geq 0\}} g(x, u_n(x)) v_n^0(x) \, dx
$$

$$
+ \liminf_{n \to \infty} \int_{\{v_n^0 < 0\}} g(x, u_n(x)) v_n^0(x) \, dx
$$

$$
\leq \liminf_{n \to \infty} \int_{0}^{\pi} g(x, u_n(x)) v_n^0(x) \, dx
$$

$$
\leq \int_{0}^{\pi} h(x) v(x) \, dx,
$$

which contradicts the condition (6).
It remains to prove that \( \{ g(x, u_n(x)) v_n(x) \} \) is bounded from below by a function in \( L^1(0, \pi) \) for a.e. \( x \in (0, \pi) \) and \( n \) large enough. By Lemma 3, it follows from (16) that there exists \( \delta > 0 \) such that

\[
\delta \| v_n \|_{H^1}^2 \leq \int_0^\pi h_n(x)(v_n^-(x) + v_n^0(x) - v_n^+(x)) \, dx
\]

\[
\leq \| h_n \|_{L^1} \| v_n^- + v_n^0 - v_n^+ \|_C
\]

\[
\leq 1/\| u_n \|_C \left( \| h \|_{L^1} + \| e \|_{L^1} \right) \| v_n^- + v_n^0 - v_n^+ \|_C.
\] (20)

Since \( v_n^0 \to v \) in \( C[0, \pi] \), \( v_n^- \to 0 \) in \( H^1_0(0, \pi) \), so that both \( v_n^- \) and \( v_n^+ \) are convergent to zero in \( H^1_0(0, \pi) \) and hence by the compact imbedding of \( H^1_0(0, \pi) \) into \( C[0, \pi] \) we also have for \( x \in [0, \pi] \)

\[
\| u_n \|_C \| v_n^+(x) \|_2 \leq \| u_n \|_C \| v_n^+ \|_2 \leq C_1 \| u_n \|_C \| v_n^+ \|_H \leq C_1 C_2 = C_3
\]

for some constants \( C_1, C_2, C_3 \geq 0 \) independent of \( n \). We may assume that \( \{ \| v_n^+ \|_C \} \) is also bounded by \( C_3 \). Then

\[
u_n(x) v_n^+(x) \geq -\| u_n \|_C \| v_n^+(x) - v_n^0(x) \|_2 / 2
\]

\[
= -\| u_n \|_C \| v_n^+(x) \|_2 / 2 \geq - C_3 / 2
\]

and so

\[
g(x, u_n(x)) v_n^0(x) = \left[ g_1(x, u_n(x))/u_n(x) \right] u_n(x) v_n^0(x) + g_2(x, u_n(x)) v_n^0(x)
\]

\[
\geq (C_3 / 2) p(x) - C_3 e(x)
\]

for a.e. \( x \in (0, \pi) \) and \( n \) large enough. This completes the proof of the theorem.

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