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Risk and Pitman closeness properties of feasible generalized double k-class estimators in linear regression models with non-spherical disturbances under balanced loss function

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Abstract

In this article, a family of feasible generalized double k-class estimator in a linear regression model with non-spherical disturbances is considered. The performance of this estimator is judged with feasible generalized least-squares and feasible generalized Stein-rule estimators under balanced loss function using the criteria of quadratic risk and general Pitman closeness. A Monte-Carlo study investigates the finite sample properties of several estimators arising from the family of feasible double k-class estimators.

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Keywords: Linear regression model; Balanced loss function; Pitman closeness; Double k-class estimator

1. Introduction

Precision of estimation is an important criterion for judging the quality of estimation procedures for the parameters in linear regression models. Appreciating the role of the goodness of fitted model, Zellner [25] has advocated that both the precision of estimation and the goodness of fit should be utilized in analyzing the performance of estimators and therefrom making a choice of estimator. Accordingly,

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he has recommended the use of balanced loss function, which is indeed a convex linear combination of two quantities measuring the precision of estimator and the goodness of fitted model. Both the quantities have quadratic loss structure and can also be viewed as reflecting the predictive ability of the estimator for the actual and average values of the study variable within the sample; see, e.g., Shalabh [19]. Thus the balanced loss function provides an interesting framework for structuring the losses in a fairly general and sufficiently flexible manner.

Taking the criterion as risk, i.e., the expected value of balanced loss function, interesting findings arising from comparison of estimators are reported by, e.g., Giles et al. [6], Ohtani [12,13], Dey et al. [4], Ohtani et al. [14], Wan [23] and Zellner [26], Toutenburg and Shalabh [21] to site a few. Such a performance criterion on being based on quadratic losses may fail to reveal the intrinsic properties of estimator as pointed out by Rao [16], and it is recommended to use a criterion that measures appropriately the clustering of estimates around true parameter value. One such criterion is the probability of closeness introduced by Pitman [15] for the scalar parameter case. An extension to the multivariate parameter case is the generalized Pitman closeness criterion presented by Rao et al. [17]; see also Keating et al. [7], Kourouklies [8], Mason et al. [9], Rao et al. [18], Srivastava and Srivastava [20] and the cited references for interesting exposition and applications.

In this paper, we consider a linear regression model with not necessarily spherical disturbances and present the family of feasible generalized double k-class estimators presented by Wan and Chaturvedi [24] for estimating the regression coefficients of a linear regression model with non-spherical disturbances. To study the efficiency properties of these estimators, a suitable balanced loss function is formulated and large sample asymptotic approximations for the risk and generalized Pitman closeness under this loss function are employed for the comparison of estimators. Some Monte-Carlo results are also presented for the dominance of several estimators arising from feasible double k-class estimators.

2. Model specification, estimators and loss functions

Consider the following linear regression model:

$$y = X\beta + \varepsilon, \tag{2.1}$$

where y is a $T \times 1$ vector of T observations on the study variable, X is a $T \times p$ matrix of T observations on p explanatory variables, β is a $p \times 1$ vector of coefficients associated with them and ε is a $T \times 1$ vector of not necessarily spherical disturbances.

The disturbance vector ε is assumed to follow a multivariate normal distribution with mean vector 0 and unknown variance covariance matrix $\sigma^2\Omega^{-1}$. Further, we assume that the elements of Ω are functions of a $q \times 1$ parameter vector θ belonging to an open subset of q dimensional Euclidean space. It is also assumed that a consistent estimator $\hat{\theta}$ of θ is available so that Ω consistently estimated by $\hat{\Omega}$.

If we estimate β in (2.1) by the method of generalized least squares and use $\hat{\Omega}$ in place of Ω, the feasible generalized least-squares (FGLS) estimator of β is given by

$$\hat{\beta} = (X'\hat{\Omega}X)^{-1}X'\hat{\Omega}y. \tag{2.2}$$

In the spirit of Ullah and Ullah [22], Wan and Chaturvedi [24] have presented the following family of feasible generalized double k-class estimators which are characterized by two non-stochastic scalars k_1 and k_2. Terming it as feasible generalized double k-class (FGKK) estimators defined by

$$\hat{\beta}_{kk} = \left[1 - \left(\frac{k_1}{T-p+2}\right)\frac{(y-X\hat{\beta})'\hat{\Omega}(y-X\hat{\beta})}{y'\hat{\Omega}y - k_2(y-X\hat{\beta})'\hat{\Omega}(y-X\hat{\beta})}\right]\hat{\beta}, \tag{2.3}$$

they have obtained the asymptotic distribution of estimators for $k_1 > 0$ and $k_2 < 1$ under certain regularity conditions choosing the performance criterion as asymptotic risk under a general quadratic loss function, they have deduced conditions for the superiority of FGKK estimators over the FGLS estimator $\hat{\beta}$ and feasible generalized Stein-rule (FGSR) estimator $\hat{\beta}_{\mathrm{SR}}$ which is specified by FGKK with $k_2 = 1$. They also reported the results of Monte-Carlo experiment comparing several estimators for the model with $AR(1)$ disturbances.

While precision of estimator as measured by the general quadratic losses is an important criterion for judging the performance and therefrom making a choice of estimator, Zellner [25] has argued that the goodness of fitted model should also be taken into account. Accordingly, he has proposed the balanced loss function which has gained considerable popularity during the recent past; see, e.g., Giles et al. [6], Ohtani [12], Ohtani et al. [14], Wan [23] and Zellner [26] for various applications.

The balanced loss function for the estimation of β by an estimator $\tilde{\beta}$ is specified by

$$L(\tilde{\beta}, \beta) = \alpha(\tilde{\beta}-\beta)'X'\Omega X(\tilde{\beta}-\beta) + (1-\alpha)(y-X\tilde{\beta})'\Omega(y-X\tilde{\beta}), \tag{2.4}$$

where α is a non-stochastic scalar lying between 0 and 1.

The first component on the right-hand side of (2.4) reflects the goodness of fitted model while the second component reflects the precision of estimation. Alternatively, they can be regarded as measuring the performance of predictions $(X\tilde{\beta})$ for the average values $(X\beta)$ and the actual values (y) of the study variable; see also Shalabh [19].

If we put $\alpha = 0$ in (2.4), then the balanced loss function reduces to the criterion of precision of estimation only whereas $\alpha = 1$ provides the criterion of goodness of fit. Any other choice of $0 < \alpha < 1$ will decide the weight to be assigned to the precision of estimation and goodness of fit. The choice of α essentially depends on the experimenter and objective of the experiment. The experimenter may decide the value of α to be assigned on the basis of his experience with similar type of studies in the past, long association with the experiment or some prior information about the experiment.

Another interesting criterion is the generalized Pitman closeness employing the loss function (2.4).

Exact expressions for both the criteria in the present context are difficult to derive. Even if one succeeds in obtaining the expressions, their intricate nature would not permit us to conduct superiority comparisons and therefore consider their approximations when T is large. For this purpose, we assume that the explanatory variables are asymptotically cooperative in the sense that the matrix $T^{-1}X'\Omega X$ approaches to a finite and non-singular matrix as $T \to \infty$.

In order to derive the asymptotic distribution of $\hat{\beta}_{kk}$, we follow Chaturvedi and Shukla [3]. Assuming k_1 to be of order $0(1)$, we define for all $j, k = 1, 2, \ldots, q$

$$\Omega_j = \frac{\partial \Omega}{\partial \theta_j}, \quad \Omega_{jk} = \frac{\partial^2 \Omega}{\partial \theta_j \partial \theta_k},$$

$$A = \frac{X'\Omega X}{T}, \quad A_j = \frac{X'\Omega_j X}{T}, \quad A_{jk} = \frac{X'\Omega_{jk} X}{T}$$

and

$$\boldsymbol{a} = \frac{X'\Omega\varepsilon}{\sqrt{T}}, \quad \boldsymbol{a}_j = \frac{X'\Omega_j\varepsilon}{\sqrt{T}}, \quad \boldsymbol{a}_{jk} = \frac{X'\Omega_{jk}\varepsilon}{\sqrt{T}}.$$

Further, we require the following regularity conditions for the validity of Edgeworth expansion of the distribution.

(i) Each matrix in the sets $A_1, A_2, \ldots, A_q$ and covariance matrix of each vector in $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_q$ converges to a finite matrix as $T \to \infty$.
(ii) $\frac{X'C^2X}{T}$ is bounded and tends to a finite matrix as T tend to infinity for all C in Ω_6.
(iii) Finally, we assume that $\hat{\theta}$ has an even function of $M\varepsilon$ where $M = I - X(X'X)^{-1}X'$ and has a stochastic expansion

$$\hat{\theta} = \theta + \frac{1}{T^{1/2}} e + O_p(T^{-1}), \tag{2.5}$$

where the distribution of e is multivariate normal such that $E(e)$ is of order $O(T^{-1})$ and $E(ee') = \Lambda + O(T^{-1})$ where the element of matrix Λ are of order $O(1)$. This assumption ensures that the estimation error of $\hat{\theta}$ can be expanded with respect to T to the required order of approximation.

Under the above assumptions, we analyze the efficiency properties of estimators with respect to the criteria of asymptotic risk and general Pitman closeness under the balanced loss function in the next two sections.

3. Risk under the balanced loss function

For comparing the FGLS, FGSR and FGKK estimators, we suppose α to be positive and k_2 smaller than 1 so that the FGKK estimator has finite moments. Now consider the following quantities:

$$D_1 = \frac{\sqrt{T}(\phi + 1 - k_2)}{2\sigma^2 k_1 \alpha \sqrt{\phi}} [L(\hat{\beta}, \beta) - L(\hat{\beta}_{kk}, \beta)], \tag{3.1}$$

$$D_2 = \frac{\sqrt{T\phi}(\phi + 1 - k_2)}{2\sigma^2 k_1 \alpha (1 - k_2)} [L(\hat{\beta}_{\mathrm{SR}}, \beta) - L(\hat{\beta}_{kk}, \beta)], \tag{3.2}$$

where

$$\phi = \frac{1}{\sigma^2 T} \beta' X' \Omega X \beta. \tag{3.3}$$

The FGKK estimator is said to be superior than the FGLS estimator with respect to the criterion of risk under the balanced loss function when $E[L(\hat{\beta}, \beta)]$ is greater than $L(\hat{\beta}_{kk}, \beta)$, i.e., $E(D_1) > 0$. Similarly, the FGKK estimator is superior to the FGSR estimator when $E(D_2) > 0$.

Theorem I. *The asymptotic approximation for $E(D_1)$ and $E(D_2)$ to order $O(T^{-1/2})$ are given by*

$$E(D_1) = \frac{1}{2\sqrt{T\phi}} \left[2p - \frac{\phi}{\phi + 1 - k_2} \left(4 + \frac{k_1}{\alpha} \right) \right], \tag{3.4}$$

$$E(D_2) = \frac{1}{2\sqrt{T\phi}} \left[-2p + \left(1 + \frac{\phi}{\phi + 1 - k_2} \right) \left(4 + \frac{k_1}{\alpha} \right) \right]. \tag{3.5}$$

Let $\rho(\hat{\beta}, \beta)$ denote the risk of the estimator $\hat{\beta}$ under the balanced loss function (2.4).

$$Q_{ij} = \frac{1}{T} X' \Omega_i \Omega^{-1} \Omega_j X - A_i A^{-1} A_j,$$

and λ_{ij} be the (i,j)th element of $\lim_{T \to \infty} E[T(\hat{\theta} - \theta)(\hat{\theta} - \theta)']$. Then following Chaturvedi and Shukla [3], up to order $O(T^{-1})$, we obtain the following expression for the risk of FGLS estimator under the balanced loss function

$$\rho(\hat{\beta}, \beta) = \sigma^2 \left[\alpha p + (1 - \alpha)(T - p) + \frac{1}{T} \sum_{i,j=1}^{q} \mathrm{tr}(A^{-1} Q_{ij}) \lambda_{ij} \right]. \tag{3.6}$$

Further, combining (3.6) and (3.4), the risk of FGKK estimator, up to order of our approximation, is given by

$$\rho(\hat{\beta_{kk}}, \beta) = \sigma^2 \left[\alpha p + (1 - \alpha)(T - p) + \frac{1}{T} \sum_{i,j=1}^{q} \mathrm{tr}(A^{-1} Q_{ij}) \lambda_{ij} \right] + \frac{\sigma^2 k_1 \alpha}{T(\phi + 1 - k_2)} \left[2p - \frac{\phi}{\phi + 1 - k_2} \left(4 + \frac{k_1}{\alpha} \right) \right]. \tag{3.7}$$

In (3.6) and (3.7), the term

$$\frac{\sigma^2}{T}\sum_{i,j=1}^{q}\mathrm{tr}(A^{-1}Q_{ij})\lambda_{ij}$$

gives the effect of estimation covariance matrix on the risk of FGLS and FGKK estimators. Hence, up to the order of our approximation, the effect of estimation of covariance matrix is the same on FGLS estimator and FGKK estimator and independent of the scalar α.

From (3.4), we observe that the FGKK estimator is better than the FGLS estimator according to the criterion of risk under balanced loss function to the order of our approximation when

$$0<k_1<2\alpha\left[(p-2)+\frac{(1-k_2)p}{\phi}\right],\quad p>2-\frac{(1-k_2)p}{\phi}. \tag{3.8}$$

An interesting implication of this result is that the FGKK estimator performs better than the GLS estimator in case of $p=2$ when k_1 is smaller than $4\alpha(1-k_2)/\phi$. This result remains true for $p=1$ also provided that

$$0<k_1<\frac{2\alpha}{\phi}(1-k_2-\phi),\quad k_2<(1-\phi). \tag{3.9}$$

Setting $\alpha=1$ in (3.6), we obtain

$$0<k_1<2\left[(p-2)+\frac{(1-k_2)p}{\phi}\right] \tag{3.10}$$

which is the condition obtained by Wan and Chaturvedi [24]. Comparing (3.8) and (3.10), we observe that the range of k_1 for the superiority of FGKK estimator over the FGLS estimator shrinks if we take the criterion as risk under balanced loss function instead of the risk under traditional quadratic error loss function measuring only the precision of estimation.

If we consider the superiority of FGSR estimator over the FGLS estimator, we find from (3.8) with $k_2=1$ that the FGSR estimator unlike FGKK estimator can never perform better than the FGLS estimator for $p=1$ and 2. Further, for $p>2$, the range of k_1 for the superiority of FGSR estimator over the FGLS estimator is smaller than the corresponding range for the superiority of FGKK estimator over the FGLS estimator.

Next, let us compare the FGSR and FGKK estimators.

From (3.5), we observe that the FGKK estimator is better than the FGSR estimator with respect to the criterion of risk under the balanced loss function to the order of our approximation when

$$k_1>2\alpha\left[(p-2)-\frac{\phi p}{2\phi+1-k_2}\right]. \tag{3.11}$$

Combining it with (3.8), we see that the FGKK is superior to both the FGLS and FGSR estimators when

$$2\alpha\left[(p-2) - \frac{\phi p}{2\phi + 1 - k_2}\right] < k_1 < 2\alpha\left[(p-2) + \frac{(1-k_2)}{\phi}\right]. \tag{3.12}$$

Thus we find that the FGKK estimator for all values of $k_1 > 0$ and $k_2 < 1$ has better performance than the GLS and GSR estimators for $p = 2$. This result continues to hold good for $p = 1$ also provided that the scalars k_1 and k_2 satisfy condition (3.9).

4. General Pitman closeness under the balanced loss function

Following Rao et al. [17], we take the performance criterion as the general Pitman closeness under the balanced loss function; see also Keating et al. [7] for an expository account. Accordingly, the GKK estimator is said to be superior to the GLS and GSR estimators when the respective probabilities $P(D_1 > 0)$ and $P(D_2 > 0)$ are greater than 0.5; see also Chaturvedi and Bhatti [2].

As it is difficult to derive the exact expressions for the probabilities $P(D_1 > 0)$ and $P(D_2 > 0)$, we consider their approximations when T is large.

Theorem II. *The asymptotic expressions for the probabilities $P(D_1 > 0)$ and $P(D_2 > 0)$, to order $O(T^{-1/2})$ are given by*

$$P(D_1 > 0) = 0.5 + \frac{1}{\sqrt{2T\pi\phi}}\left[p - 1 - \frac{\phi k_1}{2\alpha(\phi + 1 - k_2)}\right], \tag{4.1}$$

$$P(D_2 > 0) = 0.5 - \frac{1}{\sqrt{2T\pi\phi}}\left[p - 1 - \frac{(2\phi + 1 - k_2)k_1}{2\alpha(\phi + 1 - k_2)}\right]. \tag{4.2}$$

According to the criterion of generalized Pitman closeness, the FGKK estimator is better than the FGLS estimator when $P(D_1 > 0)$ exceeds 0.5, i.e., when

$$0 < k_1 < 2\alpha(p-1)\left(1 + \frac{1-k_2}{\phi}\right), \quad p > 1. \tag{4.3}$$

Setting $k_2 = 1$ in (4.3), we obtain the condition for the superiority of FGSR estimator over the FGLS estimator as follows:

$$0 < k_1 < 2\alpha(p-1), \quad p > 1 \tag{4.4}$$

see, Srivastava and Srivastava [20] who have considered the linear regression model with $\Omega = I$ and $\alpha = 1$.

Comparing (4.3) and (4.4), it is interesting to note that the range of k_1 for the superiority of FGSR estimator over the FGLS estimator is shorter than the corresponding range for the superiority of FGKK estimator over the FGLS estimator.

For the special case $p = 1$, it is obvious from (4.1) that none of the FGSR and FGKK estimators dominates the FGLS estimator.

Next, let us compare the FGSR and FGKK estimators.

It is seen from (4.2) that the FGKK estimator has better performance than the FGSR estimator with respect to the criterion of generalized Pitman closeness under the balanced loss function to the order of our approximation when

$$k_1 > 2\alpha(p-1)\left(1 - \frac{\phi}{2\phi + 1 - k_2}\right). \tag{4.5}$$

Combining it with (4.3), it is seen that the FGKK estimator is better than both the FGLS and FGSR estimators when the characterizing scalar k_1 satisfies the condition

$$2\alpha(p-1)\left(1 - \frac{\phi}{2\phi + 1 - k_2}\right) < k_1 < 2\alpha(p-1)\left(1 + \frac{1-k_2}{\phi}\right) \tag{4.6}$$

provided that p exceeds one.

5. Monte-Carlo study

The Monto-Carlo experiment was conducted on SHAZAM econometric package. It considers the following linear regression model:

$$y = x\beta + \varepsilon.$$

The observations on X are generated such that $X'X = I$. The error process is assumed to follow an $AR(1)$ process, i.e.,

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t,$$

where $|\rho| < 1$ and v_t's are i.i.d. following $N(0, \sigma_v^2)$. We set $\sigma^2 = 1$ and consider $T = 20, 30, 60; p = 4, 12, 20; \rho = -0.7, -0.5, -0.3, -0.1, 0.1, 0.3, 0.5, 0.7$ and weights of the balanced loss function $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$. The following cases of FGKK are considered:

(i) When $k_1 = 0$, i.e., FGLS.
(ii) When $k_1 = p - 2$ and $k_2 = 1$, i.e., FGSR.
(iii) When $k_1 = \frac{1}{T-p}$ and $k_2 = 1 - k_1$, i.e., feasible generalized minimum mean squared error estimator (FGMMSE), proposed by [5], see also [10,11].
(iv) When $k_1 = \frac{T-p+2}{T-p}$ and $k_2 = 1 - \frac{k_1}{T-p+2}$, i.e., adjusted feasible generalized minimum mean squared error estimator (AFGMMSE), proposed by [10].
(v) When $k_1 = \frac{(T-p+2)p}{T-p}$ and $k_2 = 1 - \frac{k_1}{T-p+2}$, i.e., feasible generalized double k-class estimator (FGKKC), suggested by Carter et al. [1].

These estimators are exposed to two criteria, viz., risk and general Pitman nearness under balanced loss function. The quantities $E(D_1), E(D_2), P(D_1 > 0)$ and $P(D_2 > 0)$ as defined in (3.4), (3.5), (4.1) and (4.2), respectively, are computed based on 2000 replications. The quantities $E(D_1)$ and $E(D_2)$ compare FGMMSE, AFGMMSE and FGKKC with FGLS and FGSR, respectively, under risk criterion whereas $P(D_1 > 0)$ and $P(D_2 > 0)$ does the same under Pitman nearness criterion. For example, if $E(D_1 > 0)$ and $P(D_1 > 0) > 0.5$ for FGMMSE, it implies that FGMMSE is superior to FGLS under risk and Pitman criteria, respectively, and vice versa. Similarly, $E(D_2 > 0)$ and $P(D_2 > 0) > 0.5$ for FGMMSE indicate the superiority of FGMMSE over FGSR under risk and Pitman nearness criteria and vice versa. Some representative computed values of $E(D_1), E(D_2), P(D_1 > 0)$ and $P(D_2 > 0)$ are represented in Tables 1–4 respectively and are plotted against ρ in Figs. 1–4.

Now, we analyze the performance of these estimators from these simulation results. Firstly, we adopt the risk criterion and analyze $E(D_1)$ (see Table 1).

When $\alpha = 0.1$, i.e., 10% weight is assigned to goodness of fit and 90% weight to the precision of estimation (see Fig. 1(a)). It can be seen clearly, that FGLS is uniformly superior to FGKKC irrespective of values of ρ. It is observed that, in general, FGMMSE and AFGMMSE dominate FGLS when $|\rho| < 0.5$ and for other values of ρ, the opposite holds true. For higher values of $|\rho|$, FGMMSE dominates AFGMMSE as well as FGKKC. The magnitude of this dominance decreases as $|\rho|$ decreases and then, in particular, performance of FGMMSE and AFGMMSE becomes almost same but still both dominates FGLS.

When weight for goodness of fit increases, i.e., $\alpha = 0.3$ (see Fig. 1(b)), then interestingly, FGKKC and FGLS are equally efficient but AFGMMSE and FGMMSE uniformly dominates FGLS. However, the magnitude of this dominance decreases as $|\rho|$ decreases. In general, AFGMMSE dominate the FGMMSE as well as FGKKC.

When goodness of fit and precision of estimation are equally important, i.e., $\alpha = 0.5$ (see Fig. 1(c)), then AFGMMSE uniformly dominates the FGMMSE and FGKKC both. The dominance of FGMMSE and FGKKC over FGLS is low for lower values of $|\rho|$ and as $|\rho|$ increases, their dominance also increases.

Surprisingly, when goodness of fit is assigned higher weight than precision of estimation, i.e., $\alpha \geqslant 0.7$ (see Fig. 1(d) and (e)), the dominance results are same as in the case when $\alpha = 0.5$. Thus the effect of assigning higher weight to precision of estimation and to goodness of fit is clearly visible in the performance of these estimators.

Now, consider the values of $E(D_2)$ (see Table 2) to study the performance of FGMMSE, AFGMMSE and FGKKC with FGSR. When $\alpha = 0.1$ (see Fig. 2(a)), FGMMSE dominates AFGMMSE and FGKKC both when $|\rho|$ is high, say, greater than 0.5. Also, the magnitude of dominance of FGMMSE over FGSR is higher for high values of $|\rho|$ and low for lower values of $|\rho|$. The AFGMMSE and FGKKC are uniformly equally efficient with respect to FGSR as well as between themselves irrespective of the values of ρ. On the other hand, FGMMSE is more efficient than AFGMMSE only for higher values of $|\rho|$, (say > 0.5). So choice of FGMMSE is a better option when precision of estimation is more important than goodness of fit.

Interestingly enough, when goodness of fit is assigned more than 30% weight, the performance of all the estimators almost stabilizes for all values of $\alpha \geqslant 0.3$ (see Fig. 2(c)–(e)). Under such condition, AFGMMSE and FGKKC are as good as FGSR. The magnitude of dominance of FGSR over FGMMSE is higher for higher values of $|\rho|$. The choice of FGSR emerges out to be a better choice under such conditions.

Table 1
Values of $E(D_1)$

α	ρ	$T=20$, $p=4$ FGM-MSE	$T=20$, $p=4$ AFG-MMSE	$T=20$, $p=4$ FGKKC	$T=30$, $p=12$ FGM-MSE	$T=30$, $p=12$ AFG-MMSE	$T=30$, $p=12$ FGKKC	$T=30$, $p=20$ FGM-MSE	$T=30$, $p=20$ AFG-MMSE	$T=30$, $p=20$ FGKKC	$T=60$, $p=20$ FGM-MSE	$T=60$, $p=20$ AFG-MMSE	$T=60$, $p=20$ FGKKC
0.1	−0.7	−78.220	−107.466	−105.472	−33.774	−69.440	−67.803	−62.579	−180.878	−170.169	−5.220	−9.670	−29.678
0.1	−0.5	−25.975	−31.295	−25.472	2.023	0.992	−17.716	−7.486	−27.037	−47.670	6.024	8.483	−14.021
0.1	−0.3	10.414	12.086	−1.277	6.522	9.082	−7.871	4.883	6.342	−18.500	9.122	12.876	−9.017
0.1	−0.1	7.319	8.481	−0.596	7.750	11.073	−4.779	7.733	13.948	−11.537	9.600	13.497	−8.809
0.1	0.1	6.027	7.106	−0.048	8.840	12.738	−4.570	8.165	14.782	−11.329	8.332	11.938	−13.235
0.1	0.3	6.486	8.171	−0.487	8.648	12.672	−7.408	5.333	7.308	−18.740	−1.583	−3.497	−29.343
0.1	0.5	−12.944	−11.783	−22.053	3.357	4.465	−18.412	−3.607	−18.146	−43.321	−70.064	−134.872	−104.636
0.1	0.7	−73.410	−105.080	−96.464	−37.838	−73.986	−70.228	−46.668	−151.135	−152.089	−1255.315	−2864.846	−1277.400
0.3	−0.7	46.306	61.633	14.370	26.233	47.326	3.773	23.821	56.121	−5.801	14.708	24.258	1.452
0.3	−0.5	13.838	16.839	4.362	12.322	19.826	2.410	11.644	28.058	0.451	10.551	15.972	1.375
0.3	−0.3	6.670	7.901	1.955	7.953	12.369	1.569	8.751	20.487	1.727	8.871	12.990	1.193
0.3	−0.1	4.354	5.163	1.105	6.797	10.476	1.415	7.999	18.501	2.183	9.126	13.330	1.307
0.3	0.1	4.339	5.250	1.050	7.044	10.819	1.501	7.851	18.136	2.094	11.046	16.607	1.492
0.3	0.3	6.038	7.555	1.448	9.165	14.432	2.032	9.135	21.809	2.172	16.801	27.447	2.155
0.3	0.5	11.940	16.262	2.826	14.818	25.378	3.069	12.998	31.576	1.453	35.393	67.941	3.986
0.3	0.7	32.798	50.435	7.341	32.429	67.534	5.902	29.292	74.684	0.294	300.105	741.032	75.216
0.5	−0.7	39.973	53.273	14.043	40.328	74.680	19.112	41.262	103.599	26.222	18.712	31.083	7.743
0.5	−0.5	11.663	14.261	4.273	14.196	23.457	6.320	15.313	38.379	9.527	11.297	17.314	4.323
0.5	−0.3	5.957	7.106	2.121	8.268	12.988	3.473	9.673	23.340	5.863	8.840	13.055	3.240

0.5	−0.1	3.997	4.808	1.285	6.501	10.033	2.617	7.890	19.074	4.816	8.961	13.201	3.293
0.5	0.1	4.122	5.013	1.297	6.885	10.681	2.804	8.034	19.182	4.901	11.814	17.886	4.546
0.5	0.3	6.014	7.674	1.816	9.206	14.831	3.876	9.843	24.214	6.154	20.725	34.112	8.511
0.5	0.5	11.569	15.555	3.407	17.450	30.360	7.708	16.539	42.157	10.659	54.230	104.382	24.320
0.5	0.7	43.053	59.271	7.820	46.757	95.470	21.284	44.986	121.107	30.928	584.731	1380.152	330.748
0.7	−0.7	39.349	51.576	14.971	45.051	84.206	24.973	50.317	127.835	41.598	20.520	33.919	10.455
0.7	−0.5	10.942	13.370	4.324	15.465	25.829	8.285	17.028	42.638	13.514	11.611	17.826	5.598
0.7	−0.3	5.414	6.499	2.061	8.245	13.167	4.188	10.074	25.098	7.796	8.851	13.146	4.119
0.7	−0.1	3.716	4.440	1.356	6.634	10.460	3.287	7.953	19.584	6.018	8.916	13.202	4.149
0.7	0.1	3.939	4.806	1.374	6.750	10.490	3.329	7.959	19.659	6.061	12.066	18.349	5.793
0.7	0.3	5.872	7.437	1.971	9.500	15.344	4.803	10.275	25.569	7.970	22.086	36.566	11.108
0.7	0.5	12.062	16.461	3.760	17.891	31.204	9.281	18.360	46.859	14.777	63.672	121.820	33.766
0.7	0.7	41.286	63.213	11.470	51.823	105.160	27.336	50.745	138.459	43.205	703.381	1632.433	437.842
0.9	−0.7	41.349	54.958	15.918	47.852	89.167	28.342	52.401	134.877	47.644	21.653	36.026	12.039
0.9	−0.5	10.881	13.354	4.465	15.340	25.629	8.869	17.735	45.631	15.715	11.807	18.150	6.311
0.9	−0.3	5.304	6.417	2.099	8.309	13.256	4.629	10.110	25.601	8.696	8.679	12.850	4.514
0.9	−0.1	3.717	4.476	1.422	6.420	10.055	3.492	7.935	19.649	6.653	8.930	13.196	4.645
0.9	0.1	3.900	4.801	1.425	6.748	10.563	3.660	7.989	19.816	6.709	12.196	18.587	6.494
0.9	0.3	5.846	7.388	2.067	9.485	15.390	5.229	10.390	26.240	8.930	23.347	38.676	12.803
0.9	0.5	12.627	17.095	4.154	18.045	31.351	10.102	18.485	48.377	16.439	67.972	130.307	38.489
0.9	0.7	41.912	64.649	12.112	56.538	114.913	31.743	55.183	148.480	50.901	807.409	1900.012	520.884

Table 2
Values of $E(D_2)$

α	ρ	$T=20$,	$p=4$		$T=30$,	$p=12$		$T=30$,	$p=20$		$T=60$,	$p=20$	
		FGM-MSE	AFG-MMSE	FGKKC	FGM-MSE	AFG-MMSE	*FGKKC*	FGM-MSE	AFG-MMSE	FGKKC	FGM-MSE	AFG-MMSE	FGKKC
0.1	−0.7	84.894	11.441	−10.049	116.442	7.946	−8.264	615.777	23.597	7.894	435.543	17.511	−17.497
0.1	−0.5	51.256	2.197	−4.993	20.996	1.556	−2.503	132.897	5.537	2.091	64.404	2.859	−4.428
0.1	−0.3	30.388	0.067	−1.087	6.162	0.496	−1.219	50.280	2.214	0.964	17.491	0.818	−2.068
0.1	−0.1	15.234	0.141	−0.172	2.481	0.221	−0.788	32.550	1.485	0.720	6.917	0.339	−1.402
0.1	0.1	8.171	0.203	−0.319	1.885	0.179	−0.747	31.952	1.463	0.718	6.281	0.310	−1.367
0.1	0.3	22.269	0.619	−0.600	4.258	0.362	−0.967	49.697	2.202	0.979	13.964	0.664	−1.927
0.1	0.5	42.289	1.127	−2.161	15.107	1.176	−1.778	114.905	4.861	1.854	52.683	2.388	−3.973
0.1	0.7	67.581	8.197	−4.422	79.445	5.747	−5.229	500.880	19.584	6.456	347.312	14.612	−14.341
0.3	−0.7	−20.199	0.374	0.429	−12.221	−0.565	−0.791	−43.689	2.222	1.326	−27.232	−0.861	−1.631
0.3	−0.5	−2.728	0.116	0.141	−5.833	−0.339	−0.245	−6.592	0.439	0.359	−11.599	−0.453	−0.472
0.3	−0.3	−0.284	0.049	0.051	−3.274	−0.202	−0.136	−0.984	0.141	0.188	−7.316	−0.307	−0.232
0.3	−0.1	−0.161	0.028	0.023	−2.542	−0.162	−0.094	−0.589	0.056	0.138	−5.632	−0.242	−0.173
0.3	0.1	−0.153	0.025	0.020	−2.459	−0.159	−0.085	−0.393	0.062	0.139	−5.770	−0.249	−0.164
0.3	0.3	−0.223	0.034	0.028	−3.155	−0.202	−0.096	−0.055	0.104	0.171	−7.251	−0.308	−0.219
0.3	0.5	−0.503	0.057	0.589	−4.973	−0.299	−0.168	−4.335	0.353	0.333	−12.476	−0.505	−0.394
0.3	0.7	−10.711	0.083	1.539	−11.006	−0.529	−0.453	−21.771	1.450	1.042	−33.305	−1.172	−1.199
0.5	−0.7	−20.862	−0.245	−0.987	−39.990	−2.386	0.760	−67.230	−1.910	0.072	−118.997	−4.508	1.535
0.5	−0.5	−8.636	−0.102	−0.155	−11.024	−0.705	0.198	−17.109	−0.518	0.039	−26.985	−1.124	0.326
0.5	−0.3	−0.265	−0.046	−0.044	−5.185	−0.344	0.081	−9.057	−0.283	0.030	−12.032	−0.521	0.123

0.5	−0.1	−0.156	−0.027	−0.029	−3.485	−0.236	0.046	−7.001	−0.221	0.024	−8.153	−0.359	0.073
0.5	0.1	−0.154	−0.026	−0.017	−3.437	−0.234	0.050	−7.065	−0.226	0.022	−8.111	−0.358	0.074
0.5	0.3	−0.236	−0.035	−0.047	−4.590	−0.311	0.077	−9.344	−0.296	0.021	−11.743	−0.513	0.128
0.5	0.5	−6.519	−0.066	−0.054	−9.330	−0.616	0.176	−18.305	−0.568	0.025	−25.734	−1.092	0.320
0.5	0.7	−12.520	−0.291	−0.253	−29.263	−1.802	0.513	−74.221	−2.174	−0.031	−103.886	−4.103	1.318
0.7	−0.7	−40.877	−0.372	−0.314	−50.535	−3.082	1.374	−119.899	−3.860	−0.503	−158.922	−6.091	2.918
0.7	−0.5	−12.610	−0.099	−0.087	−13.709	−0.891	0.406	−27.498	−0.941	−0.101	−33.610	−1.414	0.670
0.7	−0.3	−7.246	−0.043	−0.052	−5.884	−0.395	0.166	−13.759	−0.478	−0.044	−14.051	−0.613	0.278
0.7	−0.1	−2.149	−0.026	−0.038	−4.074	−0.278	0.113	−9.898	−0.346	−0.027	−9.260	−0.410	0.177
0.7	0.1	−2.151	−0.026	−0.036	−3.814	−0.264	0.107	−9.877	−0.346	−0.029	−9.141	−0.405	0.175
0.7	0.3	−5.237	−0.037	−0.042	−5.358	−0.368	0.155	−13.577	−0.475	−0.045	−13.564	−0.596	0.271
0.7	0.5	−8.554	−0.069	−0.175	−10.733	−0.721	0.306	−28.335	−0.977	−0.109	−31.056	−1.328	0.619
0.7	0.7	−30.366	−0.160	−0.274	−36.352	−2.302	0.911	−113.062	−3.651	−0.478	−136.955	−5.477	2.460
0.9	−0.7	−41.049	−0.381	−0.294	−56.524	−3.484	1.735	−141.779	−4.690	−0.781	−183.031	−7.035	3.680
0.9	−0.5	−13.613	−0.099	−0.074	−14.384	−0.943	0.492	−33.315	−1.170	−0.181	−37.586	−1.586	0.863
0.9	−0.3	−6.244	−0.042	−0.068	−6.322	−0.428	0.217	−16.036	−0.574	−0.082	−15.196	−0.665	0.363
0.9	−0.1	−2.151	−0.026	−0.041	−4.197	−0.289	0.143	−11.460	−0.414	−0.056	−9.665	−0.429	0.230
0.9	0.1	−2.152	−0.025	−0.038	−4.073	−0.283	0.139	−11.422	−0.413	−0.056	−9.745	−0.434	0.233
0.9	0.3	−5.240	−0.038	−0.067	−5.687	−0.393	0.195	−15.873	−0.572	−0.082	−14.579	−0.643	0.352
0.9	0.5	−7.589	−0.076	−0.081	−11.444	−0.776	0.378	−32.696	−1.155	−0.177	−34.713	−1.490	0.800
0.9	0.7	−32.427	−0.160	−0.162	−41.699	−2.666	1.166	−137.122	−4.582	−0.751	−153.278	−6.154	3.050

Table 3
Values of $P(D_1 > 0)$

α	ρ	$T=20$, $p=4$			$T=30$, $p=12$			$T=30$, $p=20$			$T=60$, $p=20$		
		FGM-MSE	AFG-MMSE	FGKKC	FGM-MSE	AFG-MMSE	*FGKKC*	FGM-MSE	AFG-MMSE	FGKKC	FGM-MSE	AFG-MMSE	FGKKC
0.1	−0.7	0.476	0.467	0.102	0.223	0.187	0.043	0.139	0.075	0.011	0.139	0.075	0.011
0.1	−0.5	0.615	0.609	0.136	0.468	0.423	0.074	0.404	0.257	0.019	0.404	0.257	0.019
0.1	−0.3	0.765	0.756	0.179	0.762	0.729	0.104	0.825	0.693	0.052	0.825	0.693	0.052
0.1	−0.1	0.871	0.864	0.211	0.955	0.946	0.128	0.989	0.961	0.085	0.989	0.961	0.085
0.1	0.1	0.845	0.837	0.225	0.972	0.966	0.150	0.989	0.970	0.082	0.989	0.970	0.082
0.1	0.3	0.717	0.702	0.154	0.790	0.749	0.125	0.854	0.731	0.038	0.854	0.731	0.038
0.1	0.5	0.501	0.482	0.089	0.456	0.400	0.076	0.481	0.325	0.029	0.481	0.325	0.029
0.1	0.7	0.331	0.317	0.022	0.182	0.160	0.053	0.180	0.107	0.015	0.180	0.107	0.015
0.3	−0.7	0.801	0.796	0.661	0.893	0.879	0.539	0.944	0.920	0.460	0.944	0.920	0.460
0.3	−0.5	0.909	0.906	0.748	0.970	0.967	0.647	0.992	0.985	0.591	0.992	0.985	0.591
0.3	−0.3	0.958	0.958	0.776	0.993	0.992	0.719	1.000	0.998	0.736	1.000	0.998	0.736
0.3	−0.1	0.971	0.971	0.788	1.000	1.000	0.778	1.000	1.000	0.821	1.000	1.000	0.821
0.3	0.1	0.973	0.972	0.831	1.000	0.999	0.803	1.000	1.000	0.808	1.000	1.000	0.808
0.3	0.3	0.949	0.946	0.813	0.998	0.996	0.741	1.000	0.999	0.759	1.000	0.999	0.759
0.3	0.5	0.897	0.892	0.773	0.977	0.969	0.641	0.995	0.990	0.648	0.995	0.990	0.648
0.3	0.7	0.819	0.811	0.721	0.887	0.871	0.541	0.969	0.950	0.546	0.969	0.950	0.546
0.5	−0.7	0.977	0.976	0.920	1.000	1.000	0.972	1.000	1.000	0.975	1.000	1.000	0.975
0.5	−0.5	0.976	0.975	0.906	0.999	0.999	0.975	1.000	1.000	0.982	1.000	1.000	0.982
0.5	−0.3	0.987	0.987	0.923	1.000	1.000	0.978	1.000	1.000	0.989	1.000	1.000	0.989
0.5	−0.1	0.990	0.990	0.918	1.000	1.000	0.983	1.000	1.000	0.994	1.000	1.000	0.994
0.5	0.1	0.987	0.987	0.930	1.000	1.000	0.988	1.000	1.000	0.995	1.000	1.000	0.995

0.5	0.3	0.987	0.987	0.943	1.000	1.000	0.984	1.000	1.000	0.994	1.000	1.000	0.994
0.5	0.5	0.988	0.987	0.953	1.000	1.000	0.982	1.000	1.000	0.989	1.000	1.000	0.989
0.5	0.7	0.986	0.985	0.956	0.999	0.999	0.982	1.000	1.000	0.987	1.000	1.000	0.987
0.7	−0.7	0.994	0.994	0.981	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.7	−0.5	0.995	0.995	0.970	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
0.7	−0.3	0.995	0.995	0.963	1.000	1.000	0.997	1.000	1.000	1.000	1.000	1.000	1.000
0.7	−0.1	0.993	0.993	0.953	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.7	0.1	0.996	0.996	0.967	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
0.7	0.3	0.999	0.998	0.977	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000
0.7	0.5	0.998	0.998	0.988	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.7	0.7	1.000	1.000	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	−0.7	0.999	0.999	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	−0.5	0.996	0.996	0.986	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	−0.3	0.991	0.991	0.968	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	−0.1	0.994	0.994	0.973	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	0.1	0.993	0.993	0.973	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	0.3	0.999	0.999	0.989	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	0.5	0.998	0.998	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	0.7	0.999	0.999	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 4
Values of $P(D_2)>0$

		$T=20$,	$p=4$		$T=30$,	$p=12$		$T=30$,	$p=20$		$T=60$,	$p=20$	
α	ρ	FGM-MSE	AFG-MMSE	FGKKC	FGM-MSE	AFG-MMSE	*FGKKC*	FGM-MSE	AFG-MMSE	FGKKC	FGM-MSE	AFG-MMSE	FGKKC
0.1	−0.7	0.594	0.608	0.257	0.948	0.955	0.012	0.993	0.997	0.009	0.995	0.996	0.ooo
0.1	−0.5	0.598	0.591	0.273	0.903	0.917	0.021	0.987	0.993	0.007	0.963	0.969	0.002
0.1	−0.3	0.590	0.582	0.276	0.853	0.868	0.022	0.972	0.989	0.008	0.924	0.933	0.002
0.1	−0.1	0.559	0.537	0.306	0.808	0.839	0.018	0.952	0.981	0.004	0.896	0.906	0.001
0.1	0.1	0.568	0.540	0.303	0.775	0.815	0.027	0.950	0.981	0.007	0.881	0.892	0.002
0.1	0.3	0.596	0.583	0.254	0.827	0.846	0.029	0.977	0.994	0.010	0.887	0.898	0.006
0.1	0.5	0.603	0.620	0.220	0.901	0.914	0.025	0.984	0.993	0.008	0.940	0.944	0.005
0.1	0.7	0.731	0.727	0.177	0.941	0.950	0.021	0.990	0.996	0.008	0.985	0.986	0.004
0.3	−0.7	0.242	0.255	0.387	0.373	0.416	0.208	0.242	0.255	0.192	0.378	0.402	0.137
0.3	−0.5	0.131	0.134	0.367	0.238	0.268	0.229	0.131	0.134	0.176	0.207	0.225	0.138
0.3	−0.3	0.068	0.073	0.359	0.141	0.162	0.249	0.068	0.073	0.170	0.068	0.071	0.132
0.3	−0.1	0.050	0.059	0.308	0.066	0.077	0.270	0.050	0.059	0.150	0.011	0.011	0.123
0.3	0.1	0.042	0.047	0.360	0.060	0.072	0.275	0.042	0.047	0.138	0.009	0.011	0.128
0.3	0.3	0.078	0.087	0.364	0.111	0.128	0.266	0.078	0.087	0.154	0.055	0.061	0.147
0.3	0.5	0.135	0.158	0.390	0.233	0.268	0.259	0.135	0.158	0.176	0.169	0.185	0.163
0.3	0.7	0.227	0.332	0.400	0.388	0.427	0.249	0.227	0.332	0.176	0.337	0.360	0.171
0.5	−0.7	0.034	0.042	0.897	0.013	0.016	0.857	0.072	0.094	0.897	0.002	0.002	0.926
0.5	−0.5	0.035	0.039	0.876	0.007	0.008	0.806	0.067	0.088	0.876	0.001	0.001	0.872
0.5	−0.3	0.022	0.025	0.896	0.002	0.002	0.776	0.056	0.084	0.896	0.000	0.000	0.834
0.5	−0.1	0.015	0.018	0.887	0.003	0.003	0.762	0.048	0.068	0.887	0.000	0.000	0.819
0.5	0.1	0.019	0.025	0.899	0.002	0.002	0.779	0.040	0.055	0.899	0.000	0.000	0.825

0.5	0.3	0.017	0.025	0.915	0.004	0.004	0.807	0.055	0.070	0.915	0.001	0.001	0.846
0.5	0.5	0.019	0.040	0.933	0.006	0.006	0.830	0.050	0.066	0.933	0.000	0.000	0.889
0.5	0.7	0.022	0.111	0.943	0.009	0.010	0.856	0.048	0.063	0.943	0.001	0.002	0.923
0.7	−0.7	0.010	0.017	0.977	0.001	0.001	0.993	0.002	0.003	0.977	0.000	0.000	0.999
0.7	−0.5	0.008	0.012	0.957	0.000	0.000	0.983	0.002	0.002	0.957	0.000	0.000	0.997
0.7	−0.3	0.008	0.011	0.952	0.000	0.000	0.959	0.003	0.005	0.952	0.000	0.000	0.995
0.7	−0.1	0.011	0.014	0.942	0.000	0.000	0.962	0.005	0.008	0.942	0.000	0.000	0.990
0.7	0.1	0.007	0.013	0.956	0.000	0.000	0.963	0.003	0.004	0.956	0.000	0.000	0.990
0.7	0.3	0.004	0.013	0.971	0.001	0.001	0.982	0.002	0.003	0.971	0.000	0.000	0.995
0.7	0.5	0.004	0.027	0.982	0.000	0.000	0.987	0.002	0.003	0.982	0.000	0.000	0.999
0.7	0.7	0.001	0.092	0.991	0.000	0.000	0.996	0.000	0.002	0.991	0.000	0.000	1.000
0.9	−0.7	0.002	0.017	0.994	0.000	0.000	1.000	0.000	0.000	0.994	0.000	0.000	1.000
0.9	−0.5	0.006	0.009	0.983	0.000	0.000	1.000	0.000	0.000	0.983	0.000	0.000	1.000
0.9	−0.3	0.012	0.015	0.964	0.000	0.000	0.993	0.000	0.000	0.964	0.000	0.000	1.000
0.9	−0.1	0.009	0.014	0.965	0.000	0.000	0.994	0.001	0.001	0.965	0.000	0.000	1.000
0.9	0.1	0.008	0.013	0.968	0.000	0.000	0.994	0.000	0.000	0.968	0.000	0.000	1.000
0.9	0.3	0.002	0.008	0.983	0.000	0.000	0.999	0.000	0.000	0.983	0.000	0.000	1.000
0.9	0.5	0.004	0.028	0.991	0.000	0.000	1.000	0.000	0.001	0.991	0.000	0.000	1.000
0.9	0.7	0.001	0.104	0.997	0.000	0.000	1.000	0.000	0.000	0.997	0.000	0.000	1.000

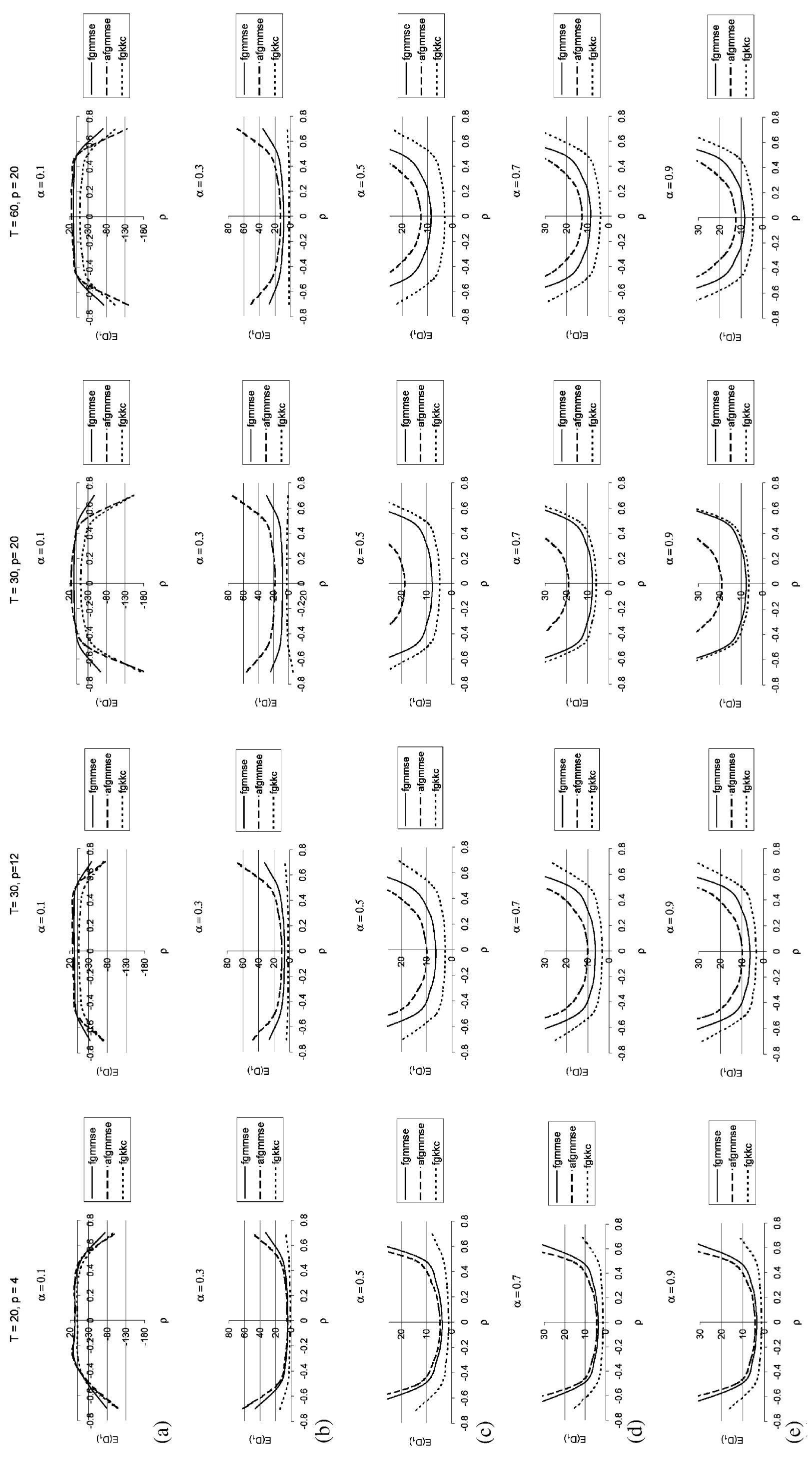

Fig. 1. Simulated values of $E(D_1)$.

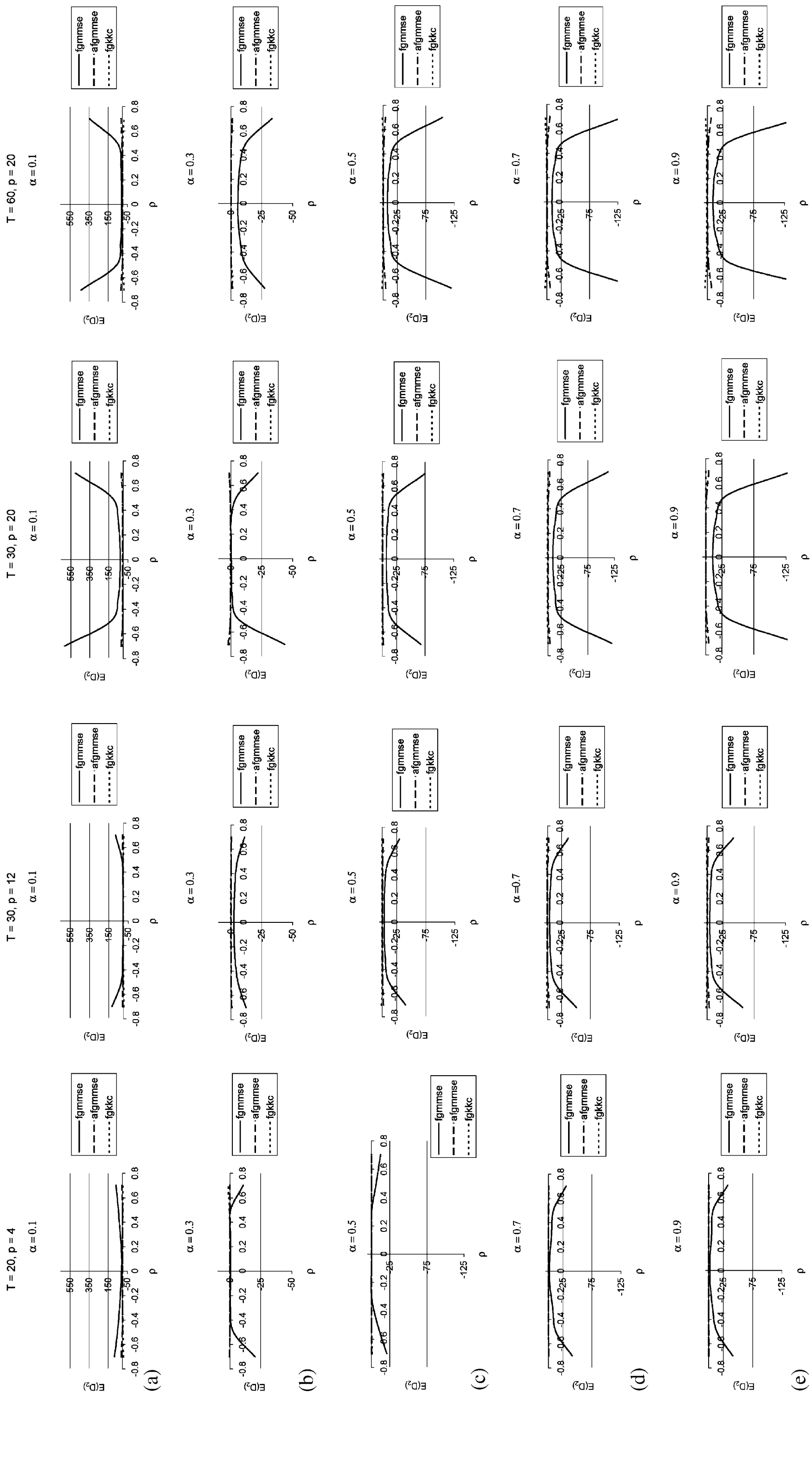

Fig. 2. Simulated values of $E(D_2)$.

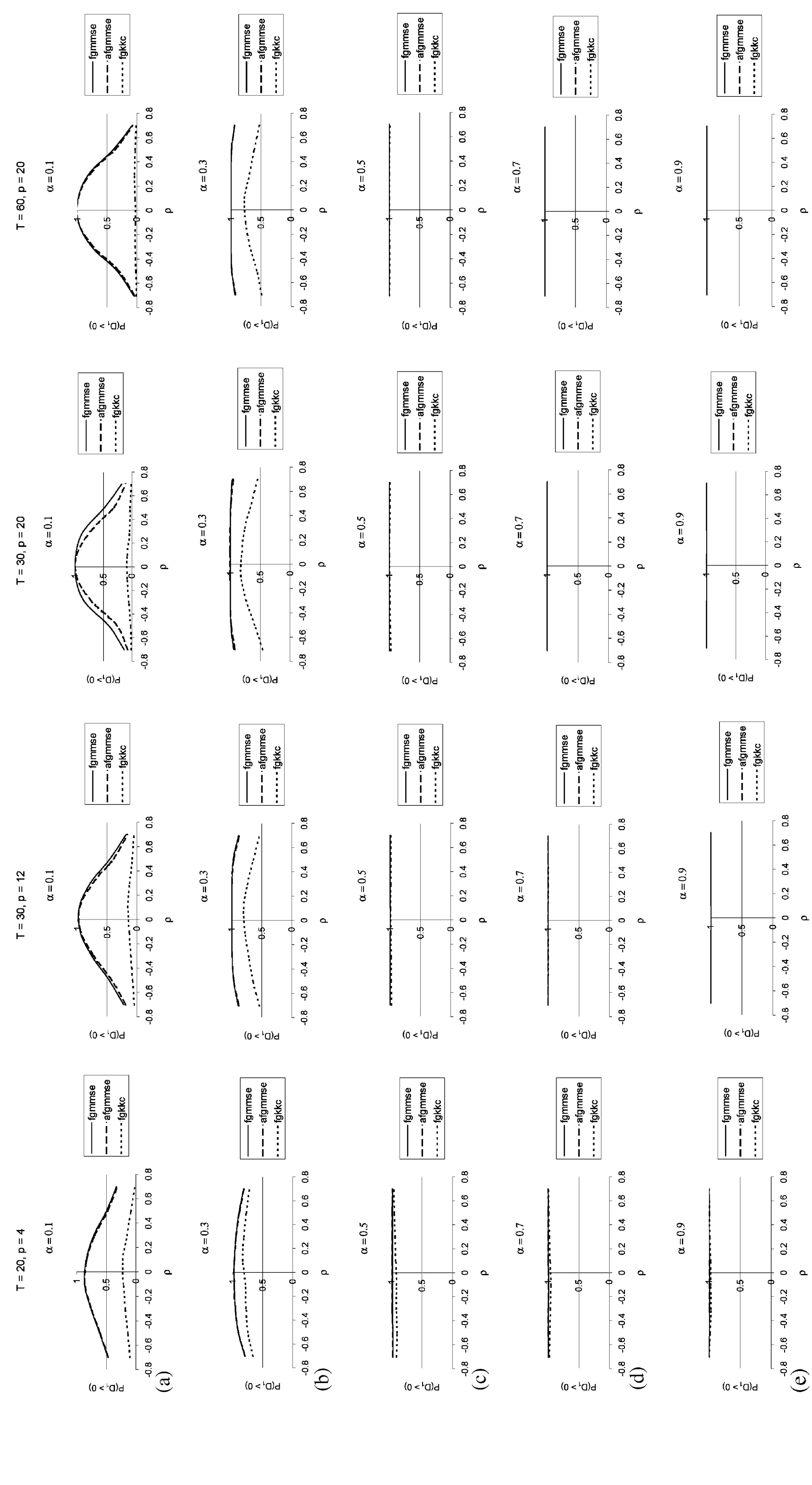

Fig. 3. Simulated values of $P(D_1 > 0)$.

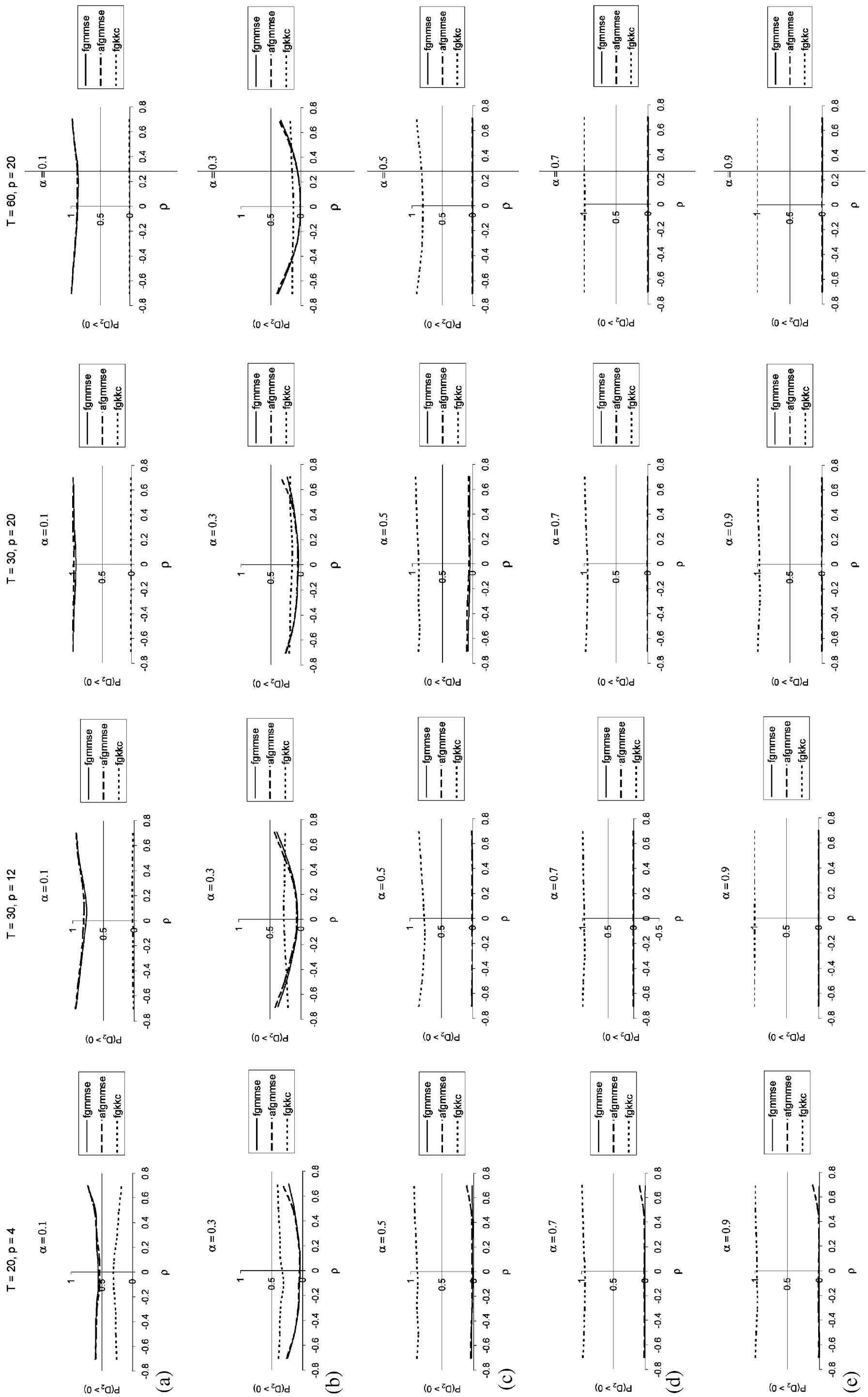

Fig. 4. Simulated values of $P(D_2>0)$.

Further, we consider the general Pitman nearness criterion to judge the performance of the three estimators, viz., FGMMSE, AFGMMSE and FGKKC with respect to FGLS and FGSR. Restricting our attention to the values of $P(D_1 > 0)$ (see Table 3), we see that when $\alpha = 0.1$, FGLS is uniformly superior to FGKKC for all values of ρ whereas FGMMSE and AFGMMSE perform better than FGLS when $|\rho|$ is low, (say < 0.5) (see Fig. 3(a)). Interestingly, again when $\alpha \geqslant 0.3$, FGLS emerges to be the least favored choice among FGMMSE, AFGMMSE and FGKKC. In particular, the dominance of FGMMSE and AFGMMSE over FGLS as well as between them almost stabilizes, (see Fig. 3(b)–(e)). Exceptionally, when $\alpha = 0.3$, FGMMSE and AFGMMSE dominates FGLS and FGKKC both. In case when $\alpha \geqslant 0.5$, even FGKKC performs as good as FGMMSE and AFGMMSE both and give no preferences as such.

Coming to the values of $P(D_2 > 0)$ (see Table 4) to compare FGMMSE, AFGMMSE and FGKKC with FGSR, we see when $\alpha = 0.1$, FGSR dominates FGKKC uniformly whereas itself dominated by FGMMSE and AFGMMSE both uniformly (see Fig. 4(a)). The performance of FGMMSE and AFGMMSE is as good as of FGSR. For $\alpha = 0.3$, (Fig. 4(b)), FGSR emerges as a better choice over other three estimators. The performance pattern stabilizes when $\alpha \geqslant 0.5$ (see Fig. 4(c)–(e)). The FGSR dominates FGKKC but is dominated by FGMMSE and AFGMMSE both. The performance of FGMMSE and AFGMMSE with respect to FGSR as well as between them does not vary much.

We would like to mention here about the numerical accuracies of (3.4), (3.5), (4.1) and (4.2). Each expression was computed for every experimental setting to that particular data set generated as per the given plan. Those values were found to be reasonably close to the values computed through simulations. This indicates that the large sample asymptotic approximations are good enough to depict the finite sample behavior of the estimators. With a view to not unnecessarily increase the length of the paper, we have not reported them inside the paper.

6. Some concluding remarks

We have analyzed the risk properties of FGLS, FGSR and FGKK estimators for the vector of regression coefficients under a balanced loss function defined by (2.4) with α different from zero.

For the purpose of comparison, we have chosen two criteria, viz., the risk and the generalized Pitman closeness under the balanced loss function, which are known to have their own qualifications and limitations. Realizing that the exact results are difficult to derive, we have considered their asymptotic approximations when the number of observations grows large.

When there is only one regression coefficient in the model, our investigation have revealed that the FGSR cannot be better than the FGLS estimator whether we take the criterion as risk or generalized Pitman closeness. Such is not the case with the FGKK estimator which has better performance than the FGLS estimator with respect to the criterion of risk provided that condition (3.9) is satisfied. On the other hand, if we choose the criterion of generalized Pitman closeness, the FGKK estimator too fails to dominate the FGLS estimator.

When there are two regression coefficients in the model, the FGSR estimator again fails to dominate the FGLS estimator with respect to risk criterion but it succeeds in dominating the FGLS estimator with respect to the criterion of generalized Pitman closeness provided that k_1 is less than 2α. Interestingly enough, the FGKK estimator performs better than the FGLS estimator with respect to risk criterion for k_1 smaller than $4\alpha(1-k_2)/\phi$ and generalized Pitman closeness criterion for k_1 smaller than $2\alpha(\phi+1-k_2)/\phi$. Further, it is seen from (3.12) and (4.6) that the FGKK estimator is better than both the FGLS and FGSR estimators according to the risk criterion when

$$0 < k_1 < \frac{4\alpha(1-k_2)}{\phi} \tag{6.1}$$

and the generalized Pitman closeness criterion when

$$2\alpha\left(1-\frac{\phi}{2\phi+1-k_2}\right) < k_1 < 2\alpha\left(1+\frac{1-k_2}{\phi}\right). \tag{6.2}$$

When there are more than two regression coefficients, the picture is not that gloomy. And the FGKK estimator, including the FGSR estimator as a particular case, performs better than the FGLS estimator under condition (3.8) in case of risk criterion and (4.3) in case of generalized Pitman closeness criterion. In fact, the FGKK estimator is found to be superior to both the FGLS and FGSR estimator when condition (3.12) in case of risk criterion and condition (4.6) in case of generalized Pitman closeness are satisfied. Looking at (3.12) and (4.6), it is observed that the FGKK estimator remains better than the FGLS and FGSR estimators with respect to both the criteria as long as condition (4.6) accompanied with $k_2 < (1-\phi)$ holds true.

Lastly, it may be remarked that the findings emerging from our investigations remain unaltered, at least to the given order of asymptotic approximation, if we replace Ω in the balanced loss function (2.4) by its consistent estimator $\hat{\Omega}$.

Acknowledgments

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Appendix

If we define

$$N = [I_p - X(X'\Omega X)^{-1}X'\Omega],$$

the estimation errors of the FGLS and FGKK estimators can be expressed as

$$(\hat{\beta} - \beta) = \xi_{-\frac{1}{2}} + \xi_{-1} + \xi_{-\frac{3}{2}} + O_p(T^{-2}), \tag{A.1}$$

$$(\hat{\beta}_{kk} - \beta) = \xi_{-\frac{1}{2}} + (\xi_1 - \eta_{-1}) + (\xi_{-\frac{3}{2}} - \eta_{-\frac{3}{2}}) + O_p(T^{-2}), \tag{A.2}$$

where

$$\begin{aligned}
\xi_{-\frac{1}{2}} &= (X'\Omega X)^{-1} X'\Omega\varepsilon, \\
\xi_{-1} &= \frac{1}{T^{1/2}} \sum^{q} e_j (X'\Omega X)^{-1} X' \left(\frac{\partial \Omega}{\partial \theta_j}\right) N\varepsilon, \\
\xi_{-\frac{3}{2}} &= \frac{1}{T} \sum_{j,k}^{q} e_j e_k (X'\Omega X)^{-1} X' \left[\frac{1}{2}\left(\frac{\partial^2 \Omega}{\partial \theta_j \partial \theta_k}\right) - \left(\frac{\partial \Omega}{\partial \theta_j}\right) X (X'\Omega X)^{-1} X' \left(\frac{\partial \Omega}{\partial \theta_k}\right)\right] N\varepsilon, \\
\eta_{-1} &= \frac{k_1}{T(\phi + 1 - k_2)} \beta, \\
\eta_{-\frac{3}{2}} &= \frac{k_1}{T(\phi + 1 - k_2)} (X'\Omega X)^{-1} X'\Omega\varepsilon + \frac{k_1 \phi}{(\phi + 1 - k_2)} \left(\frac{\varepsilon'\Omega\varepsilon}{T\sigma^2} - 1\right) \beta \\
&\quad - \frac{k_1}{T^2 \sigma^2 (\phi + 1 - k_2)^2} \beta \left[2\beta' X'\Omega\varepsilon + \frac{1}{T^{1/2}} \sum_j e_j \beta' X' \left(\frac{\partial \Omega}{\partial \theta_j}\right) X\beta\right]
\end{aligned}$$

with e_j and e_k denoting the jth and kth element of vector e defined by (2.5); see Wan and Chaturvedi [24, Section 3].

From (2.4), the balanced loss functions associated with the FGLS and FGKK estimators are given by

$$L(\hat{\beta}, \beta) = (1 - \alpha)\left[\varepsilon'\Omega\varepsilon - 2\varepsilon'\Omega X(\hat{\beta} - \beta)\right] + (\hat{\beta} - \beta)' X'\Omega X (\hat{\beta} - \beta), \tag{A.3}$$

$$L(\hat{\beta}_{kk}, \beta) = (1 - \alpha)[\varepsilon'\Omega\varepsilon - 2\varepsilon'\Omega X(\hat{\beta}_{kk} - \beta)] + (\hat{\beta}_{kk} - \beta)' X'\Omega X (\hat{\beta}_{kk} - \beta). \tag{A.4}$$

As α is assumed to be positive, using (A.1) and (A.2), we can write

$$\begin{aligned}
&\frac{T(\phi + 1 - k_2)}{2k_1 \alpha} [L(\hat{\beta}, \beta) - L(\hat{\beta}_{kk}, \beta)] \\
&\quad = \varepsilon'\Omega X\beta + \frac{1}{\alpha T^{1/2}} \sum_j^{q} e_j \beta' X' \left(\frac{\partial \Omega}{\partial \theta_j}\right) N\varepsilon - \frac{k_1 \beta' X'\Omega X\beta}{2T\alpha(\phi + 1 - k_2)} \\
&\qquad + \varepsilon'\Omega X (X'\Omega X)^{-1} X'\Omega\varepsilon + \phi \left(\frac{\varepsilon'\Omega\varepsilon}{T\sigma^2} - 1\right) \varepsilon'\Omega X\beta \\
&\qquad - \frac{\varepsilon'\Omega X\beta}{T\sigma^2(\phi + 1 - k_2)} \left[2\beta' X'\Omega\varepsilon + \frac{1}{T^{1/2}} \sum_j e_j \beta' X' \left(\frac{\partial \Omega}{\partial \theta_j}\right) X\beta\right] \\
&\qquad + O_p(T^{-1/2}).
\end{aligned} \tag{A.5}$$

If we write

$$Z = \frac{1}{\sigma^2\sqrt{T\phi}}\,\varepsilon'\Omega X\beta, \quad \phi = \frac{1}{T\sigma^2}\,\beta' X'\Omega X\beta$$

we observe that Z has a normal distribution with zero mean and unit variance.

Thus, from (A.5), we can express

$$\begin{aligned} D_1 &= \frac{\sqrt{T}(\phi + 1 - k_2)}{2\sigma^2 k_1 \alpha\sqrt{\phi}}\,[L(\hat{\beta}, \beta) - L(\hat{\beta}_{kk}, \beta)] \\ &= Z + \frac{1}{T^{1/2}}\,U_1 + O_p(T^{-1}), \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} U_1 = &\frac{1}{\sigma^2\sqrt{\phi}}\left[\frac{1}{\alpha T^{1/2}}\sum^{q} e_j \beta' X'\left(\frac{\partial\Omega}{\partial\theta_j}\right)N\varepsilon + \varepsilon'\Omega X(X'\Omega X)^{-1}X'\Omega\varepsilon\right] \\ &+ T^{1/2}\phi Z\left(\frac{\varepsilon'\Omega\varepsilon}{T\sigma^2} - 1\right) - \frac{\sqrt{\phi}}{(\phi + 1 - k_2)}\left[\frac{k_1}{2\alpha} + 2Z^2\right. \\ &\left.+ \frac{Z}{T\sigma^2\sqrt{\phi}}\sum_j e_j \beta' X'\left(\frac{\partial\Omega}{\partial\theta_j}\right)X\beta\right]. \end{aligned}$$

Using the stochastic independence of e and Z, we obtain expression (3.4) of Theorem I for $E(D_1)$ to the given order of approximation.

Similarly, using (A.4), result (3.5) of Theorem I can be derived.

For the results in Theorem II, we observe from (A.6) that the characteristic function of D_1 can be expressed as

$$\begin{aligned} \Psi(t) &= E[\exp\{itD_1\}] \\ &= E\left[\left(1 + \frac{it}{\sqrt{T}}U_1\right)\exp\{itZ\} + O_p(T^{-1})\right]. \end{aligned} \tag{A.7}$$

By virtue of normality Z,

$$E[\exp\{itZ\}] = \exp\left\{-\frac{t^2}{2}\right\}.$$

Similarly, using stochastic independence of e and z, we have

$$\begin{aligned} &E\left[\sum^{q} e_j \beta' X'\left(\frac{d\Omega}{d\theta_j}\right)N\varepsilon\exp\{itZ\}\right] = 0, \\ &E\left[Z\left(\frac{\varepsilon'\Omega\varepsilon}{T\sigma^2} - 1\right)\exp\{itZ\}\right] = 0, \end{aligned}$$

$$E\left[Z^2 \exp\{itZ\}\right] = (1 - t^2) \exp\left\{-\frac{t^2}{2}\right\},$$

$$E\left[Z \sum_j e_j \beta' X' \left(\frac{d\Omega}{d\theta_j}\right) X\beta \exp\{itZ\}\right] = 0.$$

Lastly, we find that

$$\begin{aligned} &E\left[\varepsilon'\Omega X(X'\Omega X)^{-1} X'\Omega\varepsilon \exp\{itZ\}\right] \\ &\quad = \exp\left\{-\frac{t^2}{2}\right\} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\sigma^2 v'\Omega^{\frac{1}{2}} X(X'\Omega X)^{-1} X'\Omega^{\frac{1}{2}} v + \frac{2it\sigma}{\sqrt{T\phi}} v'\Omega^{\frac{1}{2}} X\beta \right. \\ &\qquad \left. + \frac{2it\sigma}{\sqrt{T\phi}} v'\Omega^{\frac{1}{2}} X\beta - \frac{t^2}{T\phi} \beta' X'\Omega X\beta\right] \frac{\exp\left\{-\frac{v'v}{2}\right\}}{(2\pi)^{T/2}} dv \\ &\quad = \sigma^2 (p - t^2) \exp\left\{-\frac{t^2}{2}\right\}, \end{aligned}$$

where

$$v = \frac{1}{\sigma} \Omega^{\frac{1}{2}} \left(\varepsilon - \frac{it}{\sqrt{T\phi}} X\beta\right).$$

Using these results in (A.7), we get

$$\begin{aligned} \Psi(t) = &\left[1 + \frac{it}{\sqrt{T\phi}} \left\{p - \frac{\phi}{\phi + 1 - k_2} \left(\frac{k_1}{2\alpha} + 2\right)\right\} \right. \\ &\left. - \frac{it^3}{\sqrt{T\phi}} \left(1 - \frac{2\phi}{\phi + 1 - k_2}\right)\right] \exp\left\{-\frac{t^2}{2}\right\} + O(T^{-1}). \end{aligned} \tag{A.8}$$

Applying inversion theorem, we find the probability density function of D_1 as follows:

$$f(D_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(t) \exp\{-itD_1\}\, dt \tag{A.9}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-itD_1 - \frac{t^2}{2}\right\} dt \\
&\quad + \frac{1}{2\pi\sqrt{T\phi}}\left[p - \frac{\phi}{\phi+1-k_2}\left(\frac{k_1}{2\alpha}+2\right)\right] \int_{-\infty}^{\infty} it \exp\left\{-itD_1 - \frac{t^2}{2}\right\} dt \\
&\quad + \frac{1}{2\pi\sqrt{T\phi}}\left(1 - \frac{2\phi}{\phi+1-k_2}\right) \int_{-\infty}^{\infty} -it \exp\left\{-itD_1 - \frac{t^2}{2}\right\} dt \\
&= \left[1 + \frac{D_1}{\sqrt{T\phi}}\left\{p - 3 + \frac{\phi}{\phi+1-k_2}\left(4 - \frac{k_1}{2\alpha}\right)\right\}\right. \\
&\quad \left. + \frac{D_1^3}{\sqrt{T\phi}}\left(1 - \frac{2\phi}{\phi+1-k_2}\right)\right] \frac{\exp\{-\frac{1}{2}D_1^2\}}{\sqrt{2\pi}} + O(T^{-1})
\end{aligned} \tag{A.10}
$$

whence we find

$$
\begin{aligned}
P(D_1 > 0) &= \int_0^{\infty} f(D_1)\, dD_1 \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi T\phi}}\left[p - 1 - \frac{\phi k_1}{2\alpha(\phi+1-k_2)}\right] + O(T^{-1})
\end{aligned} \tag{A.11}
$$

which is the result (4.1) stated in Theorem II.

In a similar manner, using the expansion for D_2, result (4.2) of Theorem II can be derived.

References

[1] R.A.L. Carter, V.K. Srivastava, A. Chaturvedi, Selecting a double k-class estimator for regression coefficients, Statist. Probab. Lett. 18 (1993) 363–371.

[2] A. Chaturvedi, M.I. Bhatti, On the double k-class estimators in linear regression, J. Quant. Econom. 14 (1998) 38–53.

[3] A. Chaturvedi, G. Shukla, Stein-rule estimation in linear models with nonscalar error covariance matrix, Sankhya Ser. B 52 (1990) 293–304.

[4] D.K. Dey, M. Ghosh, W.E. Strawderman, On estimation with balanced loss function, Statist. Probab. Lett. 45 (1999) 97–101.

[5] R.W. Farebrother, The minimum mean square error linear estimator and ridge regression, Technometrics 17 (1975) 127–128.

[6] J.A. Giles, D.E.A. Giles, K. Ohtani, The exact risks of some pre-test and Stein-type regression estimators under balanced loss, Comm. Statist. Theory Methods 25 (1996) 2901–2924.

[7] J.P. Keating, R.L. Mason, P.K. Sen, Pitman's Measure of closeness: A Comparison of Statistical Estimators, SIAM, Philadelphia, 1993.

[8] S. Kourouklis, Improved estimation under Pitman's measure of closeness, Ann. Inst. Statist. Math. 48 (1996) 509–518.

[9] R.L. Mason, J.P. Keating, P.K. Sen, N.W. Blaylock, Comparison of linear estimators using Pitman's measure of nearness, J. Amer. Statist. Assoc. 85 (1990) 579–581.

[10] K. Ohtani, Exact small sample properties of an operational variant of the minimum mean squared error estimator, Comm. Statist. Theory and Methods 25 (1996) 1223–1231.

[11] K. Ohtani, Minimum mean squared error estimation of each individual coefficients in a linear regression model, J. Statist. Plann. Inference 62 (1997) 301–316.

[12] K. Ohtani, The exact risk of a weighted average estimator of the OLS and Stein-rule estimators in regression under balanced loss, Statist. Papers 16 (1998) 35–45.
[13] K. Ohtani, Inadmissibility of the Stein-rule estimator under the balanced loss function, J. Econ. 83 (1999) 193–201.
[14] K. Ohtani, D.E.A. Giles, J.A. Giles, The exact risk performance of a pre-test estimator in a heteroskedastic linear regression model under the balanced loss function, Econ. Rev. 16 (1997) 119–130.
[15] E.J.G. Pitman, The closest estimator of statistical parameters, Proc. Cambridge Philos. Soc. 33 (1937) 212–222.
[16] C.R. Rao, Some comments on the minimum mean square error as a criterion of estimation, in: M. Csorgo, D.A. Dawson, J.N.K. Rao, A.K.Md.E. Saleh (Eds.), Statistics and Related Topics, North-Holland, Amsterdam, 1981, pp. 123–143.
[17] C.R. Rao, J.P. Keating, R.L. Mason, The Pitman nearness criterion and its determination, Comm. Statist. Theory Methods 15 (1986) 3173–3191.
[18] C.R. Rao, V.K. Srivastava, H. Toutenburg, Pitman nearness comparisons of Stein-type estimators for regression coefficients in replicated experiments, Statist. Papers 39 (1998) 61–74.
[19] Shalabh, Performance of Stein-rule procedure for simultaneous prediction of actual and average values of study variable in linear regression model, Bulletin of the 50th Session of the International Statistical Institute, 1995, pp. 1375–1390.
[20] A.K. Srivastava, V.K. Srivastava, Pitman closeness for Stein-rule estimators of regression coefficients, Statist. Probab. Lett. 18 (1993) 85–89.
[21] H. Toutenburg, Shalabh, Estimation of regression coefficients subject to exact linear restrictions when some observations are missing and balanced loss function is used, TEST, 2003, in press.
[22] A. Ullah, S. Ullah, Double k-class estimators of coefficients in linear regression, Econometrica 46 (1978) 705–722 Errata Econometrica 49 (1978) 554.
[23] A.T.K. Wan, Risk comparison of the inequality constrained least squares and other related estimator under balanced loss, Econ. Lett. 46 (1994) 203–210.
[24] A.T.K. Wan, A. Chaturvedi, Double k-class estimators in regression models with non-spherical disturbance, J. Multivariate Anal. 79 (2001) 226–250.
[25] A. Zellner, Bayesian and non-Bayesian estimation using balanced loss functions, in: S.S. Gupta, J.O Berger (Eds.), Statistical Decision Theory and Related Topics, Springer, New York, 1994, pp. 377–390.
[26] A. Zellner, The finite sample properties of simultaneous equations' estimates and estimators Bayesian and non-Bayesian approaches, J. Econ. 83 (1998) 185–212.