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Actions on permutations and unimodality of descent polynomials

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Abstract

We study a group action on permutations due to Foata and Strehl and use it to prove that the descent generating polynomial of certain sets of permutations has a non-negative expansion in the basis $\{t^i(1+t)^{n-1-2i}\}_{i=0}^{m}$, $m = \lfloor (n-1)/2 \rfloor$. This property implies symmetry and unimodality. We prove that the action is invariant under stack sorting which strengthens recent unimodality results of Bóna. We prove that the generalized permutation patterns (13–2) and (2–31) are invariant under the action and use this to prove unimodality properties for a q-analog of the Eulerian numbers recently studied by Corteel, Postnikov, Steingrímsson and Williams.

We also extend the action to linear extensions of sign-graded posets to give a new proof of the unimodality of the (P, ω)-Eulerian polynomials of sign-graded posets and a combinatorial interpretations (in terms of Stembridge's peak polynomials) of the corresponding coefficients when expanded in the above basis.

Finally, we prove that the statistic defined as the number of vertices of even height in the unordered decreasing tree of a permutation has the same distribution as the number of descents on any set of permutations invariant under the action. On restricting to the set of stack sortable permutations we recover a result of Kreweras.

1. Introduction

The *n*-th *Eulerian polynomial*, $A_n(t) = A_{n1} + A_{n2}t + \cdots + A_{n(n-1)}t^{n-1}$, may be defined as the generating polynomial for the number of *descents* over the symmetric group $\mathfrak{S}_n$, i.e.,

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)},$$

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where $\mathrm{des}(\pi) = |\{i : a_i > a_{i+1}\}|$ and where $\pi : i \to a_i$ $(1 \leq i \leq n)$ is identified with the word $a_1 a_2 \cdots a_n$ in the distinct n letters $a_1, \ldots, a_n$ taken out of $[n] := \{1, 2, \ldots, n\}$.

In a series of papers [20,23,24] Foata and Strehl studied a group action on the symmetric group, $\mathfrak{S}_n$, with the following properties. The number of orbits is the n-th tangent number or secant number, according as n is odd or even, and if an orbit, $\mathrm{Orb}(\pi)$, of a permutation $\pi \in \mathfrak{S}_n$ is enumerated according to the number of descents then

$$\sum_{\sigma \in \mathrm{Orb}(\pi)} t^{\mathrm{des}(\sigma)} = (2t)^{\mathrm{v}(\pi)} (1+t)^{n-1-2\mathrm{v}(\pi)}, \tag{1.1}$$

where $\mathrm{v}(\pi) = |\{i : a_{i-1} > a_i < a_{i+1}\}|$. From (1.1) it follows that $A_n(t)$ has non-negative coefficients when expanded in the basis $\{t^k (1+t)^{n-1-2k}\}_{k=0}^{\lfloor (n-1)/2 \rfloor}$, a result which can also be proven analytically [14,22]. This implies that the sequence $\{A_{ni}\}_{i=0}^{n-1}$ is symmetric and unimodal, i.e., that $A_{ni} = A_{n(n-1-i)}$, $1 \leq i \leq n-1$ and

$$A_{n0} \leq A_{n1} \leq \cdots \leq A_{nc} \geq A_{n(c+1)} \geq \cdots \geq A_{n(n-1)},$$

where $c = \lfloor (n-1)/2 \rfloor$. Indeed, $\{A_{ni}\}_{i=0}^{n-1}$ is a non-negative sum of unimodal and symmetric sequences with the same center of symmetry.

We will in this paper study a slightly modified version of the Foata–Strehl action and show that interesting subsets of $\mathfrak{S}_n$ are invariant under the action. In particular we show that the set of r-stack sortable permutations is invariant under the action which strengthens the recent result of Bóna [4,5] claiming that the corresponding descent generating polynomial is symmetric and unimodal.

In Section 5 we prove that the generalized permutation patterns (13–2) and (2–31) are invariant under the modified Foata–Strehl action. This is used to prove unimodality properties for a q-analog of the Eulerian numbers recently studied by Corteel, Postnikov, Steingrímsson and Williams [17,18,36,42,50] and which appears as a translation of the polynomial enumerating the cells in the totally non-negative part of a Grassmannian [36,50], and also in the stationary distribution of the ASEP model in statistical mechanics [17,18].

We will in Section 6 define an action on the set of linear extensions of a sign-graded poset, see Section 6 for relevant definitions. This enables us to give a combinatorial interpretation in terms of Stembridge's peak polynomials of the coefficients of the (P, ω)-Eulerian polynomials when expanded in the basis $\{t^i (1+t)^{d-2i}\}_{i=0}^{\lfloor d/2 \rfloor}$, $d = |P| - r - 1$.

In Section 7 we study the statistic $\pi \to \mathrm{veh}(\pi)$ on permutations which is defined as the number of vertices of even height in the unordered increasing tree of π. We prove that veh has the same distribution as des on every subset of $\mathfrak{S}_n$ invariant under the action. This can be seen as a generalization of a result of Kreweras [32]. In Section 8 we also find a Mahonian partner for veh.

Finally, in Section 10, we discuss further directions and open problems.

2. The action of Foata and Strehl

Let $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ and let $x \in [n]$. We may write π as the concatenation $\pi = w_1 w_2 x w_4 w_5$ where w_2 is the maximal contiguous subword immediately to the left x whose letters are all smaller than x, and w_4 is the maximal contiguous subword immediately to the right of x whose letters are all smaller than x. This is the x-*factorization* of π. Define $\varphi_x(\pi) = w_1 w_4 x w_2 w_5$. Then φ_x is an involution acting on $\mathfrak{S}_n$ and it is not hard to see that φ_x

and φ_y commute for all $x, y \in [n]$. Hence for any subset $S \subseteq [n]$ we may define the function $\varphi_S : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\varphi_S(\pi) = \prod_{x \in S} \varphi_x(\pi).$$

The group $\mathbb{Z}_2^n$ acts on $\mathfrak{S}_n$ via the functions φ_S, $S \subseteq [n]$. This action was studied by Foata and Strehl in [20,23,24]. To be precise, Foata and Strehl defined the action as $C \circ \varphi_S \circ C$, where $C : \mathfrak{S}_n \to \mathfrak{S}_n$ is the involution described by $a_1 a_2 \cdots a_n \mapsto b_1 b_2 \cdots b_n$, where $b_i = n + 1 - a_i$, $1 \le i \le n$. Sometimes it is preferable to define the action on the *decreasing binary tree* of the permutation. The decreasing binary tree of a permutation of a finite subset of $\{1, 2, 3, \ldots\}$ is defined recursively as follows. The empty tree corresponds to the empty word. If π is non-empty then we may write π as the concatenation $\pi = LmR$ where m and L and R are the subwords to the left and right of m respectively. The tree corresponding to π has a root labeled m and as left subtree the tree corresponding to L and as right subtree the tree corresponding to R. This describes a bijective correspondence between the set of decreasing binary trees with labels $[n]$ and $\mathfrak{S}_n$. It is not hard to see that the tree of $\varphi_x(\pi)$ is obtained by exchanging the subtrees rooted at x, if any. Another action on permutations with similar properties was studied by Hetyei and Reiner [30] and subsequently by Foata and Han [21].

Let $\pi = a_1 a_2 \cdots a_n$ be a permutation in $\mathfrak{S}_n$ and let $a_0 = a_{n+1} = n + 1$. If $k \in [n]$ then a_k is a

valley if $a_{k-1} > a_k < a_{k+1}$,
peak if $a_{k-1} < a_k > a_{k+1}$,
double ascent if $a_{k-1} < a_k < a_{k+1}$, and
double descent if $a_{k-1} > a_k > a_{k+1}$.

Let $x \in [n]$ and let $\pi = a_1 a_2 \ldots a_n \in \mathfrak{S}_n$. We make the following observation.

- If x is a double descent then $\varphi_x(\pi)$ is obtained by inserting x between the first pair of letters a_i, a_{i+1} to the right of x such that $a_i < x < a_{i+1}$.
- If x is a double ascent then $\varphi_x(\pi)$ is obtained by inserting x between the first pair of letters a_i, a_{i+1} to the left of x such that $a_i > x > a_{i+1}$.

We modify the Foata–Strehl action in the following way. If $x \in [n]$ then

$$\varphi_x'(\pi) = \begin{cases} \varphi_x(\pi) & \text{if } x \text{ is a double ascent or double descent,} \\ \pi & \text{if } x \text{ is a valley or a peak.} \end{cases}$$

The functions are easily visualized when a permutation is represented graphically. Let $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ and imagine marbles at coordinates (i, a_i), $i = 0, 1, \ldots, n + 1$ in the grid $\mathbb{N} \times \mathbb{N}$. For $i = 0, 1, \ldots, n$ connect (i, a_i) and $(i + 1, a_{i+1})$ with a wire. Suppose that gravity acts on the marbles from above and suppose that x is not at an equilibrium. If we release x from the wire it will slide and stop when it has reached the same height again. The resulting permutation will be $\varphi_x'(\pi)$, see Fig. 1. The functions φ_x' were studied by Shapiro, Woan and Getu unaware[1] that they are essentially the same as the functions defining the Foata–Strehl action.

Again it is clear that the φ_x''s are involutions and that they commute. Hence, for any subset $S \subseteq [n]$ we may define the function $\varphi_S' : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\varphi_S'(\pi) = \prod_{x \in S} \varphi_x'(\pi).$$

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[1] The present author was also unaware of this until it was pointed out by the referee.
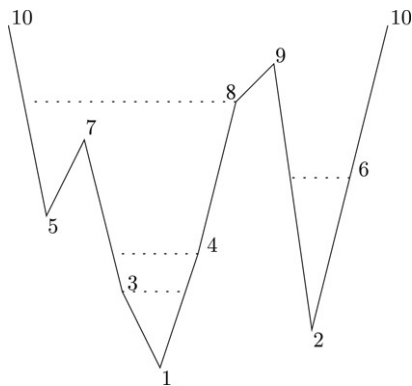
Fig. 1. Graphical representation of $\pi = 573148926$. The dotted lines indicates where the double ascents/descents move to.
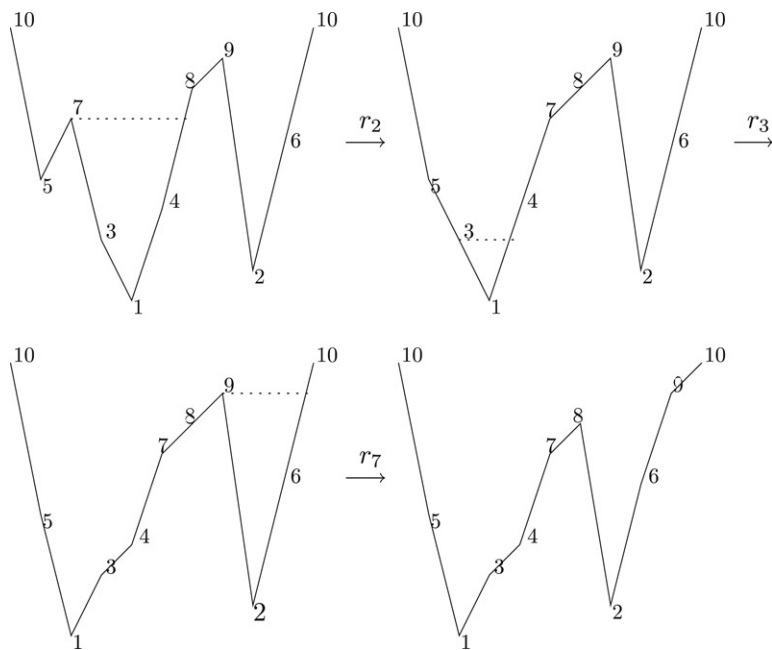

Fig. 2. Computing $S(573148926) = r_7 r_3 r_2(573148926) = 513478269$.

Hence the group $\mathbb{Z}_2^n$ acts on $\mathfrak{S}_n$ via the functions φ'_S, $S \subseteq [n]$. Subsequently we will refer to this action as the modified Foata–Strehl action, or the MFS-action for short.

3. Properties of the modified Foata–Strehl action

For $\pi \in \mathfrak{S}_n$ let $\mathrm{Orb}(\pi) = \{g(\pi) : g \in \mathbb{Z}_2^n\}$ be the orbit of π under the MFS-action. There is a unique element in $\mathrm{Orb}(\pi)$ which has no double descents and which we denote by $\hat{\pi}$. The next theorem follows from the work in [24,45], but we prove it here for completeness.

Theorem 3.1. *Let $\pi \in \mathfrak{S}_n$. Then*

$$\sum_{\sigma \in \mathrm{Orb}(\pi)} t^{\mathrm{des}(\sigma)} = t^{\mathrm{des}(\hat{\pi})}(1+t)^{n-1-2\mathrm{des}(\hat{\pi})} = t^{\mathrm{peak}(\pi)}(1+t)^{n-1-2\mathrm{peak}(\pi)},$$

where $\mathrm{peak}(\pi) = |\{i : a_{i-1} < a_i > a_{i+1}\}|$.

Proof. If x is a double ascent in π then $\mathrm{des}(\varphi'_x(\pi)) = \mathrm{des}(\pi) + 1$. It follows that

$$\sum_{\sigma \in \mathrm{Orb}(\pi)} t^{\mathrm{des}(\sigma)} = t^{\mathrm{des}(\hat{\pi})}(1+t)^a,$$

where a is the number of double ascents in $\hat{\pi}$. If we delete all double descents from $\hat{\pi}$ we get an alternating permutation

$$n + 1 > b_1 < b_2 > b_3 < \cdots > b_{n-a} < n + 1,$$

with the same number of descents. Hence $n - a = 2\mathrm{des}(\hat{\pi}) + 1$. Clearly $\mathrm{des}(\hat{\pi}) = \mathrm{peak}(\pi)$ and the theorem follows. $\square$

For a subset T of $\mathfrak{S}_n$ let

$$W(T; t) = \sum_{\pi \in T} t^{\mathrm{des}(\pi)} \quad \text{and} \quad \overline{W}(T; t) = \sum_{\pi \in T} t^{\mathrm{peak}(\pi)}.$$

Corollary 3.2. *Suppose that $T \subseteq \mathfrak{S}_n$ is invariant under the MFS-action. Then*

$$W(T; t) = 2^{-n+1}(1+t)^{n-1}\overline{W}(T; 4t(1+t)^{-2}).$$

Equivalently

$$W(T; t) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_i(T)t^i(1+t)^{n-1-2i},$$

where

$$b_i(T) = 2^{-n+1+2i}|\{\pi \in T : \mathrm{peak}(\pi) = i\}|.$$

Proof. It is enough to prove the theorem for an orbit of a permutation $\pi \in \mathfrak{S}_n$. Since the number of peaks is constant on $\mathrm{Orb}(\pi)$ the equality follows from Theorem 3.1. $\square$

Remark 3.3. If we want to prove "combinatorially" that the coefficients of $W(T; t)$ form a symmetric and unimodal sequence then we can construct an involution proving symmetry and an injection proving unimodality easily as follows.

Define $f : \mathfrak{S}_n \to \mathfrak{S}_n$ by $f = \varphi'_{[n]}$. Clearly f is an involution and restricts to any subset of $\mathfrak{S}_n$ invariant under the MFS-action. Moreover,

$$\mathrm{des}(f(\pi)) + \mathrm{des}(\pi) = n - 1, \tag{3.1}$$

so f has the desired properties. The involution f was defined differently in [4]. To find an injection

$$g_j : \{\pi \in T : \mathrm{des}(\pi) = j\} \to \{\pi \in T : \mathrm{des}(\pi) = j + 1\},$$

for $j = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor$ it suffices to find an injection from the set of subsets of cardinality k of $[m]$ to the set of subsets of cardinality $k + 1$ of $[m]$, for $1 \leq k \leq \lfloor m/2 \rfloor$. This can done as in e.g. [38].

4. Invariance under stack sorting

Much has been written on the combinatorics of the stack sorting problem (cf. [6]) since it was introduced by Knuth [31]. The stack sorting operator S can be defined recursively on permutations of finite subsets of $\{1, 2, \ldots\}$ as follows. If w is empty then $S(w) = w$ and if w is non-empty write w as the concatenation $w = LmR$, where m is the greatest element of w and L and R are the subwords to the left and right of m respectively. Then $S(w) = S(L)S(R)m$.

Let $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. Recall that $i \in [n-1]$ is a descent in π if $a_i > a_{i+1}$. If i is a descent in π we let $r_i(\pi)$ be the permutation obtained by inserting a_i between the first pair of letters a_j, a_{j+1} to the right of x such that $a_j < x < a_{j+1}$ ($a_{n+1} = n+1$). The following theorem describes a new way of computing $S(\pi)$, see Fig. 2.

Theorem 4.1. *Let $i_1 < i_2 < \cdots < i_d$ be the descents in the permutation $\pi = a_1 a_2 \cdots a_n$. Then*

$$S(\pi) = r_{i_d} r_{i_{d-1}} \cdots r_{i_1}(\pi).$$

Proof. Let $S' : \mathfrak{S}_n \to \mathfrak{S}_n$ be defined by $S'(\pi) = r_{i_d} r_{i_{d-1}} \cdots r_{i_1}(\pi)$. It is straightforward to check that S' satisfies the same recursion as S. $\square$

From the above description of S we see that $S(\varphi_x(\pi)) = S(\pi)$, hence the following corollary.

Corollary 4.2. *If $\sigma, \tau \in \mathfrak{S}_n$ are in the same orbit under the MFS-action then $S(\sigma) = S(\tau)$.*

Corollary 4.2 can also be deduced from [7, Proposition 2.1].

Let $r \in \mathbb{N}$. A permutation $\pi \in \mathfrak{S}_n$ is said to be r-stack sortable if $S^r(\pi) = 12 \cdots n$. Denote by $\mathfrak{S}_n^r$ the set of r-stack sortable permutations in $\mathfrak{S}_n$. By Corollary 4.2 we have that $\mathfrak{S}_n^r$ is invariant under the MFS-action for all $n, r \in \mathbb{N}$ so Corollary 3.2 applies.

Corollary 4.3. *For all $n, r \in \mathbb{N}$ we have*

$$W(\mathfrak{S}_n^r; t) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_i(\mathfrak{S}_n^r) t^i (1+t)^{n-1-2i},$$

where

$$b_i(\mathfrak{S}_n^r) = 2^{-n+1+2i} |\{\pi \in \mathfrak{S}_n^r : \mathrm{peak}(\pi) = i\}|.$$

An immediate consequence of Corollary 4.3 is the following theorem due to Bóna.

Theorem 4.4 (*Bóna [4,5]*). *For all $n, r \in \mathbb{N}$, the coefficients of $W(\mathfrak{S}_n^r; t)$ form a symmetric and unimodal sequence.*

An open problem posed by Bóna [4] is to determine whether the polynomial $W(\mathfrak{S}_n^r; t)$ has the stronger property of having all zeros real for $n, r \in \mathbb{N}$. This is known for $r \geq n-1$ because then $W(\mathfrak{S}_n^r; t) = A_n(t)$ and the Eulerian polynomials are known to have all zeros real (cf. [29]), and for $r = 1$ as we then get the Narayana polynomials (4.1) which are known to have all zeros real by e.g. Malo's theorem (cf. [33]). In [11] we prove real-rootedness whenever $r = 2$ or $r = n-2$. It is easy to see (cf. [9]) that if all zeros of $p(t) = \sum_{i=0}^{n} a_i t^i$ are real and $\{a_i\}_{i=0}^{n}$ is non-negative and symmetric with center of symmetry $d/2$, then

$$p(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} b_i t^i (1+t)^{d-2i},$$

where b_i, $i = 0, \ldots, \lfloor d/2 \rfloor$ are non-negative. Hence Corollary 4.3 can be seen as further evidence for a positive answer to Bóna's question.

Knuth [31] proved that the 1-stack sortable permutations are exactly the permutations that avoid the pattern 231, i.e., permutations $\pi = a_1 a_2 \cdots a_n$ such that $a_k < a_j < a_i$ for no $1 \leq i < j < k \leq n$. The set of 231-avoiding permutations in $\mathfrak{S}_n$ is denoted by $\mathfrak{S}_n(231)$. Simion [39] proved that the n-th Narayana polynomial is the descent generating polynomial of $\mathfrak{S}_n(231)$, i.e.,

$$
\begin{aligned}
W(\mathfrak{S}_n(231); t) &= \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} t^k \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1} \binom{2k}{k} \binom{n-1}{2k} t^k (1+t)^{n-1-2k},
\end{aligned} \tag{4.1}
$$

where the second equality can be derived using hypergeometric formulas, see also [41]. Hence we have the following corollary.

Corollary 4.5. *Let* $n, k \in \mathbb{N}$. *Then*

$$
|\{\pi \in \mathfrak{S}_n(231) : \mathrm{peak}(\pi) = k\}| = 2^{n-1-2k} \frac{1}{k+1} \binom{2k}{k} \binom{n-1}{2k}.
$$

5. A refinement of the Eulerian polynomials

The statistic $(2\text{--}31) : \mathfrak{S}_n \to \mathfrak{S}_n$ is an instance of a generalized permutation pattern as introduced by Babson and Steingrímsson [2]. Let $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. Then $(2\text{--}31)(\pi)$ is the number of pairs $1 \leq i < j \leq n-1$ such that $a_{j+1} < a_i < a_j$. Similarly, let $(13\text{--}2)(\pi)$ be the number of pairs $2 \leq i < j \leq n$ such that $a_{i-1} < a_j < a_i$.

Theorem 5.1. *The statistics* $(2\text{--}31)$ *and* $(13\text{--}2)$ *are constant on any orbit under the MFS-action.*

Proof. An alternative description of $(2\text{--}31)(\pi)$, $\pi = a_1 a_2 \cdots a_n$ is the number triples (a_i, a_j, a_k) such that $1 \leq i < j < k \leq n$ and $a_k < a_i < a_j$, where (a_j, a_k) is a pair of consecutive peak and valley. By consecutive we mean that there are no other peaks or valleys in between a_j and a_k. The number of such triples is invariant under the action since a_j and a_k cannot move and a_i cannot move over the peak a_j. A similar reasoning applies to $(13\text{--}2)$. $\square$

Define a (p, q)-refinement of the Eulerian polynomial by

$$
A_n(p, q, t) = \sum_{\pi \in \mathfrak{S}_n} p^{(13\text{--}2)(\pi)} q^{(2\text{--}31)(\pi)} t^{\mathrm{des}(\pi)}.
$$

These polynomials (or at least $A_n(p, 1, t)$ and $A_n(1, q, t)$) have been in focus in several recent papers [17,18,36,42,50]. A fascinating property of the polynomial $A_n(p, 1, t)$ is that it appears as a translation of the polynomial enumerating the cells in the totally non-negative part of a Grassmannian [36,50], and also in the stationary distribution of the ASEP model in statistical mechanics [17,18].

From Theorem 5.1 and Corollary 3.2 we get that

$$
A_n(p, q, t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} b_{n,i}(p, q) t^i (1+t)^{n-1-2i},
$$

where

$$b_{n,i}(p,q) = 2^{-n+1+2i} \sum_{\substack{\pi \in \mathfrak{S}_n \\ \text{peak}(\pi)=i}} p^{(13\text{–}2)(\pi)} q^{(2\text{–}31)(\pi)}. \tag{5.1}$$

Proposition 5.2. *Let $n \in \mathbb{N}$. Then*

$$A_n(p,q,t) = A_n(q,p,t).$$

Proof. Let f be as in Remark 3.3 and let $R : \mathfrak{S}_n \to \mathfrak{S}_n$ be defined by

$$R(\pi) = a_n \cdots a_2 a_1, \quad \text{if } \pi = a_1 a_2 \cdots a_n.$$

Let $\pi' = R(f(\pi))$. Then

$$(\text{des}(\pi'), (13\text{–}2)(\pi'), (2\text{–}31)(\pi')) = (\text{des}(\pi), (2\text{–}31)(\pi), (13\text{–}2)(\pi)),$$

and the proposition follows. $\square$

A further striking property of $A_n(p,q,t)$ is that

$$A_n(q,q^2,q) = A_n(q^2,q,q) = [n]_q[n-1]_q \cdots [1]_q,$$

where $[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$. This is because the statistics

$$S_1 = (13\text{–}2) + (13\text{–}2) + (2\text{–}31) + \text{des} \quad \text{and} \quad S_2 = (13\text{–}2) + (2\text{–}31) + (2\text{–}31) + \text{des}$$

are Mahonian (see Section 8), a fact due to Simion and Stanton [40], see also [2].

6. An action on the linear extensions of a sign-graded poset

Recall that a *labeled poset* is a pair (P, ω) where P is a finite poset and $\omega : P \to \mathbb{Z}$ is an injection. The *Jordan–Hölder set*, $\mathcal{L}(P, \omega)$, is the set of permutations $\pi = a_1 a_2 \cdots a_p$ ($p = |P|$) of $\omega(P)$ such that if x is smaller than y in P ($x <_P y$), then $\omega(x)$ precedes $\omega(y)$ in π. The (P, ω)-*Eulerian polynomial* is defined by

$$W(P, \omega; t) = \sum_{\pi \in \mathcal{L}(P,\omega)} t^{\text{des}(\pi)}.$$

Hence the n-th Eulerian polynomial is the (P, ω)-Eulerian polynomial of an anti-chain of size n. The (P, ω)-Eulerian polynomials have been intensively studied since they were introduced by Stanley [46] in 1972. For example, the Neggers–Stanley conjecture which asserts that these polynomials always have real zeros has attracted widespread attention [1,3,8–10,13,26,34,35, 44,48,49]. A labeled poset is *naturally labeled* if $x <_P y$ implies $\omega(x) < \omega(y)$. Neggers [34] made the conjecture for naturally labeled posets in 1978 and Stanley formulated the conjecture in its general form in 1986. However, in [10], we found a family of counterexamples to the Neggers–Stanley conjecture and subsequently Stembridge [44] found counterexamples that are naturally labeled thus disproving Neggers original conjecture.

Although the Neggers–Stanley conjecture is refuted many questions regarding the (P, ω)-Eulerian polynomials remain open. A question which is still open is whether the coefficients of $W(P, \omega; t)$ always form a unimodal sequence. It is easy to see that real-rootedness implies unimodality. This weaker property was recently established by Reiner and Welker [35] for a large

and important class of posets, namely the class of naturally labeled and *graded* posets. A poset P is graded if every saturated chain in P has the same length. Prior to [35], Gasharov [26] proved unimodality for graded naturally labeled posets of rank at most 2. In [9] we proved unimodality for (P, ω)-Eulerian polynomials of labeled posets which we call *sign-graded posets*. The class of sign-graded posets contains the class of naturally labeled graded posets.

If (P, ω) is a labeled poset we may associate signs to the edges of the Hasse diagram, $E(P)$, of P as follows. Let $\epsilon : E(P) \to \{-1, 1\}$ be defined by

$$\epsilon(x, y) = \begin{cases} 1 & \text{if } \omega(x) < \omega(y), \\ -1 & \text{if } \omega(x) > \omega(y). \end{cases}$$

It is not hard to prove that the (P, ω)-Eulerian polynomial only depends on ϵ, see [9]. A labeled poset (P, ω) is sign-graded if for every maximal chain $x_1 < x_2 < \cdots < x_k$ in P, the sum of signs

$$\sum_{i=1}^{k} \epsilon(x_{i-1}, x_i),$$

is the same. Note that this definition extends the notion of graded posets since if (P, ω) is naturally labeled then all signs are equal to one and the above sum is just the length of the chain. The common value, r, of the above sum is called the *rank* of (P, ω). One may associate a (generalized) *rank function* $\rho : P \to \mathbb{Z}$ to a sign-graded poset by

$$\rho(x) = \sum_{i=1}^{k} \epsilon(x_{i-1}, x_i),$$

where $x_1 < x_2 < \cdots < x_k = x$ is any saturated chain from a minimal element to x. In [9] we prove the following theorem.

Theorem 6.1 (*Brändén [9]*). *Let (P, ω) be a sign-graded poset of rank r and let $d = p - r - 1$. Then*

$$W(P, \omega; t) = \sum_{i=0}^{\lfloor d/2 \rfloor} a_i(P, \omega) t^i (1 + t)^{d-2i},$$

where $a_i(P, \omega)$, $i = 0, 1, \ldots, \lfloor d/2 \rfloor$ are non-negative integers.

From the proof of Theorem 6.1 in [9] it is not evident what the numbers $a_i(P, \omega)$ count. We will now give an alternative proof of Theorem 6.1 by extending the MFS-action to $\mathcal{L}(P, \omega)$. This will also give us an interpretation of the numbers $a_i(P, \omega)$, $i = 0, \ldots, \lfloor d/2 \rfloor$. If both (P, ω) and (P, λ) are sign-graded one can prove [9, Corollary 2.4] that up to a multiple of t the corresponding Eulerian polynomials are the same. Moreover, in [9] we prove that if (P, ω) is sign-graded then there exists a labeling μ of P such that

(1) (P, μ) is sign-graded,
(2) the rank function of (P, μ) has values in $\{0, 1\}$,
(3) all elements of rank 0 have negative labels and
(4) all elements of rank 1 have positive labels.

Such a labeling will be called *canonical*. Hence it is no restriction in assuming that the sign-graded poset is labeled canonically.

Definition 6.2. Let (P, ω) be sign-graded with ω canonical. For $x \in \omega(P)$ define a map $\psi_x : \mathcal{L}(P, \omega) \to \mathcal{L}(P, \omega)$ as follows. Let $\pi = a_1 a_2 \cdots a_p \in \mathcal{L}(P, \omega)$ and let $a_0 = a_{p+1} = 0$.
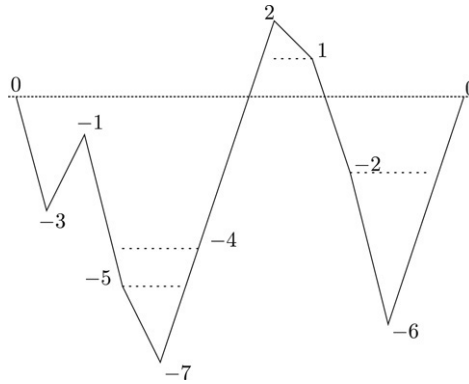
Fig. 3. The dotted lines indicates where the double ascents/descents are mapped.

- If $x < 0$ is a double descent let $\psi_x(\pi)$ be the permutation obtained by inserting x between the first pair of letters a_i, a_{i+1} to the right of x such that $a_i < x < a_{i+1}$.
- If $x < 0$ is a double ascent let $\psi_x(\pi)$ be the permutation obtained by inserting x between the first pair of letters a_i, a_{i+1} to the left of x such that $a_i > x > a_{i+1}$.
- If $x > 0$ is a double descent let $\psi_x(\pi)$ be the permutation obtained by inserting x between the first pair of letters a_i, a_{i+1} to the left of x such that $a_i < x < a_{i+1}$
- If $x > 0$ is a double ascent $\psi_x(\pi)$ be the permutation obtained by inserting x between the first pair of letters a_i, a_{i+1} to the right of x such that $a_i > x > a_{i+1}$.
- If x is a peak or a valley let $\psi_x(\pi) = \pi$.

See Fig. 3.

It is not immediate that this definition makes sense, i.e., that the resulting permutation represents a linear extension of P. Suppose that $x < 0$ is a letter of $\pi \in \mathcal{L}(P, \omega)$. Then x is a letter of a maximal contiguous subword w of π whose letters are all negative. By construction ψ_x will not move x outside of the word w. We claim that

$$\omega^{-1}(w) = \{y \in P : \omega(y) \text{ is a letter of } w\}$$

is an anti-chain. Suppose that $y_1 <_P y_2$ are elements in $\omega^{-1}(w)$. Then, since $\rho(y_1) = \rho(y_2) = 0$, there must be an element $z \in P$ such that $y_1 <_P z <_P y_2$, $\rho(z) = 1$ and $\omega(z) > 0$. This means that $\omega(z)$ is between $\omega(y_1)$ and $\omega(y_2)$ in π, so $\omega(z)$ is a letter of w contrary to the assumption that all letters of w are negative. Since $\omega^{-1}(w)$ is an anti-chain and since ψ_x does not move x outside $\omega^{-1}(w)$ we have that $\psi_x(\pi) \in \mathcal{L}(P, \omega)$. The case $x > 0$ is analogous.

We may now define a $\mathbb{Z}_2^P$-action on $\mathcal{L}(P, \omega)$ by

$$\psi_S(\pi) = \prod_{x \in S} \psi_{\omega(x)}(\pi), \quad S \subseteq P.$$

Let $\hat{\pi}$ be the unique permutation in $\mathrm{Orb}(\pi)$ such that $0\hat{\pi}0$ has no double descents.

Theorem 6.3. *Let (P, ω) be a sign-graded poset of rank r where ω is canonical and let $\pi \in \mathcal{L}(P, \omega)$. Then*

$$\sum_{\sigma \in \mathrm{Orb}(\pi)} t^{\mathrm{des}(\sigma)} = t^{\mathrm{des}(\hat{\pi})}(1+t)^{p-r-1-2\mathrm{des}(\hat{\pi})}.$$

Moreover, if $r = 0$ then peak$(\cdot)$ *is invariant under the $\mathbb{Z}_2^P$-action and* peak$(\pi) = $ des$(\hat{\pi})$ *for all* $\pi \in \mathcal{L}(P, \omega)$.

Proof. If x is a double ascent in $0\pi0$ then des$(\psi_x(\pi)) = $ des$(\pi) + 1$. It follows that

$$\sum_{\sigma \in \mathrm{Orb}(\pi)} t^{\mathrm{des}(\sigma)} = t^{\mathrm{des}(\hat{\pi})}(1 + t)^a$$

where a is the number of double ascents in π. Suppose $r = 0$. Deleting all double ascents in $\hat{\pi}$ results in an alternating permutation

$$0 > a_1 < a_2 > a_3 < \cdots > a_{p-a} < 0,$$

with the same number of peaks/descents as π. Hence $p - a = 2$peak$(\pi) + 1$.

If $r = 1$, deleting all double ascents in $\hat{\pi}$ results in an alternating permutation

$$0 > a_1 < a_2 > a_3 < \cdots < a_{p-a} > 0,$$

with the same number of descents. Hence $p - a - 2 = 2$des$(\hat{\pi})$. $\square$

Stembridge [43] developed a theory of "enriched P-partitions" in which the distribution of *peaks* in $\mathcal{L}(P, \omega)$ and the polynomial, viz.,

$$\overline{W}(P, \omega; t) = \sum_{\pi \in \mathcal{L}(P, \omega)} t^{\mathrm{peak}(\pi)},$$

play a significant role. For a canonically labeled poset (P, ω) let $(\hat{P}, \hat{\omega})$ be any canonically labeled poset such that $\hat{P}$ is obtained from P by adjoining a greatest element.

Theorem 6.4. *Let (P, ω) be a canonically labeled sign-graded poset of rank r. If $r = 0$ then*

$$W(P, \omega; t) = 2^{-p+1}(1 + t)^{p-1}\overline{W}(P, \omega; 4t(1 + t)^{-2}).$$

Equivalently,

$$a_i(P, \omega) = 2^{-p+1+2i}|\{\pi \in \mathcal{L}(P, \omega) : \mathrm{peak}(\pi) = i\}|.$$

If $r = 1$ then

$$W(P, \omega; t) = 2^{-p}t^{-1}(1 + t)^p\overline{W}(\hat{P}, \hat{\omega}; 4t(1 + t)^{-2}).$$

Equivalently,

$$a_i(P, \omega) = 2^{-p+2+2i}|\{\pi \in \mathcal{L}(\hat{P}, \hat{\omega}) : \mathrm{peak}(\pi) = i + 1\}|.$$

Proof. Note that $W(\hat{P}, \hat{\omega}; t) = t^{-r}W(P, \omega; t)$, so we may assume that $r = 0$. By Theorem 6.3 the proof follows just as the proof of Corollary 3.2. $\square$

7. Vertices of even height

To any permutation w of a finite subset of $\{1, 2, \ldots\}$ we may associate a decreasing unordered tree as follows. Let ∞ be a symbol which is greater than every letter in w. If w is empty then $T(w; \infty)$ is the tree with a single vertex labeled ∞. Otherwise write w as $w = m_1 w_1 m_2 w_2 \cdots m_k w_k$ where m_i are the left-to-right maxima of w. Then $T(w; \infty)$ is the
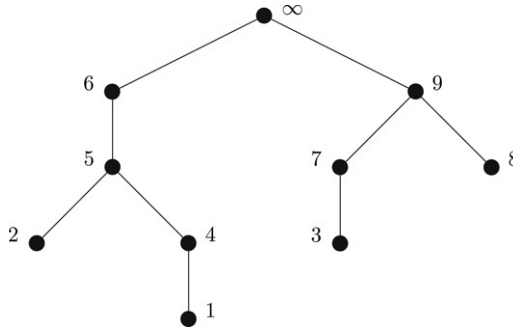
Fig. 4. The decreasing unordered tree corresponding to 652419738.

labeled tree with $T(w_i; m_i)$ as subtrees of the root, see Fig. 4. Let $veh(\pi)$ be the number of (non-root) vertices of even height in $T(\pi; \infty)$. As Fig. 4 suggests

$$veh(652419738) = |\{1, 5, 7, 8\}| = 4.$$

We will here show that veh and des have the same distribution on any subset of $\mathfrak{S}_n$ invariant under the MFS-action. For $\pi \in \mathfrak{S}_n$ and $x \in [n]$ let $r_\pi(x)$ be the number of right edges in the path from the root to x in the decreasing binary tree associated with π. It is plain to see that $r_\pi(x) + 1$ is equal to the height of x as a vertex $T(\pi; \infty)$. Let $\mathrm{Odd}(\pi)$ the set of all $x \in [n]$ for which $r_\pi(x)$ is odd. Hence $\mathrm{Odd}(\pi)$ is the set of vertices of even height in $T(\pi; \infty)$. Also, let $\mathrm{Redge}(\pi)$ be the set of vertices in the decreasing binary tree which are ends of right edges. Clearly, $des(\pi) = |\mathrm{Redge}(\pi)|$. Define $\Psi, \Phi : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\Psi(\pi) = \prod_x \varphi_x(\pi) \quad (x \in \mathrm{Odd}(\pi));$$

$$\Phi(\pi) = \prod_x \varphi_x(\pi) \quad (x \in \mathrm{Redge}(\pi)).$$

Theorem 7.1. *The transformations Ψ and Φ are inverses of each other. Moreover, if $\pi \in \mathfrak{S}_n$ then*

$$\mathrm{Odd}(\pi) = \mathrm{Redge}(\pi') \quad and$$
$$\mathrm{Redge}(\pi) = \mathrm{Odd}(\pi''),$$

where $\pi' = \Psi(\pi)$ and $\pi'' = \Phi(\pi)$.

Proof. Note that it is enough to prove the first equality since then

$$\Phi(\Psi(\pi)) = \prod_x \varphi_x \prod_y \varphi_y(\pi) \quad \left(x \in \mathrm{Redge}(\pi'), y \in \mathrm{Odd}(\pi)\right)$$

$$= \prod_x \varphi_x \prod_y \varphi_y(\pi) \quad (x, y \in \mathrm{Odd}(\pi))$$

$$= \pi,$$

because the involutions φ_x commute.

Let T and T' be the decreasing binary trees corresponding to π and π', respectively. Suppose that $x \in \mathrm{Odd}(\pi)$ and let y be its father in T. If there is a right edge between x and y in T then $y \notin \mathrm{Odd}(\pi)$, which means that there will also be a right edge between x and y in T', so that

$x \in \mathrm{Redge}(\pi')$. If there is a left edge between x and y in T then also $y \in \mathrm{Odd}(\pi)$ so that Ψ will turn this edge to a right edge and hence $x \in \mathrm{Redge}(\pi')$.

The fact that $x \notin \mathrm{Odd}(\pi)$ implies $x \notin \mathrm{Redge}(\pi)$ follows similarly.   □

Note that Ψ and Φ restricts to bijections on all subsets of $\mathfrak{S}_n$ invariant under the "proper" Foata–Strehl action, but not on subsets invariant under the modified Foata–Strehl action. Define a transformation $\Psi' : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\Psi'(\pi) = \prod_x \varphi'_x(\pi) \quad (x \in \mathrm{Odd}(\pi)).$$

Theorem 7.2. *Let $T \subseteq \mathfrak{S}_n$ be invariant under the modified Foata–Strehl action. Then $\Psi' : T \to T$ is a bijection and*

$$\mathrm{veh}(\pi) = \mathrm{des}(\Psi'(\pi)), \quad \pi \in T.$$

Proof. Since the involutions φ_x commute we may write Ψ' as $\Psi' = F \circ \Psi$ where F is defined by

$$F(\pi) = \prod_x \varphi_x(\pi) \quad (x \in \mathrm{Redge}(\pi), c(x) = 2),$$

and where $c(x)$ is the number children of x in the decreasing binary tree of π. Clearly, $\mathrm{des}(\pi) = \mathrm{des}(F(\pi))$ so it remains to prove that Ψ' is a bijection.

Let f be defined as in Remark 3.3 and let $\pi \in \mathfrak{S}_n$. Then since the involutions φ_x commute we have

$$f(\Psi'(\pi)) = \prod_y \varphi'_y \prod_x \varphi'_x(\pi) \quad (y \in [n], x \in \mathrm{Odd}(\pi))$$

$$= \prod_x \varphi'_x(\pi) \quad (x \notin \mathrm{Odd}(\pi)).$$

It follows that Ψ' can be defined recursively on the set of permutations of any finite subset of $\{1, 2, \ldots\}$ as follows. The empty word is mapped by Ψ' to itself, and if $w = LnR$ where n is the greatest element of w and L and R are the words to the left and right of n respectively then

$$\Psi'(w) = \Psi'(L)nf(\Psi'(R)),$$

where f is as in Remark 3.3. From this recursive definition it is plain to see that Ψ' is bijective.   □

Corollary 7.3. *Let $n, r \in \mathbb{N}$. Then* veh *and* des *have the same distribution over $\mathfrak{S}_n^r$.*

To every unordered decreasing tree $T(\pi; \infty)$ corresponding to a permutation $\pi \in \mathfrak{S}_n(231)$ there is a unique ordered unlabeled tree obtained by ordering the children of a vertex decreasingly from left to right and dropping the labels. Recall that a *Dyck path* of length $2n$ is a lattice path in $\mathbb{N}^2$ starting at the origin and ending at $(2n, 0)$, using steps $u = (1, 1)$ and $d = (1, -1)$, and never going below the x-axis. If we traverse the ordered tree in pre-order and write a u every time we go down an edge and write a d every time we go up an edge we obtain a Dyck path. This describes a bijection between the set of Dyck path of length $2n$ and the set of ordered trees with $n + 1$ vertices (and by the above also between the set of Dyck path of length $2n$ and $\mathfrak{S}_n(231)$). Note that a vertex of even height translates into an up-step of even height in the Dyck path, and
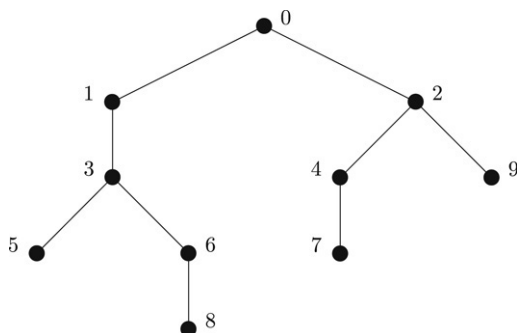
Fig. 5. The increasing unordered tree corresponding to 586317492.

a descent translates into a double up-step *uu* in the path. We have thus recovered the following classical result of Kreweras [32].

Corollary 7.4. *The statistics "up-steps at even height" and "double up-steps" have the same distribution over the set of Dyck paths of a given length.*

When restricted to $\mathfrak{S}_n(231)$ one may express veh as the following alternating sum of permutation patterns [12]

$$\mathrm{veh}(\pi) = d_1(\pi) - 2d_2(\pi) + 4d_3(\pi) - \cdots + (-2)^{n-2}d_{n-1}(\pi),$$

where $d_i(\pi)$ is the number of decreasing subsequences of length $i + 1$ in π.

8. A Mahonian partner for vertices of even height

Recall that the *descent set* of a permutation $\pi = a_1a_2 \cdots a_n$ is defined by $\mathrm{Des}(\pi) = \{i \in [n-1] : a_i > a_{i+1}\}$ and that the *major index* of π as

$$\mathrm{MAJ}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i.$$

A statistic $B : \mathfrak{S}_n \to \mathbb{N}$ is said to be *Mahonian* if it has the same distribution as MAJ on $\mathfrak{S}_n$, i.e.,

$$\sum_{\pi \in \mathfrak{S}_n} q^{B(\pi)} = [n]_q[n-1]_q \cdots [1]_q,$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$. A bistatistic (A, B) is *Euler–Mahonian* if it has the same distribution as (des, MAJ) on $\mathfrak{S}_n$. We will now redefine the statistic veh so that we can define a Mahonian partner for it. To every permutation $\pi = a_1a_2 \cdots a_n \in \mathfrak{S}_n$ we associate an increasing unordered tree, $T'(\pi)$, as follows. If b is a right-to-left minimum of π then b is a successor of the root, which is labeled 0. Otherwise b is the successor of the leftmost element a to the right of b which is smaller than b, see Fig. 5.

Let $n \in \mathbb{N}$. We (re-)define the statistic *vertices of even height*, $\mathrm{veh}' : \mathfrak{S}_n \to \mathbb{N}$, by letting $\mathrm{veh}'(\pi)$ be the number of (non-root) vertices in $T'(\pi)$ of even height. Thus $\mathrm{veh}'(586317492) = |\{3, 4, 8, 9\}| = 4$. We define the *even vertex set*, $\mathrm{EV}(\pi)$, as the set of indices $1 \le i \le n$ such that the vertex a_i is of even height.

The *complement*, π^c, of a permutation $\pi = a_1a_2 \cdots a_n$ is the permutation $b_1b_2 \cdots b_n$ on the same letters as π such that $a_i < a_j$ if and only if $b_i > b_j$ for all $1 \le i < j \le n$. We define

a transformation Θ on permutations of any finite subset of $\{1, 2, \ldots\}$ recursively as follows. The empty permutation is mapped onto itself and if π is the concatenation $\sigma m \tau$ where m is the smallest letter in π, then $\Theta(\pi) = \Theta(\sigma^c) m \Theta(\tau)$. It is clear that Θ restricted to the symmetric group is a bijection.

$$\Theta(586317492) = \Theta(6358)1\Theta(7492)$$
$$= 63\Theta(58)1\Theta(794)2$$
$$= 635819742.$$

If $S \subset \mathbb{Z}$ and $k \in \mathbb{Z}$ let $S + k := \{s + k : s \in S\}$.

Theorem 8.1. *Let $n \in \mathbb{N}$. For all permutations $\pi \in \mathfrak{S}_n$ we have*

$$\mathrm{EV}(\Theta(\pi)) = \mathrm{Des}(\pi).$$

Proof. The proof is by induction over the length $n = |\pi|$ of π. The case $n = 0$ is clear. Suppose that $n > 0$. Then we can write $\pi \in \mathfrak{S}_n$ as the concatenation $\sigma 1 \tau$.

Let $k = |\sigma|$. If $1 \leq i \leq k$ then clearly $i \in \mathrm{EV}(\sigma)$ if and only if $i \notin \mathrm{EV}(\pi)$. Hence

$$\mathrm{EV}(\pi) = ([k] \setminus \mathrm{EV}(\sigma)) \cup (\mathrm{EV}(\tau) + k + 1) \quad \text{and}$$
$$\mathrm{Des}(\pi) = ([k] \setminus \mathrm{Des}(\sigma^c)) \cup (\mathrm{Des}(\tau) + k + 1),$$

since $[n] \setminus \mathrm{Des}(\pi) = \{n\} \cup \mathrm{Des}(\pi^c)$ for all π of length n. Using induction we get

$$\mathrm{EV}(\Theta(\pi)) = \mathrm{EV}(\Theta(\sigma^c)1\Theta(\tau))$$
$$= ([k] \setminus \mathrm{EV}(\Theta(\sigma^c))) \cup (\mathrm{EV}(\Theta(\tau)) + k + 1)$$
$$= ([k] \setminus \mathrm{Des}(\sigma^c)) \cup (\mathrm{Des}(\tau) + k + 1)$$
$$= \mathrm{Des}(\pi). \quad \square$$

It is desirable to find a bijection which is not defined recursively and which proves Theorem 8.1.

We may now define a Mahonian partner for veh$'$. The statistic sum of indices of vertices even height, SIVEH : $\mathfrak{S}_n \to \mathbb{N}$, is defined by

$$\mathrm{SIVEH}(\pi) = \sum_{i \in \mathrm{EV}(\pi)} i.$$

Corollary 8.2. *For all $n \in \mathbb{N}$ the bistatistic* (veh$'$, SIVEH) *is Euler–Mahonian on $\mathfrak{S}_n$.*

9. Gal's conjecture on γ-polynomials

Recall that the h-polynomial of a simplicial complex Δ of dimension $d - 1$ is the polynomial $h_\Delta(t) = h_0(\Delta) + h_1(\Delta)t + \cdots + h_d(\Delta)t^d$ defined by the polynomial identity

$$\sum_{i=0}^{d} h_i(\Delta)t^i(1 + t)^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta)t^i,$$

where $f_i(\Delta)$, $-1 \leq i \leq d - 1$ is the number of faces of Δ of dimension i. If Δ is a simplicial homology sphere then the *Cohen–Macaulay property* and the *Dehn–Sommerville equations*

imply that $\{h_i(\Delta)\}_{i=0}^d$ is non-negative and symmetric. Hence one may define the γ-polynomial of Δ, $\gamma_\Delta(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i(\Delta) t^i$, by

$$h_\Delta(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i(\Delta) t^i (1+t)^{d-2i}.$$

A simplicial complex Δ is *flag* if the minimal non-faces of Δ have cardinality two. The following conjecture generalizes the Charney–Davis conjecture [15].

Conjecture 9.1 (*Gal [25]*). *If Δ is a flag simplicial homology sphere of dimension $d-1$, then*

$$\gamma_i(\Delta) \geq 0, \quad 0 \leq i \leq \lfloor d/2 \rfloor.$$

It is desirable to find a combinatorial, geometrical or ring-theoretical description of the numbers $\gamma_i(\Delta)$. In [35] Reiner and Welker associated to any graded naturally labeled poset (P, ω) a simplicial polytopal sphere, $\Delta_{\mathrm{eq}}(P)$, whose h-polynomial is the (P, ω)-Eulerian polynomial. Hence, Theorem 6.3 gives a combinatorial description of the γ-polynomial of $\Delta_{\mathrm{eq}}(P)$ and verifies Conjecture 9.1 for $\Delta_{\mathrm{eq}}(P)$.

In [37] Postnikov, Reiner and Williams extended the MFS-action to give a combinatorial interpretation of the γ-polynomials of tree-associahedra which confirms Conjecture 9.1 in this case. Also, Chow [16] has given a combinatorial interpretation of the γ-polynomials of the Coxeter complexes of type B and D and confirming Conjecture 9.1 for these complexes.

10. Further directions and open problems

Let $\mathcal{I}_n$ be the set of involutions in $\mathfrak{S}_n$ and let

$$I_n(t) = \sum_{\pi \in \mathcal{I}_n} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{n-1} I_{n,k} t^k.$$

Brenti has conjectured that the sequence $\{I_{n,k}\}_{k=0}^{n-1}$ has no internal zeros and is log-concave, i.e.,

$$I_{n,k}^2 \geq I_{n,k+1} I_{n,k-1}, \quad 1 \leq k \leq n-2,$$

see [19] where progress on this conjecture was made. Motivated by Brenti's conjecture Guo and Zeng [28] proved the weaker statement that $\{I_{n,k}\}_{k=0}^{n-1}$ is unimodal. Also, Strehl [47] proved symmetry for $\{I_{n,k}\}_{k=0}^{n-1}$ and the following conjecture was made in [28].

Conjecture 10.1 (*Guo–Zeng [28]*). *Let $n \in \mathbb{N}$. Then*

$$I_n(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} a_{n,i} t^i (1+t)^{n-1-2i},$$

where $a_{n,i} \in \mathbb{N}$ for $0 \leq i \leq \lfloor (n-1)/2 \rfloor$.

Gessel [27] has conjectured a fascinating property of the joint distribution of descents and inverse descents.

Conjecture 10.2 (*Gessel [27]*). *Let $\tau \in \mathfrak{S}_n$. Then*

$$\sum_{\pi \in \mathfrak{S}_n} s^{\text{des}(\pi)} t^{\text{des}(\pi^{-1}\tau)} = \sum_{k,j} c_n(\tau; k, j)(s+t)^k (st)^j (1+st)^{n-k-1-2j}, \tag{10.1}$$

where $c_n(\tau; k, j) \in \mathbb{N}$ for all $k, j \in \mathbb{N}$.

Symmetry properties imply that an expansion such as (10.1) with $c_n(\tau; k, j) \in \mathbb{Z}$, $k, j \in \mathbb{N}$ exists. Moreover, $c_n(\tau; k, j)$ only depends on the number of descents of τ. In light of Conjectures 10.1 and 10.2 there might be another $\mathbb{Z}_2^n$-action on permutations which also behaves well with respect to the inverse permutation.

Recall the definition of $A_n(p, q, t)$ of Section 5. The first non-trivial examples are

$$A_3(p, q, t) = (1+t)^2 + (p+q)t$$
$$A_4(p, q, t) = (1+t)^3 + (p+q)(p+q+2)t(1+t)$$
$$A_5(p, q, t) = (1+t)^4 + (p+q)((p+q)^2 + 2(p+q) + 3)t(1+t)^2$$
$$\qquad\qquad + (p+q)^2(p^2 + pq + q^2 + 1)t^2.$$

Conjecture 10.3. *Let $b_{n,i}(q)$ be defined by (5.1). Then $(p+q)^i \mid b_{n,i}(p, q)$ for all $0 \le i \le \lfloor (n-1)/2 \rfloor$.*

Acknowledgements

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