Minimum cost source location problem with local
3-vertex-connectivity requirements

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Abstract

Let $G = (V, E)$ be a simple undirected graph with a set $V$ of vertices and a set $E$ of edges. Each vertex $v \in V$ has a demand $d(v) \in \mathbb{Z}_+$ and a cost $c(v) \in \mathbb{R}_+$, where $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of nonnegative integers and the set of nonnegative reals, respectively. The source location problem with vertex-connectivity requirements in a given graph $G$ requires finding a set $S$ of vertices minimizing $\sum_{v \in S} c(v)$ such that there are at least $d(v)$ pairwise vertex-disjoint paths from $S$ to $v$ for each vertex $v \in V - S$. It is known that if there exists a vertex $v \in V$ with $d(v) \geq 4$, then the problem is NP-hard even in the case where every vertex has a uniform cost. In this paper, we show that the problem can be solved in $O(|V|^4 \log^2 |V|)$ time if $d(v) \leq 3$ holds for each vertex $v \in V$.

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1. Introduction

Problems of selecting the best location of facilities in a given network to satisfy a certain property are called location problems [14]. Recently, the location problems with requirements measured by a network-connectivity were studied extensively [1,2,5,7,10–13,17,19,20].

Connectivity and/or flow-amount are very important factors in applications in the control and design of multimedia networks. In a multimedia network, some vertices of the network, such as the so-called mirror servers, may have the function of offering the same services for users. Let us call a vertex that can offer the service $i$ a source, and let $S$ be a set of sources, where we can locate more than one source in a network. A user at a vertex $v$ can use the service $i$ by communicating with at least one source $s \in S$ through a path between $s$ and $v$. The flow-amount (which is the capacity

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of the paths between $S$ and $v$) affects the maximum data amount that can be transmitted from $S$ to a user at a vertex $v$. Also, the edge-connectivity or the vertex-connectivity between a source set $S$ and a vertex $v$ measures the robustness of the service against network failures. The concept of such connectivity and/or flow-amount between a vertex and a set of specified vertices was given by Ito [9], in the context of considering the design of a reliable telephone network with plural switching apparatuses. Moreover, recently, not only location problems but also connectivity augmentation problems based on this type of connectivity have been studied [6,8,15].

In this paper, we consider the problem of finding the best location of a source set $S$ under connectivity and/or flow-amount requirements from each vertex. We introduce the source location problem, which is formulated as follows.

**Problem 1 (Source Location Problems).**

**Input:** A graph $G = (V, E)$ with a set $V$ of vertices and a set $E$ of edges capacitated by nonnegative reals, a cost function $c : V \rightarrow R_+$ (where $R_+$ denotes the set of nonnegative reals), and a demand function $d : V \rightarrow R_+$.

**Output:** A vertex set $S \subseteq V$ such that $\psi(S, v) \geq d(v)$ holds for every vertex $v \in V - S$ and $\sum_{v \in S} c(v)$ is the minimum, where $\psi(S, v)$ is a measurement based on the edge-connectivity, the vertex-connectivity or the flow-amount between $S$ and a vertex $v$. □

For such measurements $\psi(S, v)$, one may consider the minimum capacity $\lambda(S, v)$ of an edge cut $C \subseteq E$ that separates $v$ from $S$, the minimum size $\kappa(S, v)$ of a vertex cut $C \subseteq V - S - v$ that separates $S$ and $v$, or the maximum number $\hat{k}(S, v)$ of paths between $S$ and $v$ such that no pair of paths has a common vertex in $V - v$.

Source location problems with $\psi = \lambda$ in undirected graphs were treated by Tamura et al. [19,20], Ito et al. [12, 13] and Arata et al. [1]. They gave polynomial time algorithms for uniform costs $c(v) = 1$, $v \in V$, while Sakashita et al. [18] showed that the problem with general costs $c(v)$, $v \in V$ is strongly NP-hard. In directed graphs, Ito et al. [11] showed that the problem is strongly NP-hard even if the cost function is uniform, while Bárász et al. [2] showed that the problem for a measurement “$\lambda^+(S, v) \geq \ell$ and $\lambda^-(S, v) \geq k$” and uniform costs can be solved in polynomial time, where $\lambda^+(S, v)$ (resp., $\lambda^-(S, v)$) is the maximum number of edge-disjoint directed paths from $S$ to $v$ (resp., from $v$ to $S$).

Ito et al. [10] treated the source location problem for undirected graphs with unit capacities, a measurement “$\kappa(S, v) \geq k$ and $\lambda(S, v) \geq \ell$ for all $v \in V - S$”, and uniform costs $c(v) = 1$, $v \in V$. They presented an $O(m + n^2 + n \min(m, \ell n) \min(\ell, n))$ time algorithm for $k \leq 2$ and showed the NP-hardness of the problem for $k \geq 3$ even if $\ell = 0$, where $n = |V|$, $m = |\{u, v\} | (u, v) \in E|$.

Thus, the problems with $\psi = \kappa$ are intractable, but Nagamochi et al. [17] showed that the problem with $\psi = \hat{k}$ and uniform demands $d(v) = k$, $v \in V$ is polynomially solvable. For this problem, they gave an $O(\min\{k, \sqrt{n}\} mn)$ time algorithm for digraphs and an $O(\min\{k, \sqrt{n}\} kn^2)$ time algorithm for undirected graphs (notice that if $\psi = \kappa$ or $\psi = \hat{k}$, then unit edge capacities can be assumed without affecting the problem). Furthermore, they showed that the source location problem for a measurement “$\hat{k}^+(S, v) \geq \ell$ and $\hat{k}^-(S, v) \geq k$” in digraphs can be solved in polynomial time, where $\hat{k}^+(S, v)$ (resp., $\hat{k}^-(S, v)$) is the maximum number of directed paths from $S$ to $v$ (resp., from $v$ to $S$) such that no pair of paths has a common vertex in $V - v$. For the problem with $\psi = \hat{k}$, uniform costs $c$, and general demands $d$ in undirected graphs, Ishii et al. [7] gave a linear time algorithm in the case of $d \in \{0, 1, 2, 3\}$ and showed that it is NP-hard if there exists a vertex $v \in V$ with $d(v) \geq 4$.

In this paper, we show that the problem with $\psi = \hat{k}$ and general demands $d \in \{0, 1, 2, 3\}$ in undirected graphs is polynomially solvable even if the cost function $c$ is general. By doing this, we clear the border between NP-hard and polynomially solvable classes of the problem with $\psi = \hat{k}$ in undirected graphs.

Here, we summarize our method, after reviewing the existing algorithms for the problem with $\psi = \hat{k}$ in undirected graphs. Nagamochi et al. [17] showed that the problem with uniform demands enjoys a matroidal property and an optimal solution can be found by a greedy method. On the other hand, the problem with general demands does not satisfy such a good property. However, for the problem with uniform costs and $d \in \{0, 1, 2, 3\}$, Ishii et al. [7] showed that the cardinality of a minimal feasible solution $S'$ obtained by a greedy method is at most twice the optimal cardinality for almost all instances. Based on the information on the $S'$, their method finds an optimal solution by replacing some two vertices in $S'$ with one vertex. In this paper, for the problem with general costs and $d \in \{0, 1, 2, 3\}$, our method first finds a minimal feasible solution $S'$ by the same greedy method as the one in [7]. Based on the information on $S'$, we show that we can reduce the problem to some special case of the hitting set problem [4], which can be solved by computing the weighted matroid intersection problems [3] $\text{poly}(|V|, |E|)$ times, where $\text{poly}(|V|, |E|)$ denotes some polynomial in $|V|$ and $|E|$.
Given an undirected graph \( G = (V, E) \), where each vertex \( v \in V \) with \( d(v) = 0, 1, 2, 3 \) is drawn as a square, a triangle, a circle, and a star, respectively, and the cost associated with each vertex is omitted. (b) A set \( S \) of black vertices is a source set; there are at least \( d(v) \) paths between \( S \) and each vertex \( v \in V - S \) such that no pair of paths has a common vertex in \( V - v \).

The rest of the paper is organized as follows. Some definitions and preliminaries are described in Section 2. Also in Section 2, we state our main result that the problem with general costs and \( d \in \{0, 1, 2, 3\} \) is polynomially solvable. In Section 3, we describe an algorithm for solving the problem, prove its correctness, and discuss the time complexity of our algorithm. Finally, we give some concluding remarks in Section 4.

2. Preliminaries

Let \( G = (V, E) \) be a simple undirected graph with a set \( V \) of vertices and a set \( E \) of edges, where we denote \(|V|\) by \( n \) and \(|E|\) by \( m \). A singleton set \( \{x\} \) may be simply written as \( x \), and "\( \subseteq \)" implies proper inclusion while "\( \subset \)" means "\( \subsetneq \)" or "\( = \)". A vertex set and an edge set of the graph \( G \) is denoted by \( V(G) \) and \( E(G) \), respectively. For a vertex subset \( V' \subseteq V \), \( G[V'] \) means the subgraph induced by \( V' \). For a vertex set \( X \subseteq V \), \( N_G(X) \) is defined as a set of all vertices in \( V - X \) which are adjacent to some of vertices in \( X \). A partition \( \mathcal{X} = \{X_1, \ldots, X_\kappa\} \) of the vertex set \( V \) means a family of nonempty mutually disjoint subsets of \( V \) whose union is \( V \), and a subpartition of \( V \) means a partition of a subset \( V' \) of \( V \). For a vertex set \( Y \subseteq V \) and a family \( \mathcal{X} \) of vertex sets, \( Y \) covers \( \mathcal{X} \) if each \( X \in \mathcal{X} \) satisfies \( X \cap Y \neq \emptyset \).

For a vertex \( v \in V \) and a vertex set \( X \subseteq V - \{v\} \) in \( G \), we denote by \( \check{k}_G(X, v) \) the maximum number of paths from \( v \) to \( X \) such that no pair of paths has a common vertex in \( V - v \). For a vertex \( v \in V \) and a vertex set \( X \subseteq V \) with \( v \in X \), let \( \check{k}_G(X, v) = \infty \). By Menger’s theorem, the following lemma holds.

**Lemma 2.** For a vertex \( v \in V \) and a vertex set \( X \subseteq V - \{v\} \), \( \check{k}_G(X, v) \geq k \) holds if and only if \(|N_G(W)| \geq k \) holds for every vertex set \( W \subseteq V - X \) with \( v \in W \).

In this paper, each vertex \( v \in V \) in \( G = (V, E) \) has a nonnegative integer demand \( d(v) \) and a nonnegative real cost \( c(v) \). For a cost function \( c : V \to R_+ \) and a set \( X \subseteq V \) of vertices, \( c(X) \) is defined as \( \sum_{v \in X} c(v) \). A vertex set \( S \subseteq V \) is called a source set if it satisfies

\[
\check{k}_G(S, v) \geq d(v) \quad \text{for all vertices } v \in V - S,
\]

and we call each vertex \( v \in S \) a source. In this paper, we consider the following source location problem with local \( k \)-vertex-connectivity requirements in an undirected graph (shortly, \( kLV-CSLP \)). Fig. 1 gives an instance of 3LV-CSLP.

**Problem 3 (**\( kLV-CSLP \)).

**Input:** An undirected graph \( G = (V, E) \), a demand function \( d : V \to \{0, 1, \ldots, k\} \), and a cost function \( c : V \to R_+ \).

**Output:** A source set \( S \subseteq V \) minimizing \( c(S) \).

The main result of this paper is described as follows.

**Theorem 4.** Given an undirected graph \( G = (V, E) \), a demand function \( d : V \to \{0, 1, 2, 3\} \), and a cost function \( c : V \to R_+ \), 3LV-CSLP can be solved in \( O(n^4 \log^2 n) \) time.

In the rest of this section, we introduce several properties for a general \( kLV-CSLP \), which will be used in the subsequent sections. For a vertex set \( X \subseteq V \), \( d(X) \) denotes the maximum demand among all vertices in \( X \), i.e.,
Let $W$ be a minimal deficient set w.r.t. $S$. Every minimal deficient set $W$ w.r.t. $v$ if and only if $S$ is a source set. □

Lemma 6 (7). Every minimal deficient set $W$ w.r.t. $v \in W$ induces a connected graph. □

Lemma 7. Each minimal deficient set $W$ w.r.t. $v \in W$ satisfies $|N_G(W)| = d(v) - 1$ if $|W| \geq 2$.

Proof. Let $W$ be a deficient set with $|W| \geq 2$, $v \in W$, and $|N_G(W)| < d(v) - 1$. For a vertex $u \in W - \{v\}$, $W' = W - \{u\}$ satisfies $|N_G(W')| \leq |N_G(W)| + 1$ and hence it is also a deficient set with $v \in W'$. □

Lemma 8. Let $W$ be a minimal deficient set w.r.t. $v \in W$. Then for each vertex $x \in N_G(W)$, $(x, v) \in E(G)$ holds or $G[W \cup \{x\}]$ has at least two internally vertex-disjoint paths between $v$ and $x$.

Proof. Otherwise there exists a partition $\{W_1, y, W_2\}$ of $W \cup \{x\}$ such that we have $v \in W_1$, $x \in W_2$, and $N_G(W_1) \cap W = \{y\} = N_G(W_2) \cap W$. By $|N_G(W_1)| \leq |N_G(W)|$, $W_1$ is also a deficient set, contradicting the minimality of $W$. □

For two vertex sets $X$ and $Y$, we say that $X$ and $Y$ intersect each other, if none of $X \cap Y$, $X - Y$, and $Y - X$ is empty. For two vertex sets $X$ and $Y$, the following holds.

$$|N_G(X)| + |N_G(Y)| \geq |N_G(X \cap Y)| + |N_G(X \cup Y)|.$$ 

(2)

Lemma 9. Let $W_i$, $i = 1, 2$ be a minimal deficient set w.r.t. $w_i \in W_i$ with $|N_G(W_i)| \leq 2$. If $W_1$ and $W_2$ intersect each other and $N_G(W_1 \cup W_2) \neq \emptyset$ holds, then one of the following (i)–(iv) holds.

(i) $|N_G(W_1 \cup W_2)| = 1$, $N_G(W_1) \subseteq W_2$, and $N_G(W_2) - W_1 \neq \emptyset$.

(ii) $|N_G(W_1 \cup W_2)| = 1$, $N_G(W_2) \subseteq W_1$, and $N_G(W_1) - W_2 \neq \emptyset$.

(iii) $|N_G(W_1 \cup W_2)| = 1$ and $N_G(W_1) - W_2 = N_G(W_2) - W_1 \neq \emptyset$.

(iv) $|N_G(W_1 \cup W_2)| = 2$, $|N_G(W_1)| = |N_G(W_2)| = 2$, $N_G(W_1) \cap W_2 = \{w_2\}$, and $N_G(W_2) \cap W_1 = \{w_1\}$.

Proof. Lemma 6 says that $W_1 \cap N_G(W_2) \neq \emptyset \neq W_2 \cap N_G(W_1)$ holds. Hence we have $|N_G(W_1 \cap W_2)| \geq 2$. By (2), $|N_G(W_1 \cup W_2)| \leq 2$ holds. The case of $|N_G(W_1 \cup W_2)| = 1$ implies (i), (ii), or (iii). Assume that $|N_G(W_1 \cup W_2)| = 2$ holds. Then (2) implies that $|N_G(W_1)| = |N_G(W_2)| = |N_G(W_1 \cap W_2)| = 2$ and $|N_G(W_1) \cap W_2| = |N_G(W_2) \cap W_1| = 1$ hold. By Lemma 7, we have $d(w_1) = d(w_2) = 3$. From the minimality of $W$, $w_i \in W_i - W_j$ holds for $i, j = \{1, 2\}$. By Lemma 8, we have $\{w_i\} = W_i \cap N_G(W_j)$. □

3. Algorithm

In this section, we give an algorithm for solving 3LV-CSLP. If a given graph is disconnected, then we can consider the problem separately for each connected component. Hence we suppose that $G$ is a connected graph. Also assume that there exists a vertex $v \in V$ with $d(v) \geq 2$ since the problem with $d : V \to \{0, 1\}$ is trivial. Here we propose an algorithm, named 3-LVC-CSLP($x$), for finding a source set $S$ such that $S$ contains a given vertex $x \in V$ and $c(S)$ is minimized. Note that an optimal solution to 3LV-CSLP can be obtained by executing algorithm 3-LVC-CSLP($x$) for each vertex $x \in V$.

We first sketch algorithm 3-LVC-CSLP($x$), which consists of two steps. The first step is a greedy method to find a minimal feasible solution $S_0$ and a family $W_0$ of minimal deficient sets w.r.t. some $s \in S_0$. We start with a source set $S_0 = V$ and a family $W_0 := \emptyset$ of minimal deficient sets, and pick up vertices $v \in V - \{x\}$, one by one, in nondecreasing order of their demands. If $S_0 - \{v\}$ remains a source set, update $S_0 := S_0 - \{v\}$; otherwise we have a minimal deficient set $W$ w.r.t. $v$ with $W \cap S_0 = \{v\}$ and update $W_0 := W_0 \cup \{W\}$ (note that Lemma 5 says that such $W$ exists).

In the second step, we reduce the problem to a problem of finding a vertex set covering specified deficient sets obtained from $W_0$. First, we decompose $V - \{x\}$ into a subpartition $\{X_1, \ldots, X_p\}$ of $V - \{x\}$ based on the information
on \( S_0 \) and \( \mathcal{W}_0 \). For each \( X_i \), we pick up \( O(|X_i|) \) pairs \( \{Y_i^1, Y_i^2\} \) of subpartitions of \( X_i \) which consist of specified deficient sets obtained from \( \mathcal{W}_0 \), compute a vertex set \( S'_i \) with the minimum cost covering \( Y_i^1 \) and \( Y_i^2 \), and obtain the vertex set \( S_i \) with \( c(S_i) = \min_j |c(S'_j)| \). Note that the problem of finding a vertex set with the minimum cost covering two subpartitions is a weighted matroid intersection problem [3], and it can be solved in \( O(|X_i|^2 \log^2 |X_i|) \) time [16]. Finally, we output \( S_1 \cup \cdots \cup S_p \cup \{x\} \) as an optimal solution. The key point of this method is that we can obtain an optimal source set in the original problem, by combining the vertex \( x \) and vertex sets \( S_i \) obtained from \( V_i \) locally.

A more precise description of Step I in the algorithm is given as follows. Step II is very complicated, and hence the details will be mentioned after describing Step I and analyzing properties of \( S_0 \) and \( \mathcal{W}_0 \).

**Algorithm 3-LVC_CSLP(x)**

**Input:** An undirected connected graph \( G = (V, E) \), a demand function \( d : V \to \{0, 1, 2, 3\} \), a cost function \( c : V \to R_+ \), and a vertex \( x \in V \).

**Output:** A source set \( S \) with \( x \in S \) minimizing \( c(S) \).

**Step I (I-0)** Number vertices of \( V \) such as \( d(v_1) \leq \cdots \leq d(v_n) \).

(I-1) Initialize \( j := 1 \), \( S_0 := V \), and \( \mathcal{W}_0 := \emptyset \).

(I-2) If \( v_j = x \) holds, then go to Step (I-4).

(I-3) If \( S_0 - \{v_j\} \) satisfies (I) then let \( S_0 := S_0 - \{v_j\} \). Otherwise select a minimal deficient set \( W' \subseteq V -(S_0 - \{v_j\}) \) w.r.t. \( v_j \), and let \( \mathcal{W}_0 := \mathcal{W}_0 \cup \{W'\} \).

(I-4) If \( j < n \), then \( j := j + 1 \) and go to Step (I-2). Otherwise go to Step II.

**Step II** Find a vertex set \( Y^* \) with the minimum cost which covers a family \( Y^* \) of specified deficient sets obtained from \( \mathcal{W}_0 \), and output \( Y^* \cup \{x\} \) as an optimal solution. The details are given in Section 3.2.

Note that in the case where \( S_0 - \{v_j\} \) does not satisfy (I) in Step I-3, there exists a minimal deficient set \( W' \subseteq V -(S_0 - \{v_j\}) \) w.r.t. \( v_j \). Before deleting \( v_j \) from \( S_0 \), \( S_0 \) is feasible and hence by Lemma 5, every deficient set contains a source in \( S_0 \). On the other hand, \( S_0 - \{v_j\} \) is infeasible. Again by Lemma 5, there is a deficient set \( W' \) with \( W' \cap (S_0 - \{v_j\}) \neq \emptyset \) such that \( W' - \{v_j\} \) is not deficient. Moreover, all vertices in \( W' - \{v_j\} \) have been already deleted, and \( d(v_j) = \max_{v \in W'} d(v) = d(W') \) holds by the sorting in Step I-0. It follows that there is a minimal deficient set \( W' \) w.r.t. \( v_j \) satisfying \( W' \subseteq V -(S_0 - \{v_j\}) \).

Let \( S_0 \) and \( \mathcal{W}_0 \) be a source set and the family of corresponding deficient sets obtained after \( v_n \) is checked in Step I, respectively. Fig. 2 shows \( S_0 \) and \( \mathcal{W}_0 \) obtained by applying Step I to \( G \) in Fig. 1(a). Note that \( x \in S_0 \) holds. Then \( S_0 \) and \( \mathcal{W}_0 \) can be characterized as follows.

**Definition 10.** For a source set \( S \), we say that a deficient set \( W \) has *property* \( (P) \) w.r.t. \( S \), if \( W \) satisfies \( W \cap S = \{s\} \) and \( d(W) = d(s) \), and it is minimal w.r.t. \( s \). Moreover, we say that a family \( \mathcal{W} \) of deficient sets has *property* \( (P) \) w.r.t. \( S \) if each \( W \in \mathcal{W} \) has property \( (P) \) w.r.t. \( S \) and every two sets \( W_1 \) and \( W_2 \) in \( \mathcal{W} \) satisfy \( W_1 \cap W_2 \cap S = \emptyset \).

**Lemma 11.** Let \( S_0 \) and \( \mathcal{W}_0 \) be a source set and a family of minimal deficient sets obtained by Step I of algorithm 3-LVC_CSLP(x), respectively. Then the following statements (i)-(iii) hold. (i) \( d(s) \in \{2, 3\} \) holds for each \( s \in S_0 - \{x\} \).
We have \( \{x\} = S_0 - \bigcup_{W \in \mathcal{W}_0} W \). (iii) \( \mathcal{W}_0 \) has property (P) w.r.t. \( S_0 \).

**Proof.** (i) By \( d(V) \geq 2 \), each vertex \( v \) with \( d(v) \leq 1 \) is always deleted from the current \( S_0 \) at Step I-3. (ii) By Steps I-2 and I-3, we can see that no \( W \in \mathcal{W} \) contains \( x \), and each \( s \in S_0 - \{x\} \) is contained in some deficient set in \( \mathcal{W}_0 \). (iii) It suffices to show that each \( W \in \mathcal{W}_0 \) has property (P) w.r.t. \( S_0 \), since Step I-3 implies that each \( s \in S_0 - \{x\} \) is contained in exactly one set in \( \mathcal{W}_0 \). At Step I-3, assume that \( v_j \) cannot be deleted. As observed in the paragraph immediately after Algorithm 3-LVC-CSLP(x), the algorithm finds a minimal deficient set \( W' \) w.r.t. \( v_j \) with \( W' \cap S_0 = \{v_j\} \) and \( d(W') = d(v_j) \); \( W' \in \mathcal{W}_0 \) has property (P) w.r.t. \( S_0 \). \( \square \)

After analyzing properties of a source set \( S \) and a family \( \mathcal{W} \) of deficient sets which has property (P) w.r.t. \( S \) in Section 3.1, we give the procedure of Step II for the details in Section 3.2.

### 3.1. Property (P)

Through this section, let \( S \) be a source set and a family \( \mathcal{W} \) of minimal deficient sets have property (P) w.r.t. \( S \). Let \( S_1 = S - \bigcup_{W \in \mathcal{W}} W (S_1 = \emptyset \) may hold\). Assume that each \( s \in S - S_1 \) satisfies \( d(s) \in \{2, 3\} \). Note that \( S_0 \) and \( \mathcal{W}_0 \) obtained in Step I of algorithm 3-LVC_CSPL(x) correspond to \( S \) and \( \mathcal{W} \) with \( S_1 = \{x\} \), respectively. We here show several lemmas, some of which generalize observations given in [7] slightly.

First, we observe the properties of deficient sets in \( \mathcal{W} \) which intersect each other. The following lemma shows that each vertex is contained in at most two sets in \( \mathcal{W} \) and each set in \( \mathcal{W} \) intersects at most two sets in \( \mathcal{W} \), in the case of \( |S| \geq 4 \) or \( V \neq \bigcup_{W \in \mathcal{W}} W \).

**Lemma 12.** Let \( S \) be a source set and let a family \( \mathcal{W} \) of minimal deficient sets \( W_j \) have the property (P) w.r.t. \( S \) such that \( S \cap W_i = \{s_i\} \).

(i) If \( |W_j \cap N_G(W_i)| = 1 \) holds for \( W_i, W_j \in \mathcal{W} \), then \( W_j \cap N_G(W_i) = \{s_j\} \).

(ii) Assume that \( |S| \geq 4 \) or \( V \neq \bigcup_{W \in \mathcal{W}} W \) hold. Let \( W_i, W_h, W_j \) be three distinct sets in \( \mathcal{W} \) such that \( W_i \cap W_h, W_i \cap W_j, W_h \cap W_j \neq \emptyset \). Then we have \( W_h \cap W_i \cap W_j = \emptyset \) and \( N_G(W_i) = \{s_h, s_j\} \) (hence, the number of \( W \in \mathcal{W} - \{W_i\} \) with \( W_i \cap W \neq \emptyset \) is at most two).

**Proof.** (i) By the property (P), we have \( s_j \notin W_i \). By Lemma 6, \( N_G(W_j) \cap W_i \neq \emptyset \) holds. Lemma 8 and these imply that \( W_j \cap N_G(W_i) = \{s_j\} \).

(ii) For each \( W_i \in \mathcal{W} \), we have \( |N_G(W_i)| \in \{1, 2\} \), since \( W_i \neq V \), \( G \) is connected and \( d(s) \in \{2, 3\} \) for \( s \in S - S_1 \). Denote \( N_G(W_i) \) and \( N_G(W_j) \) by \( \{x_i, y_i\} \) and \( \{x_j, y_j\} \), respectively (possibly \( x_i = y_i \) or \( x_j = y_j \) hold). We first claim that \( N_G(W_i \cup W_j \cup W_h) \neq \emptyset \) holds. If \( N_G(W_i \cup W_j \cup W_h) = \emptyset \) would hold, i.e., \( V = W_i \cup W_j \cup W_h \) would hold, then we would have \( |S| = 3 \) by the property (P), which contradicts the assumption that \( |S| \geq 4 \) or \( V \neq \bigcup_{W \in \mathcal{W}} W \) hold. By Lemma 6, we have \( W_i \cap N_G(W_i) \neq \emptyset \neq W_h \cap N_G(W_i) \).

We consider the following two cases separately.

Case 1. \( |W_j \cap \{x_i, y_i\}| = 1 \) or \( |W_h \cap \{x_i, y_i\}| = 1 \): Assume that \( W_j \cap \{x_i, y_i\} = \{x_i\} \) holds without loss of generality. Then we have \( x_i = s_j \) by (i). Since \( s_j \notin W_h \) holds by the property (P), we have \( W_h \cap \{x_i, y_i\} = \{y_i\} \), from which we have \( y_i = s_h \). If another \( W_k \in \mathcal{W} - \{W_i, W_j, W_h\} \) satisfies \( W_k \cap W_i \neq \emptyset \), then the property (P) implies \( s_k \in W_k - W_i \) and \( N_G(W_k) \cap W_k = \{s_h, s_j\} \cap W_k = \emptyset \), from which \( G[W_k] \) is not connected, contradicting Lemma 6. So there is no set \( W_k \in \mathcal{W} - \{W_i, W_j, W_h\} \) with \( W \cap W_k \neq \emptyset \). From \( N_G(W_i) \subseteq W_h \cup W_j \), let \( x_j \in W_h \cup W_j \) and \( x_j \in W_i \) without loss of generality. Again by (i) and \( N_G(W_j) \cap W_i = \{x_j\} \), we see \( x_j = s_i \). Hence by the property (P), we have \( s_i = x_j \notin W_h \). Therefore, \( \{s_j, s_i\} = N_G(W_i \cap W_j) \subseteq V - W_h \) and Lemma 6 imply that \( W_i \cap W_j \cap W_h = \emptyset \) holds.

Case 2. \( \{x_i, y_i\} \subseteq W_j \) and \( \{x_j, y_j\} \subseteq W_h \): We have \( W_j \cap W_h \neq \emptyset \) and \( N_G(W_j) \cap W_i \neq \emptyset \neq N_G(W_h) \cap W_i \) by Lemma 6. Without loss of generality, we can assume \( y_j \in N_G(W_i \cup W_j \cup W_h) \) by \( N_G(W_j) \subseteq W_h \cup W_i \cup W_j \) by \( N_G(W_j) \cap W_i \neq \emptyset \), we have \( x_j \in N_G(W_j) \cap W_i \). Then we have \( s_i = x_j \) by (i). Hence \( W_h \cap N_G(W_j) = \emptyset \) holds, since the property (P) implies that \( s_i = x_j \notin W_h \). This contradicts Lemma 6. \( \square \)

We decompose \( \mathcal{W} \) into subfamilies \( \mathcal{X}_i \) in the following manner. Let \( S'_i \subseteq S - S_1, i = 1, \ldots, q \) be a connected component of the graph \( H = (S - S_1, \{s_j, s_i\} \{W_j, W_k\} \subseteq \mathcal{W}, W_j \cap W_k \neq \emptyset, S \cap W_j = \{s_j\}, S \cap W_k = \{s_k\}) \). A family of deficient sets in \( \mathcal{W} \) corresponding to the sources in \( S'_i \) is denoted by \( \mathcal{X}_i \). In Fig. 2, each of \( \{W_1, W_2\}, \{W_3, W_4, W_5, W_6\}, \{W_7, W_8, W_9\} \), and \( \{W_{10}\} \) corresponds to \( \mathcal{X}_i \) for some \( i \).

We define a family of deficient sets called a *chain* as follows.
Definition 13. A family $\mathcal{W}' = \{W_1, \ldots, W_t\}$ ($t \geq 1$) of deficient sets is called a chain if it satisfies the following conditions (a) and (b).

(a) $W_i \cap W_{i+1} \neq \emptyset$ holds for $i = 1, \ldots, t - 1$ if $t \geq 2$.
(b) $W_i \cap W_h = \emptyset$ holds for two distinct $i, h \in \{1, 2, \ldots, t\}$ with $2 \leq |i - h| \leq t - 2$ if $t \geq 2$. □

In Fig. 2, each $X_i$ is a chain. Lemma 12 indicates that each $X_i$ is a chain in the case of $|S| \geq 4$ or $V \neq \bigcup_{W \in \mathcal{W}} W$.

Lemma 14. Let $S$ be a source set and let a family $\mathcal{W}$ of minimal deficient sets have the property (P) w.r.t. $S$ in $G$. If $|S| \geq 4$ or $V \neq \bigcup_{W \in \mathcal{W}} W$ hold, then each $X_i$ is a chain. □

Let $X_i = \{W_1, W_2, \ldots, W_{|X_i|}\}$ and $W_i \cap S = \{s_j\}$, where $W_i \cap W_j \neq \emptyset$ holds for each $j \in \{2, 3, \ldots, |X_i|\}$.

Lemma 12 implies that $N_G(W_i) = \{s_{j-1}, s_{j+1}\}$, $j \in \{2, 3, \ldots, |X_i| - 1\}$, holds. Chains can be divided into three types as follows.

Definition 15. Let $S$ be a source set and let a family $\mathcal{W}$ of minimal deficient sets have the property (P) w.r.t. $S$ in $G$. Let $X = \{W_1, \ldots, W_t\} \subseteq \mathcal{W}$ be a chain satisfying (a) and (b) in Definition 13 and $W_i \cap S = \{s_i\}$ for each $i$. Then $X$ is called type (A) if it satisfies the conditions (i) and (ii), type (B) if it satisfies neither (i) nor (ii), and type (C) otherwise. (In the case of type (C), assume that $X$ satisfies (i) and does not satisfy (ii) without loss of generality.)

(i) $t \geq 2$. There exists $Z' \subseteq V$ with $W_1 \cup W_2 \subseteq Z'$, $|N_G(Z'_1)| = 1$, $N_G(Z'_1) \cap N_G(W_2) \neq \emptyset$ and $Z'_1 \cap S = \{s_1, s_2\}$.
(ii) $t \geq 3$. There exists $Z'_1 \subseteq V$ with $W_{t-1} \cup W_t \subseteq Z'_1$, $|N_G(Z'_t)| = 1$, $N_G(Z'_t) \cap N_G(W_{t-1}) \neq \emptyset$ and $Z'_t \cap S = \{s_{t-2}, s_t\}$ (note that if $t \geq 3$ holds then we have $N_G(Z'_1) = \{s_3\}$ and $N_G(Z'_t) = \{s_{t-2}\}$ by $N_G(W_2) = \{s_1, s_3\}$, $N_G(W_{t-1}) = \{s_{t-2}, s_t\}$, and Lemma 12).

In Fig. 3, $\{W_4, W_5, W_6\}$ is a chain of type (C) with $Z'_1 = Z_2$ and each of $\{W_1, W_2\}$, $\{W_3\}$, $\{W_7, W_8, W_9\}$, and $\{W_{10}\}$ is of type (B). Note that if $X_i$ is a chain of type (A) or a chain of type (B) with $|X_i| \geq 3$ and $W_i \cap W_i' \neq \emptyset$, then Lemma 12(ii) implies that we have $\bigcup_{W \in X_i} W \supseteq S$, i.e., $\mathcal{W} = X_i$. We here assume that each chain is of type (B) with $|X_i| \leq 2$ or $W_i \cap W_i' = \emptyset$, or type (C) (the above two special cases can be treated similarly; however, we omit the details). Note that the case of $S_1 \neq \emptyset$ always satisfies this assumption. For each chain $X_i$ of type (C), let $Z_i$ be a vertex set corresponding to $Z'_1$ in Definition 15(i); $W_i \cup W_i' \subseteq Z_i$, $|N_G(Z_i)| = 1$, $N_G(Z_i) \cap N_G(W_i') \neq \emptyset$ and $Z_i \cap S = \{s_i, s_i'\}$. For each chain $X_i$ of type (B) satisfying $|X_i| \leq 2$, $|W_i| \geq 2$, and $d(\bigcup_{W \in X_i} W) = 3$, let $Z_i$ be a minimal vertex set that satisfies $\bigcup_{W \in X_i} W \subseteq Z_i$, $|N_G(Z_i)| = 1$, and $S \cap Z_i = S \cap (\bigcup_{W \in X_i} W)$ such that no vertex set $Z'' \subseteq Z_i$ satisfies this property if such a vertex set exists, with $Z_i = \emptyset$ otherwise. Note that if $S_1 \neq \emptyset$ or $|S| \geq 3$, then such $Z_i$ is uniquely determined, since if there were another $Z'' \neq Z_i$ of the same property, it would follow that $V = Z_i \cup Z''$, $S_1 = \emptyset$, and $|S| \leq 2$. For any other chain $X_i$ of type (B), let $Z_i = \emptyset$. Fig. 3 shows $Z_1$ and $Z_2$ for the chains $X_1 = \{W_1, W_2\}$ and $X_2 = \{W_4, W_5, W_6\}$.

In the sequel, we show that we can construct a source set $\bigcup_i S_i$ by combining a set $S_i$ of vertices covering some family of deficient sets constructed from $X_i$. However, $\bigcup_i S'_i$ obtained from choosing $S'_i$ directly as a vertex set covering $X_i$ may not be a source set. For example, in Fig. 3, a vertex $v \in W_1 \cap W_2$ can cover the chain $\{W_1, W_2\}$, but $Z_1 - v$ still remains a deficient set. To overcome this, we define deficient sets not in $\mathcal{W}$ to be covered for each
chain as follows. For a chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$ and $d(s_i^1) = 2$ (resp., $d(s_i^1) = 3$) and a vertex $u \in W_i^i - \{s_i^1\}$, let $Z_i(u) \subseteq Z_i - \{u\}$ denote a minimal deficient set w.r.t. $s_i^1$ (resp., $s_i^1$) and $Z_i(s_i^1) = \emptyset$ (note that each chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$ satisfies $d(\bigcup_{W \in \mathcal{X}_i} W) = 3$). Then we can observe that $Z_i(u)$ is uniquely determined.

**Lemma 16.** Let $S$ be a source set and let a family $\mathcal{W}$ of minimal deficient sets have the property $(P)$ in $G$. Let $\mathcal{X}_i$ be a chain with $Z_i \neq \emptyset$ and $u \in W_i^i - \{s_i^1\}$ be a vertex. Then, $Z_i(u)$ is unique with respect to a given $Z_i$.

**Proof.** See Appendix. □

For each chain $\mathcal{X}_i$ with $Z_i = \emptyset$, let $\mathcal{Y}_i^+ = \{W_{i,j-1}^j | j = 1, 2, \ldots, |\mathcal{X}_i|/2\}$ and $\mathcal{Y}_i^- = \{W_{i,j}^j | j = 1, 2, \ldots, |\mathcal{X}_i|/2\}$. For each chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$ and a vertex $u \in W_i^i$, we define two families $\mathcal{Y}_i^+(u)$ and $\mathcal{Y}_i^-(u)$ of deficient sets as the following (a) and (b).

(a) $\mathcal{Y}_i^+(u) = \{W_{i,j-1}^j | j = 2, \ldots, |\mathcal{X}_i|/2\}$ and $\mathcal{Y}_i^-(u) = \{W_{i,j}^j | j = 2, \ldots, |\mathcal{X}_i|/2\}$ if $d(s_i^1) = 2$.

(b) $\mathcal{Y}_i^+(u) = \{W_{i,j-1}^j | j = 2, \ldots, |\mathcal{X}_i|/2\}$ and $\mathcal{Y}_i^-(u) = \{W_{i,j}^j | j = 2, \ldots, |\mathcal{X}_i|/2\}$ if $d(s_i^1) = 3$.

The following lemma shows that given a vertex $u_i \in W_i^i$ for each chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$, we can find a source set by finding a vertex set $V$ which covers $\bigcup_{i:Z_i \neq \emptyset}(\mathcal{Y}_i^+(u_i) \cup \mathcal{Y}_i^-(u_i)) \cup \bigcup_{i:Z_i = \emptyset}(\mathcal{Y}_i^+ \cup \mathcal{Y}_i^-)$). Note that under the assumption in the following lemma, for each chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$, $Z_i$ is uniquely determined.

**Lemma 17.** Let $S$ be a source set and let a family $\mathcal{W}$ of minimal deficient sets have property $(P)$ w.r.t. $S$ in $G$. Assume that $|S| \geq 5$ or $S_1 \neq \emptyset$ hold. Let $u_i$ be a vertex chosen arbitrarily from $W_i^i$ in each chain $\mathcal{X}_i \subseteq \mathcal{W}$ with $Z_i \neq \emptyset$. Let $Y$ be a vertex set which covers each deficient set in $\bigcup_{i:Z_i \neq \emptyset}(\mathcal{Y}_i^+(u_i) \cup \mathcal{Y}_i^-(u_i)) \cup \bigcup_{i:Z_i = \emptyset}(\mathcal{Y}_i^+ \cup \mathcal{Y}_i^-)$). Then $S' = S_1 \cup Y \cup \{u_i | Z_i \neq \emptyset\}$ is a source set.

(In the above statement, note that $S_1 \neq \emptyset$ implies that $V - \bigcup_{i:Z_i \neq \emptyset} X_i \geq S_1 \neq \emptyset$ hold, where $X_i = (\bigcup_{W \in \mathcal{X}_i} W) \cup Z_i$ for a chain $\mathcal{X}_i$).

**Proof.** See Appendix. □

Assume that $|S| \geq 5$ or $S_1 \neq \emptyset$ hold; the assumption of Lemma 17 holds. Note that if $|S| \leq 4$, then $V \neq \bigcup_{i:Z_i \neq \emptyset} X_i$, where $X_i = (\bigcup_{W \in \mathcal{X}_i} W) \cup Z_i$. Based on Lemma 17, we next show that we can find a source set with the minimum cost among all source sets containing $S_1$, by finding a vertex set with the minimum cost covering a family of sets to be covered for each chain (the proof is given as the proof of Lemma 20). For this, we here assume that for any chain $\mathcal{X}_i$ with $|\mathcal{X}_i| = 1$, $d(W_i^i) = 3$, and $|\mathcal{X}_i| \geq 2$, if there exists $W \in \mathcal{W} - \{W_i^i\}$ with $N_G(W_i^i) \subseteq W$, then any set $X$ with $|N_G(X)| \leq 1$ and $W_i^i \subseteq X$ satisfies $W \subseteq X \subseteq S$. The following Lemma 18 shows that this assumption is possible; if $\mathcal{W}$ violates the assumption, then we can obtain another family $\mathcal{W}'$ satisfying the assumption by replacing some sets in $\mathcal{W}$. Under this assumption, Lemma 19 shows that the family of vertex sets $\bigcup_{i:Z_i \neq \emptyset} X_i = Z_i$ is a subpartition of $V$.

**Lemma 18.** Let $S$ be a source set and a family $\mathcal{W}$ of minimal deficient sets have property $(P)$ w.r.t. $S$ in $G$. Let $W_i \in \mathcal{W}$ satisfy $d(W_i) = 3$ and $|W_i| \geq 2$. Assume that $|S| \geq 3$ or $V \neq \bigcup_{i:Z_i \neq \emptyset} X_i$ hold, that there exists $W \in \mathcal{W} - \{W_i\}$ with $W_i \cap W = \emptyset$ and $N_G(W_i) \subseteq W$, and that there exists a set $X$ with $|N_G(X)| \leq 1$, $W_i \subseteq X$, and $W \cap S \cap X = \emptyset$. Then there exists a set $W' \subseteq W \cup W_i$ with $W' \cap W \neq \emptyset$ and $S \cap W_i = S \cap W'$ such that $W'$ has property $(P)$ w.r.t. $S$; $\mathcal{W}' = (\mathcal{W} - \{W_i\}) \cup \{W'\}$ also has property $(P)$ w.r.t. $S$. Moreover, such a $W'$ can be computed in $O(|W_i| \cup W)$ time.

**Proof.** See Appendix. □

**Lemma 19.** Let $S$ be a source set and let a family $\mathcal{W}$ of minimal deficient sets have the property $(P)$ in $G$. For a chain $\mathcal{X}_i$, let $X_i = (\bigcup_{W \in \mathcal{X}_i} W) \cup Z_i$. Assume that $|S| \geq 5$ or $V \neq \bigcup_{i:Z_i \neq \emptyset} X_i$ hold. Then the family of vertex sets $X_i$ is a subpartition of $V$.

**Proof.** From the construction of $\mathcal{W}_i$, a family of vertex sets $X_i' = \bigcup_{W \in \mathcal{X}_i} W$ is a subpartition of $V$. Note that every two distinct sets $Z_i, Z_j$ are pairwise disjoint, since if $Z_i \cap Z_j \neq \emptyset$ holds, then $|N_G(Z_i)| = |N_G(Z_j)| = 1$ implies that $V = Z_i \cup Z_j$ holds, contradicting $|S| \geq 5$, or $V \neq \bigcup_{i:Z_i \neq \emptyset} X_i$ hold (note that each of $Z_i$ and $Z_j$ contains at most two sources in $S$). Hence, it suffices to show that each $Z_i$ with $N_G(Z_i) = N_G(X_i)$ and $N_G(Z_i) - N_G(X_i') \neq \emptyset$ is disjoint with any $W \in \mathcal{W} - \mathcal{X}_i$. Let $Z_i$ satisfy $N_G(Z_i) = N_G(X_i)$ and $N_G(Z_i) - N_G(X_i') \neq \emptyset$, and $\{z_i\} = N_G(Z_i)$
Lemma 20. Let $S$ and $W$ satisfy the assumption of Lemma 17. For a chain $X_i$ with $Z_i \neq \emptyset$ and a vertex $u \in W_i$, let $S_i(u)$ be a vertex set with the minimum cost which covers $\gamma_i^+(u) \cup \gamma_i^-(u)$, and $S_i^+$ be a vertex set $S_i(u^*) \cup \{u^*\}$ with $c(S_i(u^*) \cup \{u^*\}) = \min_{u \in W_i} c(S_i(u) \cup \{u\})$. For a chain $X_i$ with $Z_i = \emptyset$, let $S_i^+$ be a vertex set with the minimum cost which covers $\gamma_i^+ \cup \gamma_i^-$. Then $S_i \cup \left( \bigcup_j S_i^+ \right)$ is a source set with the minimum cost among source sets containing $S_i$.

Proof. Let $S_{opt}$ be a source set with the minimum cost among source sets containing $S_1$. By Lemma 17, $S_1 \cup \left( \bigcup_j S_i^+ \right)$ is feasible. By Lemma 19, it suffices to show that for each chain $X_i$, we have $c(S_i^+) \leq c(S_{opt} \cap X_i)$, where $X_i = \left( \bigcup_{W \in X_i} W \cup Z_i \right)$.

Let $X_i$ be a chain with $Z_i = \emptyset$. From the feasibility of $S_{opt}$, we have $S_{opt} \cap W \neq \emptyset$ for each $W \in X_i$. From the minimality of $c(S_i^+)$, we have $c(S_i^+) \leq c(S_{opt} \cap X_i)$.

Let $X_i$ be a chain with $Z_i \neq \emptyset$. From the feasibility of $S_{opt}$, we have $S_{opt} \cap W \neq \emptyset$; let $u' \in S_{opt} \cap W_i$. Then by Lemma 16, we have a unique deficient set $Z_i(u')$ and a family $\gamma_i^+(u') \cup \gamma_i^-(u')$ of deficient sets to be covered by any feasible solution. Hence from the minimality of $c(S_i(u'))$, we see that $c(S_i(u')) \leq c((S_{opt} - \{u'\}) \cap X_i)$. □

Before closing this section, we give the following lemma, which will be used to analyze the complexity of the algorithm 3-LVC-CSLP(x).

Lemma 21. Let $S$ be a source set and let a family $W$ of minimal deficient sets have the property $(P)$ in $G$. Assume that $|S| \geq 4$ or $V \neq \bigcup_{W \in W} W$ hold. Then for a chain $X_i$ with $Z_i \neq \emptyset$, each of $\gamma_i^+(u)$ and $\gamma_i^-(u)$ is a subpartition of $V$ for any $u \in W_i$.

Proof. See Appendix. □

3.2. Step II

The procedure for Step II is given as follows. Let $X_i$, $Z_i$, $W_i$, $\gamma_i^+$, $\gamma_i^-$, $\gamma_i^+(u)$, and $\gamma_i^-(u)$ be defined as Section 3.1, regarding the source set $S_0$, the family $W_0$, and the set $\{x\} \subseteq S_0$ as $S$, $W$, and $S_1$, respectively.

Step II (II-0) Execute the following procedure (II-1) and (II-2) for each chain $X_i \subseteq W_0$.

(II-1) If $Z_i = \emptyset$ holds, then compute a vertex set $S_i^+$ with the minimum cost which covers $\gamma_i^+ \cup \gamma_i^-$.

(II-2) Otherwise execute the following procedure (II-2) for each $u \in W_i$.

(II-2-0) Compute a set $S_i(u)$ with the minimum cost covering $\gamma_i^+(u) \cup \gamma_i^-(u)$.

(II-2-1) Let $S_i^+$ be a vertex set $S_i(u) \cup \{u\}$ such that $c(S_i(u) \cup \{u\}) = \min_{u' \in W_i} c(S_i(u') \cup \{u'\})$.

(II-3) Output $\{x\} \cup \bigcup_j S_i^+$ as an optimal solution and halt. □

Through the procedure, $S_1 = \{x\} \neq \emptyset$ holds. Lemmas 11 and 20 together with this prove the correctness of algorithm 3-LVC-CSLP(x).

3.3. Complexity

We now analyze the complexity of algorithm 3-LVC-CSLP(x). As shown in [7], Step I can be computed in linear time. We consider the time complexity of Step II. By Lemma 21, we can see that for each chain $X_i$, we compute a
vertex set with the minimum cost, which covers two subpartitions of $X_i$ at most $|W_i| \times |V_i|$ times. This problem of covering two subpartitions can be formulated as follows.

**Problem 22. Input:** A finite set $V$, a cost function $c : V \to R_+$, and two subpartitions $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of $V$.

**Output:** A subset $S$ of $V$ minimizing $c(S)$ such that each $Y \in \mathcal{Y}_1 \cup \mathcal{Y}_2$ satisfies $S \cap Y \neq \emptyset$.

In [16, Theorem 8], it was shown that Problem 22 can be solved in $O((|V|^2 \log^2 |V|)$ time by the minimum cost flow algorithm. Hence, Step II can be implemented to run in $O\left(\sum_i |W_i| \times |X_i|^2 \log^2 |X_i|\right) = O(n^3 \log^2 n)$ time (note that each chain $X_i$ obtained in Step I is of type (B) or type (C) by $S_1 = \{x\} \neq \emptyset$). Therefore, the total time complexity of algorithm 3-LVC,CSLP(x) is $O(n^3 \log^2 n)$.

An optimal solution to Problem 3 can be obtained by executing algorithm 3-LVC,CSLP(x) for each vertex $x \in V$. Consequently, it can be found in $O(n^4 \log^2 n)$ time. Summarizing the argument given so far, Theorem 4 is now established.

**Remark 1.** We can consider a variant of algorithm 3-LVC,CSLP(x) in which a vertex $x$ is not given. As observed in Section 3.1, if a source set $S_0$ obtained in Step I satisfies $|S_0| \geq 5$, then such an algorithm works. However, in the case of $|S_0| \leq 4$, it is difficult to characterize $S_0$ and $\mathcal{V}_0$ as Section 3.1.

4. Concluding remarks

In this paper, given an undirected graph $G = (V, E)$, a demand function $d : V \to \{0, 1, 2, 3\}$, and a cost function $c : V \to R_+$, we have considered the problem of finding a source set $S \subseteq V$ minimizing $c(S)$ for which there exist $d(v)$ paths between every vertex $v \in V \setminus S$ and $S$ such that no pair of paths has a common vertex in $V \setminus v$. We have shown that the problem can be solved in $O(n^4 \log^2 n)$ time.

For general demands $d \geq 4$, the problem is NP-hard, even if every vertex has a uniform cost, as shown in [7]. The design of approximation algorithms for these problems remains a work for the future.

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Appendix

**Proof of Lemma 16.** We first consider the case of $d(s_i^1) = 3$. Assume by contradiction that there exist two distinct minimal deficient sets $W'$ and $W''$ in $Z_i \setminus \{u\}$ w.r.t. $s_i^1$. From $s_i^1 \in W' \cap W''$ and the minimality of $W'$ and $W''$, $W'$ and $W''$ intersect each other. By $u \notin W' \cup W''$ and the connectedness of $G$, $N_G(W' \cup W'') \neq \emptyset$ holds. Lemma 9 implies that we have $|N_G(W' \cup W'')| = 1$. The connectedness of $G[W_i^1]$, $u \notin W' \cup W''$, and $s_i^1 \in W' \cap W''$ imply that we have $N_G(W' \cup W'') \subseteq W_i^1$. By Lemma 8, no neighbor of $W_i^1$ can be contained in $V \setminus W' \setminus W''$, i.e., $V \setminus W' \setminus W'' \subseteq W_i^1$ holds. This means $V = W' \cup W'' \cup W_i^1$, contradicting $N_G(Z_i) \neq \emptyset$ and $W_i^1 \cup W' \cup W'' \subseteq Z_i$.

We next consider the case of $d(s_i^1) = 2$. Note that $|N_G(W_i^1)| = 1$, $N_G(W_i^1) \subseteq W_i^2$, and $Z_i = W_i^1 \cup W_i^2$ hold. By $N_G(Z_i) \neq \emptyset$, $N_G(W_i^1) \setminus Z_i \neq \emptyset$ holds. Assume by contradiction that there exist two distinct minimal deficient sets $W'$ and $W''$ in $Z_i \setminus \{u\}$ w.r.t. $s_i^2$. From $s_i^2 \in W' \cap W''$ and the minimality of $W'$ and $W''$, $W'$ and $W''$ intersect each other. By $u \notin W' \cup W''$ and the connectedness of $G$, $N_G(W' \cup W'') \neq \emptyset$ holds. Lemma 9 implies that we have $|N_G(W' \cup W'')| = 1$. From the minimality of $W_2$, $W' \setminus W_2 \neq \emptyset$ or $W'' \setminus W_2 \neq \emptyset$ holds, and hence $(W' \cup W'') \cap W_i^1 \neq \emptyset$ holds. The connectedness of $G[W_i^1], u \notin W' \cup W''$, and $|N_G(W' \cup W'')| = 1$ imply that we have $N_G(W' \cup W'') \subseteq W_i^1$. Assume that $N_G(W' \cup W'') \subseteq W_i^1 \setminus W_i^2$ holds. Then the connectedness of $G[W_i^1]$ indicates that $W_i^2 \subseteq W' \cup W''$ holds. Hence $N_G(W_i^2) \subseteq W' \cup W'' \cup N_G(W' \cup W'') \subseteq Z_i$, which contradicts that $N_G(W_i^2) \setminus Z_i \neq \emptyset$ holds. Assume that $N_G(W' \cup W'') \subseteq W_i^1 \setminus W_i^2$ holds. By $|N_G(W' \cup W'')| = 1$ and Lemma 8, no neighbor of $W_i^2$ can be contained in $V \setminus W' \setminus W''$, which contradicts that $W' \cup W'' \subseteq Z_i$ and $N_G(W_i^2) \setminus Z_i \neq \emptyset$ holds. □
Proof of Lemma 17. Assume by contradiction that $S^*$ is not a source set. Let $W^*$ be a minimal deficient set w.r.t. a vertex $w^* \in W^*$ with $W^* \cap S^* = \emptyset$ (such $W^*$ exists by Lemma 5). Note that $S \cap W^* \neq \emptyset$ holds since $S$ is a source set. Also note that each $W \in \mathcal{W}$ satisfies $S^* \cap W \neq \emptyset$ from the construction of $S^*$ (note that for each chain $\mathcal{X}_i$ with $Z_i \neq \emptyset$, if $d(s_i) = 2$ and $u_i \in W^*_1 - W_i^2$ hold, then $Z_i \cup \{u_i\} \subseteq W_i^3$ holds by Lemma 16).

Let $W_1 \in \mathcal{W}$ be a deficient set with $W_1 \cap S = \{s_1\}$ and $s_1 \in W^*$ and $\mathcal{X}_1$ be a chain containing $W_1$ without loss of generality. Now we have $W^* - W_1 \neq \emptyset$ (resp., $W_1 - W^* \neq \emptyset$) since $W^* \subseteq W_1$ and $s_1 \in W^*$ would contradict the minimality of $W_1$ (resp., we have $S^* \cap (W_1 - W^*) \neq \emptyset$). Let $N_G(W_1) = \{x_1, y_1\}$ and $N_G(W^*) = \{x^*, y^*\}$, where $x_1 \in W^*$ and $x^* \in W_1$ (note that $W^* \cap N_G(W_1) \neq \emptyset \neq W_1 \cap N_G(W^*)$ holds by Lemma 6). Let $Z^* = W_1 \cup W^*$.

Claim 23. (i) $|N_G(Z^*)| = 1$.
(ii) $V - \bigcup_{W \in \mathcal{W}} W - Z^* \neq \emptyset$ holds, or some $W \in \mathcal{W}$ is disjoint with $Z^*$. Hence, if a minimal deficient set $W \in \mathcal{W}$ intersects with $Z^*$, then $N_G(W) - Z^* \neq \emptyset$ holds.

Proof. For proving $|N_G(Z^*)| = 1$, it suffices to show that $|N_G(Z^*)| > 0$, since if $|N_G(Z^*)| > 0$, then one of the statements (i)–(iii) in Lemma 9 holds for $W \cap W^* \subseteq s_1$ for $s_1 \in W^*$. Now, from the definition of $S_1$, we have $S_1 \subseteq V - \bigcup_{W \in \mathcal{W}} W - Z^*$. Hence, in the case of $S_1 \neq \emptyset$, $|N_G(Z^*)| > 0$ and (ii) hold (note that $G$ is connected).

We consider the case of $|S| \geq 5$. Every $W \in \mathcal{W}$ satisfies $W - W^* \neq \emptyset$ by $(W - W^*) \cap S^* \neq \emptyset$. Since $|N_G(W^*)| \leq 2$ holds and no three sets $W_i, W_j, W_k$ in $\mathcal{W}$ satisfy $W_i \cap W_j \cap W_k \neq \emptyset$ by Lemma 12(ii), at most four sets in $\mathcal{W}$ have an intersection with $W^*$. The connectedness of $G$, $|S| \geq 5$ and this implies that $|N_G(Z^*)| > 0$ and some $W' \in \mathcal{W}$ is disjoint with $Z^*$. □

Let $\{z^*\} = N_G(Z^*)$. There are the following two possible cases (I) and (II). (I) Every $W \in \mathcal{W} - \{W_1\}$ satisfies $S \cap W \cap W^* = \emptyset$. (II) Some set $W \in \mathcal{W} - \{W_1\}$ satisfies $S \cap W \cap W^* \neq \emptyset$.

(I) From the assumption (I), we have $Z^* \cap S = \{s_1\}$. If $d(w^*) = 2$ held, then we would have $|N_G(W^*)| \leq 1$ and $N_G(W^*) \subseteq W_1$, and Lemma 9 implies that $|N_G(W_1)| = 2$ and $d(s_1) = 3$ would hold, from which $|N_G(W^*)| \leq d(W^*) - 2$ holds, which contradicts Lemma 7. Hence $d(w^*) = 3$ holds. Then $w^* \neq s_1$ would imply that $Z^* - \{s_1\}$ is a deficient set by $|N_G(Z^*)| = 1$, contradicting $|S \cap Z^*| = 1$ and the feasibility of $S$. Hence we have $w^* = s_1$. Therefore, $d(s_1) = 3$ and $|N_G(W_1)| = |N_G(W_1)| = 2$ hold.

Assume that $X_1 = \{W_1\}$ holds. By Lemma 8 and $s_1 = w^*$, we have $(s_1, z^*) \in E$ or there are at least two internally vertex-disjoint paths between $s_1$ and $z^*$ in both cases of $z^* = y_1$ and $z^* = y^*$. Hence, we can see $Z_1 = Z^*$ from the construction of $Z_1$ and $Z^* \cap S = \{s_1\}$. Moreover, $Z_1 \neq \emptyset$ and $S^* \cap W^* = \emptyset$ imply that $u_1 \in W_1 - W^*$ holds. Since $W^*$ is a deficient set with $s_1 \in W^*$ and $u_1 \notin W^*$, we have $Z_1 \cup \{u_1\} \subseteq W^*$ from the minimality of $Z_1(\{u_1\})$ and Lemma 16, which contradicts $S^* \cap W^* = \emptyset$.

Assume that $|X_1| \geq 2$ holds. Let $W_2 \in \mathcal{W}$ satisfy $W_1 \cap W_2 \neq \emptyset$ and $\{s_2\} = S \cap W_2$. We have $s_2 \notin Z^*$ by $S \cap Z^* = \{s_1\}$, and $N_G(W_2) - Z^* \neq \emptyset$ by Claim 23. Hence, by $|N_G(W_2) \cap W_1| = 1$ and Lemma 12(i), we have $N_G(W_2) \cap Z^* = N_G(W_2) \cap W_1 = \{s_1\}$. Moreover, Lemma 8 implies that $z^* = s_2$ holds, and hence $|N_G(Z^* \cup W_1)| = 2$ holds.

By $s^* = s_2$ and $S \cap Z^* = \{s_1\}$, we see that each $W \in \mathcal{W} - \{W_1, W_2\}$ is disjoint with $Z^*$. This implies that $W_1^1 = W_1, W_2^1 = W_2$, and $Z_1 = Z^* \cup W_2$ hold. Hence we see $u_1 \in W_1 - W^*$. Since $W^*$ is a deficient set with $s_1 \in W^*$ and $u_1 \notin W^*$, we see $Z_1 \cup \{u_1\} \subseteq W^*$ from Lemma 16, which contradicts $S^* \cap W^* = \emptyset$.

(II) Let $W_2 \in \mathcal{W} - \{W_1\}$ satisfy $S \cap W_2 \cap W^* = \{s_2\}$. We claim that

\[(W_1 - W_2) \cup (W_2 - W_1) \subseteq W^*, \text{ i.e., } W_2 \subseteq Z^*\]  

(A.1) holds. If $W_2 - Z^* \neq \emptyset$ hold without loss of generality, then Claim 23 would imply that $N_G(W_2) - Z^* \neq \emptyset$ would hold, which contradicts $s_2 \in W^*, |N_G(Z^*)| = 1$, and Lemma 8. So we have $W_2 \subseteq Z^*$ and similarly $W_1 \subseteq W_2 \cup W^*$. By (A.1) and $W_1 - W_2 \neq \emptyset \neq W_2 - W^*$, we have $W_1 \cap W_2 \neq \emptyset$. By $W_1 \cap S^* \neq \emptyset \neq W_2 \cap S^* \neq \emptyset \neq W_2 \cap S^* \neq \emptyset$, we have $W_1 \cap S^* = S^* \cap W_1 \cap W_2$. Moreover, we can see that any $W \in \mathcal{W} - \{W_1, W_2\}$ satisfies $W \cap S \cap W^* = \emptyset$, since if there exists $W_3 \in \mathcal{W} - \{W_1, W_2\}$ with $W_3 \cap S \cap W^* \neq \emptyset$, then a vertex in $W_3 - W^*$ is also contained in $W_1 \cap W_2$, contradicting Lemma 12(ii).

Assume that $X_1$ is of type (C) and $W_1 = W_1^1$ and $W_2 = W_2^1$ hold without loss of generality. $u_1 \in W_1 - W^*$ holds. We have $W_1 \cup W_2 \subseteq Z_1 \subseteq Z^*$ since $|N_G(Z_1)| = 1$ and $N_G(Z_1) \subseteq N_G(W_2)$ hold from the definition of $Z_1$. Let $N_G(Z_1) \cap N_G(W_2) = \{y_2\}$. Assume $y^* \neq Z_1$ holds. Lemma 9 implies that we have $x^* \in W_1 \cap W_2$. Note that $N_G(Z_1 \cap W^*) = \{x^*, y_2\}, |N_G(W_2)| = 2$, and $d(s_2) = 3$ hold by (A.1). Then in both cases of $d(s_1) = 2$ and $d(s_1) = 3$, $Z_1 \cap W^*$ is a deficient set by $\{s_1, s_2\} \subseteq Z_1 \cap W^*$ and so we have $Z_1(\{u_1\}) \subseteq Z_1 \cap W^*$ from Lemma 16, contradicting
S^* \cap W^* = \emptyset. Assume that y^* \in Z_1 holds. By (A.1), we have N_G(W^*) \subseteq W_1 \cap W_2, N_G(W_1 - W_2) \subseteq W_2, and N_G(W_2 - W_1) \subseteq W_1. This means that |N_G(W_1) - W_2| \subseteq N_G(W_1 - W_2) holds. Hence |N_G(W_1)| \subseteq |N_G(W_1) - W_2| = 2 holds. From this and Claim 23, we have W^* \subseteq W_1 \cup W_2. From the construction of Z_1, we have Z_1 = W_1 \cup W_2. Now u \in W_1 - W^* holds, and W^* is a deficient set with W^* \subseteq Z_1 - \{u_1\}. Hence Z_1(u_1) \subseteq W^* holds, contradicting S^* \cap W^* = \emptyset.

Assume that \lambda_1 is not a chain of type (C) or \lambda_1 is a chain of type (C) with \{W_1, W_2\} \neq \{W_1, W_2\}. Then, from the definition of chains and Lemma 12(i), we have N_G(W_1) - W_2 \neq \emptyset \neq N_G(W_2) - W_1, N_G(W_1) \cap W_1 = \{s_1\}, N_G(W_1) \cap W_2 = \{s_2\}, and d(s_1) = d(s_2) = 3. We claim that \lambda_1 \subseteq \{W_1, W_2\} holds. If there exists a deficient set W_3 \in \mathcal{V} - \{W_1, W_2\} with W_2 \cap W_3 \neq \emptyset without loss of generality, then S \cap W_2 \cap Z^* = \emptyset and Claim 23 imply that N_G(W_3) - W^* \neq \emptyset would hold and hence \lambda_1 would be of type (C) with \{W_1, W_2\} = \{W_1, W_2\}, a contradiction. Let \{y_1, y_2\} = N_G(W_1) - W_j hold for \{i, j\} = \{1, 2\}. Since \lambda_1 is not of type (C), we have y_1 \neq y_2 and y_1 \neq z^* \neq y_2. Hence z^* \in N_G(W^*) holds and Lemma 9 implies that x^* \in W_1 \cap W_2 holds. Let Z' be a vertex set in Z^* with W_1 \cup W_2 \subseteq Z' and |N_G(Z')| = 1 such that no Z'' \subset Z' satisfies this property (such Z' exists by |N_G(Z^*)| = 1 and W_1 \cup W_2 \subseteq Z^*). Then we see that Z' = Z holds since Z' \cap S = (W_1 \cup W_2) \cap S holds. Now note that Z^* - W^* \neq \emptyset \neq W_1 \cap W_2 and N_G(W^*) \cap (W_1 \cup W_2) = \{x^*\} hold by (A.1) and x^* \in W_1 \cap W_2. Hence, N_G(Z_1 \cap W^*) = \{z_1, x^*\} holds, where \{z_1\} = N_G(Z_1), and Z_1 \cap W^* is a deficient set by \{s_1, s_2\} \subseteq Z_1 \cap W^*. We have u_1 \in Z_1 \cap W^* \neq \emptyset \neq W_1 \cap W_2 by Z_1 \neq \emptyset and W^* \cap S^* = \emptyset, and hence Z_1(u_1) \subseteq Z_1 \cap W^* holds, contradicting S^* \cap W^* = \emptyset.

Proof of Lemma 18. Let \{s_1, s_2\} = S \cap W_1 and \{s_1\} = S \cap W_2. Since |S| \geq 3 or V \neq \bigcup_{W \in W} W, we have N_G(W) = W_1 \neq \emptyset, from which we have |N_G(W)| \leq 2, |N_G(W) \cap W_1| = 1 holds. Lemma 12(i) says that \{s_1\} = N_G(W) \cap W_1 holds. Note that by |W_1| \geq 2 and Lemma 7, we have |N_G(W_1)| = 2 holds. By this, Lemma 8, s \notin X, and |N_G(X)| = 1, we see that X \cap W_1 \neq \emptyset and N_G(X) = \{s\}; N_G(X \cap W_1) \\subseteq \{s, s_1\} and N_G((X \cap W_1) \cup W_1) = \{s\}. Moreover, we can observe that any minimal deficient set W' \subseteq (X \cap W_1) \cup (W_1 - \{v\}) w.r.t. s_1 satisfies X \cap W \cap W' \neq \emptyset for a vertex v \in W_1 - \{s_1\} (note that (X \cap W_1) \cup (W_1 - \{v\}) is a deficient set by |N_G((X \cap W_1) \cup W_1) - \{s\}| = 1 and d(s_1) = d(W_1 - \{v\}) = 3). By X \cap W \cap S = \emptyset, any such W' has property (P) with respect to S. It is not difficult to see that such a W' can be found in O(|W_1|) time by computing a biconnected component containing s_1 in G[W_1 \cup W].

Proof of Lemma 21. From the definition of chains and Z_i(u) \subseteq Z_i, it suffices to show that we have Z_i(u) \cap W^*_i = \emptyset for a chain \lambda_i with |\lambda_i| \geq 3 and d(s_i') = 3 and a vertex u \in W^*_i. In the case of |N_G(W^*_i) \cap W^*_j| = 1, it is not difficult to see that Z_i(u) \cap W^*_j = \emptyset, and hence Z_i(u) \cap W^*_i = \emptyset. Assume that N_G(W^*_i) \subseteq W^*_j, i.e., Z_i = W^*_i \cup W^*_j. Then we have N_G(W^*_j) - W^*_j \neq \emptyset, since otherwise \forall W = \{W_1, W_2, W_3\} and V = W_1 \cup W_2 \cup W_3 would hold, contradicting that \{s_1\} \subseteq 4 or V \neq \bigcup_{W \in W} W holds. Hence |N_G(W^*_j) \cap W^*_j| = 1 holds. By Lemma 12(i), we have \{s_1\} = N_G(W^*_i) \cap W^*_j. This implies that Z' = Z_i - W^*_j \subseteq \{s_1\} satisfies N_G(Z') = \{s_2\}. Now, by W^*_i \cap W^*_j \neq \emptyset and u \in W^*_i - \{s_1\}, we see that N_G(Z' - \{u\}) = \{u, s'_2\} and s'_1 \in Z' - \{u\}. This implies that Z_i(u) \subseteq Z' - \{u\} \subseteq V - W^*_i holds.


