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IIB backgrounds with five-form flux

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Abstract

We investigate all $N = 2$ supersymmetric IIB supergravity backgrounds with non-vanishing five-form flux. The Killing spinors have stability subgroups $Spin(7) \ltimes \mathbb{R}^8$, $SU(4) \ltimes \mathbb{R}^8$ and G_2. In the $SU(4) \ltimes \mathbb{R}^8$ case, two different types of geometry arise depending on whether the Killing spinors are generic or pure. In both cases, the backgrounds admit a null Killing vector field which leaves invariant the $SU(4) \ltimes \mathbb{R}^8$ structure, and an almost complex structure in the directions transverse to the lightcone. In the generic case, the twist of the vector field is trivial but the almost complex structure is non-integrable, while in the pure case the twist is non-trivial but the almost complex structure is integrable and associated with a relatively balanced Hermitian structure. The G_2 backgrounds admit a time-like Killing vector field and two spacelike closed one-forms, and the seven directions transverse to these admit a co-symplectic G_2 structure. The $Spin(7) \ltimes \mathbb{R}^8$ backgrounds are pp-waves propagating in an eight-dimensional manifold with holonomy $Spin(7)$. In addition we show that all the supersymmetric solutions of simple five-dimensional supergravity with a time-like Killing vector field, which include the AdS_5 black holes, lift to $SU(4) \ltimes \mathbb{R}^8$ pure Killing spinor IIB backgrounds. We also show that the LLM solution is associated with a co-symplectic co-homogeneity one G_2 manifold which has principal orbit $S^3 \times S^3$.

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1. Introduction

Supersymmetric IIB backgrounds with "active" five-form fluxes and with vanishing one- and three-form field strengths have been extensively investigated in the context of string theory, branes and black holes. Examples of such backgrounds are the Freund–Rubin space $AdS_5 \times S^5$ [1], the D3-brane [2] and the maximally supersymmetric plane wave [3] solutions which have been instrumental in the formulation and understanding of the AdS/CFT correspondence [4]. More recently, in the same context many new solutions of increasing complexity have been found preserving some supersymmetry. These include the bubbling solutions of [5] and the lift of AdS_5 black holes [6] to IIB supergravity [7,8].

Motivated by the above widespread applications, we present a systematic investigation of all supersymmetric IIB supergravity backgrounds with active five-form flux F. This is based on our solution [9,10] of the Killing spinor equations of IIB supergravity [1,11,12] for one Killing spinor, using the spinorial geometry technique of [13]. Although, the supersymmetric backgrounds with F flux are special cases of the $N=1$ IIB backgrounds, there are some differences. Unlike generic IIB supersymmetric backgrounds, backgrounds with (only) F flux always preserve an even number of supersymmetries. So the backgrounds we shall investigate will preserve at least two supersymmetries, $N \geqslant 2$. In addition, the vanishing of one- and three-form fluxes imposes additional conditions on the geometry. It turns out that the geometry of the supersymmetric backgrounds with F flux is rather restricted and the five-form field strength F takes a simple form.

In analogy with the generic $N=1$ IIB backgrounds, the $N=2$ supersymmetric IIB backgrounds with F flux can be separated into three classes distinguished by the stability subgroups of the Killing spinors in $Spin(9,1)$. These are $Spin(7) \ltimes \mathbb{R}^8$, $SU(4) \ltimes \mathbb{R}^8$ and G_2. We find that the $Spin(7) \ltimes \mathbb{R}^8$ backgrounds are pp-waves with rotation and null F flux propagating on a holonomy $Spin(7)$ manifold whose metric depends on a wave profile coordinate.

The geometry of the $SU(4) \ltimes \mathbb{R}^8$ backgrounds is more subtle. These are further divided into two subclasses, the generic and the pure spinor backgrounds, which have distinct geometries. The Killing spinors of the former backgrounds do not obey additional conditions apart from those imposed by $SU(4) \ltimes \mathbb{R}^8$ invariance up to a possible conjugation with a $Spin(9,1)$ transformation. The Killing spinors of the latter are pure $SU(4) \ltimes \mathbb{R}^8$-invariant spinors. One consequence of this is that the stability subgroups of the spinors in $Spin(9,1)$ are not sufficient to characterize uniquely the geometry of the supersymmetric backgrounds. The geometry rather depends on the embedding of the Killing spinor bundle into the spinor bundle of IIB supergravity up to a $Spin(9,1)$ rotation as has been explained in [16]. In both cases, the spacetime admits a null Killing vector field X with non-vanishing twist or rotation, and an almost complex structure with compatible $(4,0)$-form leading to an $SU(4)$ structure in the directions transverse to the lightcone. In the generic case, the twist takes values in $\mathbb{R}^8$ and so it is trivial, the almost complex structure is not integrable, the W_1, W_4 and W_5 classes associated with the $SU(4)$ structure are determined in terms of functions of the spacetime, and W_2 is related to W_3. In the pure spinor case, the twist takes values in $\mathfrak{su}(4) \oplus_s \mathbb{R}^8$ and so it is not trivial, the almost complex structure is integrable, i.e., $W_1 = W_2 = 0$, $W_4 = W_5$ is given in terms of the twist of X, and W_3 is not restricted by the Killing spinor equations. We refer to these conditions on the W classes[1] as a *relatively balanced* $SU(4)$ *Hermitian structure*. If one imposes the additional condition that the twist of X is trivial,

[1] A balanced Hermitian structure is one for which the Lee form W_4 of the Hermitian form ω vanishes. In the present context, it is the difference of the Lee forms constructed from the fundamental $SU(4)$ forms ω and $\mathrm{Re}\,\chi$ that vanishes.

then the geometric conditions can be re-expressed as $d(e^H\omega^3) = d(e^H\chi) = 0$, where ω and χ are the fundamental $SU(4)$ forms and H is a spacetime function. In both cases, most of the components of the five form field strength are determined in terms of the geometry. The field equations that remain to be imposed to find solutions are the E_{--} component of the Einstein equations, and the Bianchi identity of F. Examples of IIB solutions that admit a pure Killing spinor are the D3-brane and its intersections as well as the solutions which are obtained from uplifting all $1/4$-supersymmetric solutions of minimal five-dimensional supergravity for which the Killing spinor generates a timelike Killing vector. We show how the constraints on the five-dimensional solutions obtained in [14] are sufficient to ensure that the pure spinor constraints in IIB supergravity are satisfied when the solution is uplifted using the Ansatz given in [15]. The null Killing vector field X has trivial twist for the D3-branes and their intersections, while X has non-trivial twist for the uplifts of five-dimensional solutions.

The tangent space of IIB backgrounds with G_2-invariant spinors is the orthogonal sum of the trivial bundle of rank three and a vector bundle of rank seven corresponding to the "transverse directions". One of the directions along the trivial bundle is a time-like Killing vector field which also leaves invariant the G_2 structure, and the duals of the other two directions are associated with closed spacelike one-forms. The seven transverse directions admit a *co-symplectic or co-calibrated G_2 structure*, i.e., $X_2 = X_4 = 0$ in terms of G_2 classes. Moreover X_1, which is in the trivial representation of G_2, is expressed in terms of the covariant derivatives of the closed one-forms. The class X_3 is not restricted by the Killing spinor equations. All the components of F are expressed in terms of the geometry. In addition the Bianchi identity of F implies all the field equations. One of the consequences of the above geometric properties is that not all Killing spinor vector bilinears are Killing. This has also been the case for other $N = 1$ IIB backgrounds.

We use the relation between supersymmetry and geometry that we have described to propose a constructive method of finding IIB solutions utilizing families of G_2 co-symplectic manifolds. As an example we explore such a construction based on the classification of co-symplectic G_2 manifolds of co-homogeneity one [19]. We also uncover the co-symplectic geometry of the bubbling AdS solutions of [5]. In particular, we show that these are associated with a special family of co-homogeneity one co-symplectic G_2 manifolds that preserves an $SO(4) \times SO(4)$ symmetry whose principal orbit is $S^3 \times S^3$.

This paper is organized as follows: In Section 2, we describe the general geometric properties of the supergravity backgrounds and define the "transverse spaces" of the spacetimes with Killing spinors which have compact or non-compact stability subgroups in $Spin(9,1)$. In Section 3, we solve the Killing spinor equations of generic $SU(4) \ltimes \mathbb{R}^8$ backgrounds and describe their geometry. In Section 4, we solve the Killing spinor equations of backgrounds that admit a pure Killing spinor, and show that D3-branes and their intersections as well as the uplifts of supersymmetric solutions of minimal five-dimensional gauged supergravity admitting a timelike Killing vector field are examples of such backgrounds. These include the AdS_5 black holes. In Section 5, we show that the $Spin(7) \ltimes \mathbb{R}^8$ backgrounds are pp-waves. In Section 6, we solve the Killing spinor equations of G_2 backgrounds and relate their geometry to co-symplectic G_2 manifolds. In Section 7, we describe how families of co-calibrated G_2 manifolds can be used to construct solutions of IIB supergravity with emphasis on those of co-homogeneity one. We find that the bubbling AdS solutions are such an example. In section eight, we give our conclusions. In Appendix A, we present the linear systems associated with the $N = 2$ IIB supersymmetric backgrounds. In Appendices B and C, we summarize some results on null and G_2 structures in ten dimensions.

2. Geometry and supersymmetry

The Killing spinors of IIB supergravity are complex positive chirality Weyl spinors, $S^+_{\mathbb{C}}$. In the absence of one-form P and three-form G field strengths, $P = G = 0$, the Killing spinor equations are linear over the complex numbers. This means that if ϵ is a Killing spinor, then $i\epsilon$ is also Killing. Therefore IIB backgrounds with non-vanishing five-form flux preserve an even number of supersymmetries. This in particular implies that if one sets $P = G = 0$ in the $N = 1$ backgrounds of [9] and [10], one can obtain the conditions for the most general $N = 2$ backgrounds with five-form F fluxes. As a consequence, there are three classes of $N = 2$ supersymmetric backgrounds with F fluxes distinguished by the stability subgroups $Spin(7) \ltimes \mathbb{R}^8$, $SU(4) \ltimes \mathbb{R}^8$ and G_2 of the Killing spinors in $Spin(9, 1)$.

The maximal number of $SU(4) \ltimes \mathbb{R}^8$- and G_2-invariant spinors in $S^+_{\mathbb{C}}$ is four. If the structure group of spacetime is one of these groups, then it admits a rank four subbundle $\mathcal{I}$ of the spin bundle $S^+_{\mathbb{C}}$ spanned by the invariant spinors. Since we are investigating backgrounds that admit two $SU(4) \ltimes \mathbb{R}^8$- or G_2-invariant Killing spinors, the Killing spinor bundle $\mathcal{K}$ is a subbundle of $\mathcal{I}$. Choosing a basis of spinors in $\mathcal{I}$, one can write the embedding of $\mathcal{K}$ in $\mathcal{I}$. As we shall demonstrate confirming the analysis in [10], the conditions on the geometry of spacetime imposed by supersymmetry depend on the embedding of $\mathcal{K}$ in $\mathcal{I}$ up to a $Spin(9, 1)$ automorphism of $S^+_{\mathbb{C}}$. Before we proceed, we shall explain how the geometries of the different embeddings of $\mathcal{K}$ in $\mathcal{I}$ can be identified. Although the spinors in $\mathcal{I}$ are not Killing, nevertheless they are well-defined on the spacetime because of the reduction of its structure group. Consequently, one can use a basis in $\mathcal{I}$ to construct the form spinor bi-linears that describe the $SU(4) \ltimes \mathbb{R}^8$ structure of the spacetime. As we shall see the restrictions on the geometry of the spacetime that arise from the Killing spinor equations can be written as conditions on the covariant derivatives of these bi-linears. Moreover, the Killing spinor equations imply that (some of) the components of the five-form flux are also written in terms of these bi-linears and their covariant derivatives.

Killing spinors with isotropy group $K \ltimes \mathbb{R}^8$, $K = Spin(7), SU(4)$, are associated with a null Killing vector field X. In the complement of the zero locus of X, the cotangent bundle T^*M of the spacetime M, $\dim M = 10$, admits a real trivial rank one null subbundle I spanned by the associated one-form κ to X, and a subbundle $P = \{\alpha \in \Gamma(T^*M) \mid i_X\alpha = 0\}$ of rank 9. Moreover, one has that

$$0 \to I \to P \to \mathcal{T}^\star \to 0, \tag{2.1}$$

where $\mathcal{T}^\star$ is of rank 8 and its dual $\mathcal{T}$ is identified as the bundle of the "transverse directions" to the lightcone. Observe that P is not canonically decomposed in $\mathcal{T}^\star$ and I. It is also possible to define the bundle of higher-degree "transverse" forms of the spacetime. We shall not explain this construction further here because it can be found in the appendices of [30].

A basis of $SU(4) \ltimes \mathbb{R}^8$-invariant Killing spinors is $\{1, e_{1234}\}$ and it can be shown, up to a local $Spin(9, 1)$ transformation,[2] that the $SU(4) \ltimes \mathbb{R}^8$-invariant Killing spinors $(\epsilon, i\epsilon)$ are given by

$$\epsilon = (f - g_2 + ig_1)1 + (f + g_2 + ig_1)e_{1234}, \quad f, g_2 \neq 0, \tag{2.2}$$

where f, g_1, g_2 are real spacetime functions which describe the embedding of $\mathcal{K}$ in $\mathcal{I}$. Our spinor conventions can be found in [9,10]. If $g_2 = 0$, then the spinor is $Spin(7) \ltimes \mathbb{R}^8$ invariant. We shall show that there are two classes of $N = 2$ backgrounds with $SU(4) \ltimes \mathbb{R}^8$-invariant Killing

[2] As we shall see there is some residual symmetry left which we use to simplify the spinors further.

spinors. One such class are the *generic* backgrounds for which there is no restriction on the spacetime functions f, g_1 and g_2, i.e., the embedding of $\mathcal{K}$ in $\mathcal{I}$ is generic. However, there is another class of supersymmetric backgrounds with different geometry for which $g_1 = 0$ and $f = \pm g_2$. The Killing spinors are pure, i.e., they are annihilated by a maximally isotropic subspace. Since the geometry of these two classes is different, the isotropy group of the spinors in $Spin(9,1)$ and the number of supersymmetries are not sufficient to characterize the supersymmetric backgrounds. The $N = 2$ backgrounds with $Spin(7) \ltimes \mathbb{R}^8$-invariant Killing spinors can be thought of as a special case of $N = 2$ backgrounds with $SU(4) \ltimes \mathbb{R}^8$-invariant Killing spinors. As we have mentioned, these arise by setting $g_2 = 0$ in (2.2).

It is clear from the results of [10] that if the isotropy group of the Killing spinors is G_2, the tangent bundle of the spacetime decomposes as

$$T^*M = I^3 \oplus \mathcal{T}^*, \tag{2.3}$$

where I^3 is the trivial vector bundle of rank three. Moreover, one direction in I^3 is spanned by a time-like Killing vector field. As in the previous case, $\mathcal{T}$, which has rank 7, is the "transverse" bundle or the "transverse directions" of the spacetime. Unlike the previous cases, the decomposition of T^*M is the orthogonal decomposition with respect to the spacetime metric. However, it is not always the case that there is a submanifold B in M such that the restriction of $\mathcal{T}$ on B is its tangent bundle.

One can take the rank 4 bundle $\mathcal{I}$ of G_2 invariant spinors to be spanned by $(1 + e_{1234}, e_{51} + e_{5234})$. Moreover, up to a $Spin(9,1)$ gauge transformation, the G_2-invariant Killing spinors of $N = 2$ backgrounds can be written as

$$\epsilon = f(1 + e_{1234}) + ig(e_{51} + e_{5234}), \quad f, g \neq 0, \tag{2.4}$$

where f, g are real spacetime functions. In this case, all the embeddings of $\mathcal{K}$ in $\mathcal{I}$ give the same geometry.

3. Generic $N = 2$ $SU(4) \ltimes \mathbb{R}^8$ backgrounds

3.1. Conditions on the geometry

The linear system associated with the Killing spinor equations in this case has been presented and solved in Appendix A.1. The solution has been found using the property that for these backgrounds the functions f, g_1, g_2 which determine the Killing spinors are generic. Here, we shall investigate the consequences that the supersymmetry conditions have on the geometry. To analyze the geometry and fluxes, we introduce the pseudo-Hermitian frame $(e^+, e^-, e^\alpha, e^{\bar{\alpha}})$, $\alpha = 1, 2, 3, 4$, adapted to the description of spinors in terms of forms and write the metric and fluxes as

$$\begin{aligned} ds^2 &= 2e^- e^+ + \delta_{ij} e^i e^j = 2\big(e^- e^+ + \delta_{\alpha\bar{\beta}} e^\alpha e^{\bar{\beta}}\big), \\ F &= e^+ \wedge \Phi + e^- \wedge \Psi + e^+ \wedge e^- \wedge \mathcal{X} + {}^*\big[e^+ \wedge e^- \wedge \mathcal{X}\big], \end{aligned} \tag{3.1}$$

where Φ is an anti-self dual, and Ψ is a self-dual, four-form in the eight directions transverse to the light-cone directions, and $\mathcal{X}$ is a three-form, $i, j = 1, 2, 3, 4, 6, 7, 8, 9$. The last term in the expression for F is required by the self-duality[3] of F, ${}^*F = F$, and it is completely determined by the spacetime metric and $\mathcal{X}$.

Choosing a basis $\{1, e_{1234}\}$ in the space of $SU(4) \ltimes \mathbb{R}^8$-invariant spinors $\mathcal{I}$, one can show that the spacetime admits the form spinor bi-linears

$$e^-, \quad e^- \wedge \omega, \quad e^- \wedge \chi, \tag{3.2}$$

where

$$\begin{aligned}
&\omega = -e^1 \wedge e^6 - e^2 \wedge e^7 - e^3 \wedge e^8 - e^4 \wedge e^9 = -i\delta_{\alpha\bar\beta} e^\alpha \wedge e^{\bar\beta}, \\
&\chi = \left(e^1 + ie^6\right) \wedge \left(e^2 + ie^7\right) \wedge \left(e^3 + ie^8\right) \wedge \left(e^4 + ie^9\right),
\end{aligned} \tag{3.3}$$

are the fundamental Hermitian and (4, 0) $SU(4)$ forms, respectively.

To continue, it is convenient to carry out the description of the geometry in the gauge $f^2 + g_1^2 + g_2^2 = 1$. Moreover, it is convenient to separate the conditions that arise from supersymmetry into those that involve the light-cone directions and those that involve the transverse directions only. To describe the former, let (e_+, e_-, e_i) be the co-frame of (e^+, e^-, e^i), $e^A(e_B) = \delta^A{}_B$. Then the conditions in Appendix A.1 can be rewritten as

$$\mathcal{L}_X g = 0, \qquad \mathcal{L}_X\left(e^- \wedge \omega\right) = \mathcal{L}_X\left(e^- \wedge \chi\right) = 0, \qquad de^- = \frac{1}{2} e^- \wedge dH, \tag{3.4}$$

where $X = e_+$ and $H = \log(1 - 4f^2 g_2^2)$. Therefore X is Killing and leaves the $SU(4) \ltimes \mathbb{R}^8$ structure invariant. The last condition in (3.4) follows from the torsion free condition of the metric $de^- + \Omega^- = 0$ and the conditions in Appendix A.1. It implies that the rotation of X is trivial, i.e., $e^- \wedge de^- = 0$.

The remaining geometric conditions along the transverse directions can be expressed as

$$\begin{aligned}
&2(W_3)_{\bar\alpha\beta\gamma}\left[f^2 - (g_2 + ig_1)^2\right] + (W_2)_{\bar\alpha\bar\delta_1\bar\delta_2} \epsilon^{\bar\delta_1\bar\delta_2}{}_{\beta\gamma} = 0, \\
&(W_4 - 2W_5)_{\bar\alpha} = -\partial_{\bar\alpha} \log\left[e^{\frac{1}{2}H} \frac{1 - 2g_2^2 - 2ig_1g_2}{1 - 2g_2^2 + 2ig_1g_2}\right], \\
&(W_1)_{\bar\alpha_1\bar\alpha_2\bar\alpha_3} = i\frac{1 - 2g_2^2 - 2ig_1g_2}{8f^2g_2^2} \epsilon^\beta{}_{\bar\alpha_1\bar\alpha_2\bar\alpha_3} \partial_\beta H, \\
&(W_4)_\alpha = \frac{1}{8f^2g_2^2} \partial_\alpha H,
\end{aligned} \tag{3.5}$$

where W_1, W_2, W_3 and W_5 are the Gray–Hervella classes, see [17,18], which can be expressed in terms of fundamental $SU(4)$ forms ω and χ as described in Appendix B.2. Since W_1 and W_2 do not necessary vanish, one concludes that the almost complex structure that arises from the metric and ω in the transverse directions is *not* integrable.

There is an additional condition that arises from the Killing spinor equations which restricts the functions f, g_1, g_2 that determine the Killing spinors. This is most easily expressed by adapting a coordinate u along X, $X = \frac{\partial}{\partial u}$, and introduce coordinates (u, v, y^I) on the spacetime M such that the metric is written as

[3] Our form conventions are ${}^*G_{A_1\ldots A_{10-\ell}} = \frac{1}{\ell!} G_{A_1\ldots A_\ell} \epsilon^{A_1\ldots A_\ell}{}_{A_1\ldots A_{10-\ell}}$, the spacetime volume form is $d\,\mathrm{vol}_M = e^0 \wedge \cdots \wedge e^9$, and the orientation of the transverse directions is given by $d\,\mathrm{vol} = e^1 \wedge \cdots \wedge e^4 \wedge e^6 \wedge \cdots \wedge e^9$.

$$ds^2 = 2e^- e^+ + \delta_{ij} e^i e^j,$$
$$e^- = dv + m_i e^i,\ e^+ = du + V\,dv + n_i e^i,\ e^i = e^i{}_I\,dy^I, \tag{3.6}$$

where all components depend on v, y^I. The Killing spinor equations then imply that f, g_1, g_2 depend only on the coordinates y, v and the ratio

$$\xi(v) = \frac{(f + ig_1)^2 - g_2^2}{(f - ig_1)^2 - g_2^2}, \qquad \xi^* = \xi^{-1}, \tag{3.7}$$

depends only on v. To summarize, the Killing spinor equations of generic $SU(4) \ltimes \mathbb{R}^8$-invariant spinors imply the geometric conditions (3.4) and (3.5), and (3.7).

The conditions that arise from the transverse directions are not particularly illuminating and we have not found a way to simplify them. It is more straightforward to understand the conditions along the light-cone directions (3.4). In particular, since X has trivial twist there is a coordinate, which we again denote by v, such that the metric can be written as

$$ds^2 = 2e^{-\frac{1}{2}H}\,dv\left(du + V\,dv + n_i e^i\right) + \delta_{ij} e^i e^j, \tag{3.8}$$

where all components depend on v, y^I. This concludes the investigation of the geometry.

3.2. Fluxes

To find the conditions on the fluxes imposed by supersymmetry, we decompose the forms Φ, Ψ and $\mathcal{X}$ in (3.1) that determine the five-form field strength into $SU(4)$ representations. Using the fact that Φ is anti-self-dual in the transverse directions, one can show that

$$\Phi = \omega \wedge \alpha + \frac{1}{2} s \bar{\wedge} \operatorname{Re}\chi, \tag{3.9}$$

where α is a traceless $(1,1)$-form, $\alpha_{\beta\gamma} = \alpha_\beta{}^\beta = 0$, s is a $(2,0)$ and $(0,2)$ symmetric tensor, $s_{ij} = s_{ji}$, $s_{\alpha\bar\beta} = 0$ and $\bar{\wedge}$ denotes inner derivation.[4] In turn, one finds that

$$\alpha_{\beta\bar\gamma} = \frac{i}{2}\Phi_{\beta\bar\gamma\delta}{}^\delta, \qquad s_{\bar\alpha\bar\beta} = \frac{1}{6}\Phi_{\gamma_1\gamma_2\gamma_3(\bar\alpha}\epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar\beta)}. \tag{3.10}$$

Similarly using the self-duality of Ψ, one can write either

$$\Psi = \frac{1}{2}\operatorname{Re}(p\chi) + \frac{1}{2} w \bar{\wedge} \operatorname{Re}\chi + q\omega \wedge \omega + \hat\Psi^{2,2}, \tag{3.11}$$

or

$$\Psi = \frac{1}{2}\operatorname{Re}(p\chi) + \beta \wedge \omega + q\omega \wedge \omega + \hat\Psi^{2,2}, \tag{3.12}$$

where

$$p = \frac{1}{4!}\Psi_{\alpha_1\dots\alpha_4}\epsilon^{\alpha_1\dots\alpha_4}, \qquad w_{\bar\alpha\bar\beta} = -\frac{1}{6}\Psi_{\gamma_1\gamma_2\gamma_3[\bar\alpha}\epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar\beta]}, \qquad q = -\frac{1}{24}\Psi_\alpha{}^\alpha{}_\beta{}^\beta,$$
$$\beta_{\alpha_1\alpha_2} = \frac{i}{2}\Psi_{\alpha_1\alpha_2\beta}{}^\beta, \qquad i\beta_{\alpha_1\alpha_2} = \frac{1}{2} w_{\bar\gamma_1\bar\gamma_2}\epsilon^{\bar\gamma_1\bar\gamma_2}{}_{\alpha_1\alpha_2} \tag{3.13}$$

and $\hat\Psi^{2,2}$ is a traceless $(2,2)$-form.

[4] Let π be a k-form, then $s\bar{\wedge}\pi = \frac{1}{(k-1)!} s^j{}_{i_i}\pi_{j i_2\dots i_k} e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_k}$.

Moreover, the three-form $\mathcal{X}$ can be written as

$$\mathcal{X} = \frac{1}{2} v \bar{\wedge} \operatorname{Re}\chi + \omega \wedge \gamma + \hat{\mathcal{X}}^{2,1} + \hat{\mathcal{X}}^{1,2}, \tag{3.14}$$

where

$$v_{\bar{\alpha}} = \frac{1}{6} \mathcal{X}_{\beta_1\beta_2\beta_3} \epsilon_{\bar{\alpha}}{}^{\beta_1\beta_2\beta_3}, \qquad \gamma_\alpha = \frac{i}{3} \mathcal{X}_{\alpha\beta}{}^{\beta}, \tag{3.15}$$

and $\hat{\mathcal{X}}^{2,1}$ and $\hat{\mathcal{X}}^{1,2}$ are traceless (2, 1)- and (1, 2)-forms, respectively.

The supersymmetry conditions imply restrictions on the various irreducible representations of $SU(4)$ that appear in the above decompositions. In particular, it turns out that by inspecting the conditions in Appendix A.1 one obtains

$$\Phi = 0, \tag{3.16}$$

and so

$$F = e^- \wedge \Psi + e^+ \wedge e^- \wedge \mathcal{X} + {}^*\big[e^+ \wedge e^- \wedge \mathcal{X}\big]. \tag{3.17}$$

The remaining conditions[5] give

$$\begin{aligned}
p &= \frac{i}{8fg_2}\Big(\partial_-\big[f^2 - (g_2 - ig_1)^2\big] - \frac{i}{8}\nabla_- \operatorname{Re}\chi \cdot \operatorname{Im}\chi\big[f^2 - (g_2 - ig_1)^2\big]\Big),\\
\beta_{\alpha_1\alpha_2} &= -\frac{i}{8fg_2}\Big((\nabla_-\omega)_{\alpha_1\alpha_2} + \frac{1}{2}\big[f^2 - (g_1 - ig_2)^2\big](\nabla_-\omega \cdot \operatorname{Re}\chi)_{\alpha_1\alpha_2}\Big),\\
q &= \frac{1}{24fg_2}\Big(2g_2\partial_- g_1 - 2g_1\partial_- g_2 + \frac{1}{8}\nabla_- \operatorname{Re}\chi \cdot \operatorname{Im}\chi\Big),\\
v_\alpha &= -\frac{i}{8}\big[f^2 - (g_2 + ig_1)^2\big]\partial_\alpha \log\frac{1 + 2fg_2}{1 - 2fg_2},\\
\mathcal{X}^{2,1} + \mathcal{X}^{1,2} &= \frac{1}{8fg_2}\omega \wedge \big(de^-\big)_{-i} e^i - \frac{fg_2}{2}\big(d\omega^{2,1} + d\omega^{1,2}\big).
\end{aligned} \tag{3.18}$$

Observe that $\Psi^{2,2}$ is not restricted by the Killing spinor equations. One can substitute the above expressions into the formula for F. We shall not do this here because it does not lead to a simplification for the expression of F. However, as we shall show, F is simplified in some special cases.

3.3. Special cases

A large class of backgrounds consists of those for which the transverse metric is independent of v. Using the torsion free condition for the frame (e^-, e^+, e^i), one finds that

$$\nabla_-\omega^{2,0} = \nabla_-\chi^{4,0} = 0, \tag{3.19}$$

[5] We use $(\alpha \cdot \beta)_{j_1\dots j_\ell} = \frac{1}{k!}\alpha_{i_1\dots i_k}\beta^{i_1\dots i_k}{}_{j_1\dots j_\ell}$.

provided that[6] $de^+ \in \mathfrak{su}(4) \oplus \mathbb{R}^8$. Assuming also that Killing spinors are taken to be independent of v as well, we have that

$$p = q = \beta = 0. \tag{3.20}$$

In such a case, the flux can be written as

$$F = \frac{1}{2} e^+ \wedge e^- \wedge v \bar{\wedge} \operatorname{Re} \chi + \frac{1}{8 f g_2} e^+ \wedge \omega \wedge de^- \\ - \frac{f g_2}{2} e^+ \wedge e^- \wedge \left(d\omega^{2,1} + d\omega^{1,2}\right) + {}^*\left[e^+ \wedge e^- \wedge \mathcal{X}\right] + e^- \wedge \hat{\Psi}^{2,2}. \tag{3.21}$$

To construct solutions in this case, one must find almost Hermitian manifolds with an $SU(4)$ structure which satisfy the conditions (3.5), and then write the metric as (3.8). One must also impose the closure of F, $dF = 0$ and the $E_{--} = 0$ component of the Einstein equations.

4. Backgrounds with pure $SU(4) \ltimes \mathbb{R}^8$ invariant Killing spinors

4.1. Geometry and fluxes

In the solution of the Killing spinor equations with $SU(4) \ltimes \mathbb{R}^8$ invariant Killing spinors, a special case arises whenever the Killing spinor is pure [9]. In particular, the Killing spinors (2.2) are pure if one sets

$$g_1 = 0, \qquad f = \pm g_2 = \frac{h}{2}. \tag{4.1}$$

We consider the case $f + g_2 = 0$, the investigation of the other case is similar. The solution of the linear system can be found in Appendix A.2.1. Here we shall investigate the conditions on the geometry and write the fluxes in a closed form.

It is convenient to investigate the geometry in the gauge $h = 1$. First write the metric and fluxes as in (3.1) using the pseudo-Hermitian frame $(e^-, e^+, e^\alpha, e^{\bar{\alpha}})$. Next consider the form bilinears (3.2) and observe that the conditions on the geometry that involve light-cone directions can be written as

$$\mathcal{L}_X g = \mathcal{L}_X\left(e^- \wedge \omega\right) = \mathcal{L}_X\left(e^- \wedge \chi\right) = 0, \quad de^- \in \mathfrak{su}(4) \oplus \mathbb{R}^8, \tag{4.2}$$

i.e., $X = e_+$ is Killing vector field and preserves the $SU(4) \ltimes \mathbb{R}^8$ structure. Unlike the generic $SU(4) \ltimes \mathbb{R}^8$ case, $e^- \wedge de^- \neq 0$. The remaining geometric conditions along the transverse directions are

$$\Omega_{\alpha,\beta\gamma} = 0, \qquad \Omega_{\bar{\alpha},\beta}{}^{\beta} + \Omega_{\beta,\bar{\alpha}}{}^{\beta} = 0, \qquad \Omega_{\bar{\alpha},\beta}{}^{\beta} = -\Omega_{-,+\bar{\alpha}}, \tag{4.3}$$

which can be recast in terms of the $SU(4)$ structures, see Appendix B.2, as

$$W_1 = W_2 = 0, \qquad W_4 = W_5, \qquad (W_4)_i = \left(de^-\right)_{-i}. \tag{4.4}$$

The vanishing of W_1, W_2 can be interpreted as integrability of the almost complex structure along the transverse directions. In particular, one can show using the torsion free conditions of

[6] With this notation we mean that there are forms $\lambda = \lambda_i e^i$ and $\mu = \frac{1}{2}\mu_{ij} e^i \wedge e^j$ such that $de^+ = e^- \wedge \lambda + \mu$, where μ is $(1,1)$ and traceless.

the frame, that (e^-, e^α) span an integrable distribution of co-dimension 5. As we have seen, this is unlike what happens in the generic $SU(4) \ltimes \mathbb{R}^8$ backgrounds. The conditions (4.2) and (4.4) constitute the full set of restrictions that supersymmetry imposes on the geometry of spacetime.

Next let us turn to the flux F. Using (3.1) and the results of Appendix A.2.1 F can be written, after some work, as

$$\begin{aligned} F = &-\frac{1}{4}e^+ \wedge d\left(e^- \wedge \omega\right) + {}^*\left[e^+ \wedge e^- \wedge \mathcal{X}\right] + \frac{i}{4}e^- \wedge \nabla_- \omega^{2,0} \wedge \omega \\ &- \frac{i}{4}e^- \wedge \nabla_- \omega^{0,2} \wedge \omega - \frac{1}{2^2 \cdot 4!}e^- \wedge \omega \wedge \omega [\nabla_- \operatorname{Re}\chi \cdot \operatorname{Im}\chi] \\ &+ e^- \wedge \hat{\Psi}^{2,2}, \end{aligned} \tag{4.5}$$

where $\alpha \cdot \beta = \frac{1}{k!}\alpha_{i_1\ldots i_k}\beta^{i_1\ldots i_k}$. It is clear that all the components of F are determined in terms of the geometry apart from those of $\hat{\Psi}^{2,2}$ which is not restricted by the Killing spinor equations. In this case, Φ may not vanish.

One can adapt coordinates along the Killing vector field X, $X = \partial/\partial u$, and write the metric as (3.6). However unlike the generic case no further simplification is possible because the twist of X may not be trivial, i.e., $e^- \wedge de^- \neq 0$. If one imposes $e^- \wedge de^- = 0$, one finds additional restrictions on the geometry that are *not* implied by the Killing spinor equations.

4.2. *Special cases and examples*

4.2.1. *Special cases*

As in the class of generic backgrounds, one can take the transverse metric to be independent of v and impose $de^+ \in \mathfrak{su}(4) \oplus \mathbb{R}^8$ to find (3.19). In such cases, F can be written as

$$F = -\frac{1}{4}e^+ \wedge d\left(e^- \wedge \omega\right) + {}^*\left[e^+ \wedge e^- \wedge \mathcal{X}\right] + e^- \wedge \hat{\Psi}^{2,2}. \tag{4.6}$$

Further simplification of the metric and fluxes occurs when the rotation of X is trivial, i.e., $e^- \wedge de^- = 0$. This is equivalent to requiring that there is a one-form λ such that $de^- = \lambda \wedge e^-$ or equivalently $de^- \in \mathbb{R}^8$. Then the Frobenius theorem implies that there is a function $H = H(y, v)$ such that $e^- = e^{H(y,v)}\, dv$ for some coordinate v which is related to that denoted with the same symbol in (3.6) by a coordinate transformation. The metric takes the form

$$ds^2 = 2e^H\, dv\left(du + V\, dv + n_I\, dy^I\right) + g_{IJ}(y)\, dy^I\, dy^J. \tag{4.7}$$

A large class of known backgrounds have metric and fluxes given by (4.7) and (4.6), respectively. These include the D3-brane and intersecting D3-brane configurations as we shall see below.

Moreover, the geometric conditions are also simplified. In particular one finds that the geometric conditions in (4.4) become

$$W_1 = W_2 = 0, \qquad (W_4)_I = (W_5)_I = -\partial_I H. \tag{4.8}$$

The spacetime in this case can be reconstructed from an eight-dimensional Hermitian manifold with an $SU(4)$ structure for which the W_4 and W_5 classes satisfy the conditions above. Assuming that H depends only on y, $H = H(y)$, and setting $W_1 = W_2 = 0$ in (B.12) which expresses the exterior derivatives of ω and χ in terms of the W classes, we find that the remaining geometric conditions can be rewritten as

$$d\left(e^H \omega^3\right) = d\left(e^H \chi\right) = 0. \tag{4.9}$$

Observe that the rescaled forms $\omega_0 = e^{\frac{1}{3}H}\omega$ and $\chi_0 = e^H \chi$ for $dH \neq 0$ are not canonically normalized, so such Hermitian manifolds do not contain a Calabi–Yau in their conformal class. The expressions for the metric (4.7) and the flux (4.6), and the conditions (4.9) are the full content of the Killing spinor equations for the case that the transverse metric and H are independent of v.

Of course additional conditions are imposed on the backgrounds from the Bianchi identity of F, $dF = 0$ and the vanishing of the E_{--} component of the Einstein equations, $E_{--} = 0$. In what follows, we consider examples which admit a null Killing vector with a trivial and a non-trivial twist.

4.2.2. D3-branes and intersecting branes

The D3-brane [2] and its intersections [20] are examples of backgrounds with pure $SU(4) \ltimes \mathbb{R}^8$-invariant Killing spinors. The metric of the former is

$$ds^2 = h^{-\frac{1}{2}} ds^2(\mathbb{R}^{3,1}) + h^{\frac{1}{2}} ds^2(\mathbb{R}^6), \tag{4.10}$$

where h is a (multi-centred) harmonic function of $\mathbb{R}^6$. (h should be distinguished from the function h that multiplies the Killing spinor which we have set equal to 1.) The background preserves 16 supersymmetries. Clearly, this metric is of the form (4.7). To rewrite the metric as (3.6), set

$$ds^2 = 2h^{-\frac{1}{2}} dx\, dz + h^{-\frac{1}{2}} ds^2(\mathbb{R}^2) + h^{\frac{1}{2}} ds^2(\mathbb{R}^6) \tag{4.11}$$

and change coordinates as

$$u = x, \qquad v = h^{-\frac{1}{2}} z \tag{4.12}$$

to find

$$\begin{aligned} ds^2 &= 2e^- e^+ + h^{-\frac{1}{2}} ds^2(\mathbb{R}^2) + h^{\frac{1}{2}} ds^2(\mathbb{R}^6), \\ e^- &= dv + \frac{1}{2} v\, d\log h, \ e^+ = du. \end{aligned} \tag{4.13}$$

Observe that the pure spinor 1 satisfies the projection condition that arises in the D3-brane Killing spinor equations. It is then easy to find that

$$\omega = h^{-\frac{1}{2}} \omega(\mathbb{R}^2) + h^{\frac{1}{2}} \omega(\mathbb{R}^6), \tag{4.14}$$

where $\omega(\mathbb{R}^2)$ and $\omega(\mathbb{R}^6)$ are the constant Kähler forms on $\mathbb{R}^2$ and $\mathbb{R}^6$, respectively. Substituting all these into the flux and taking $\hat{\Psi}^{2,2} = 0$, one can easily show that

$$\begin{aligned} F &= -\frac{1}{4} h^{-\frac{3}{2}} du \wedge dv \wedge \omega(\mathbb{R}^2) \wedge dh + {}^*\big[e^+ \wedge e^- \wedge \mathcal{X}\big] \\ &= \frac{1}{4} dx \wedge dz \wedge \omega(\mathbb{R}^2) \wedge dh^{-1} + {}^*\big[e^+ \wedge e^- \wedge \mathcal{X}\big]. \end{aligned} \tag{4.15}$$

This is precisely the five-form flux of the D3-brane. Moreover observe that the space transverse to the lightcone directions is Hermitian, the closure conditions (4.9) are satisfied and the rotation of X is trivial $e^- \wedge de^- = 0$. The latter also follows from a direct inspection of the D3-brane metric. One consequence of these conditions is that $H_3 \times S^5$, which is the near-horizon geometry of the D3-brane restricted on the transverse directions to the lightcone, admits a relatively balanced $SU(4)$ Hermitian structure.

One can also show that this class includes the delocalized D3-brane intersections. We shall demonstrate this for the configuration of two intersecting D3-branes at a string [20]. The computation is straightforward for the remaining cases. The metric is [21]

$$ds^2 = 2(h_1h_2)^{-\frac{1}{2}}\,dx\,dz + h_1^{-\frac{1}{2}}h_2^{\frac{1}{2}}\,ds_1^2\big(\mathbb{R}^2\big) + h_1^{\frac{1}{2}}h_2^{-\frac{1}{2}}\,ds_2^2\big(\mathbb{R}^2\big) + h_1^{\frac{1}{2}}h_2^{\frac{1}{2}}\,ds^2\big(\mathbb{R}^4\big), \qquad (4.16)$$

where h_1, h_2 are (multi-centred) harmonic functions of the transverse space $\mathbb{R}^4$. Changing coordinates as

$$u = x, \qquad v = (h_1h_2)^{-\frac{1}{2}}z \qquad (4.17)$$

the metric can be written in the standard light-cone form with

$$e^+ = du, \qquad e^- = dv + \frac{1}{2}v\,d\log(h_1h_2) = (h_1h_2)^{-\frac{1}{2}}\,dz. \qquad (4.18)$$

The Hermitian form can be chosen as

$$\omega = h_1^{-\frac{1}{2}}h_2^{\frac{1}{2}}\omega_1\big(\mathbb{R}^2\big) + h_1^{\frac{1}{2}}h_2^{-\frac{1}{2}}\omega_2\big(\mathbb{R}^2\big) + h_1^{\frac{1}{2}}h_2^{\frac{1}{2}}\omega\big(\mathbb{R}^4\big). \qquad (4.19)$$

Substituting this into the expression for the flux, one finds that

$$F = \frac{1}{4}dx \wedge dz \wedge \omega_1\big(\mathbb{R}^2\big) \wedge dh_1^{-1} + \frac{1}{4}dx \wedge dz \wedge \omega_2\big(\mathbb{R}^2\big) \wedge dh_2^{-1} + {}^*\big[e^+ \wedge e^- \wedge \mathcal{X}\big]. \qquad (4.20)$$

Similarly observe that the transverse space is Hermitian, the closure conditions (4.9) are satisfied and the rotation of X is trivial $e^- \wedge de^- = 0$.

One can easily extend the above results to include D3-brane configurations with a null rotation by taking $n \neq 0$ and superposed with a pp-wave $V \neq 0$. One can also allow $\hat{\Psi}^{2,2} \neq 0$ which may lead to resolved D3-brane configurations.

4.2.3. *Uplifted five-dimensional solutions*

Solutions of minimal gauged five-dimensional supergravity which admit a timelike Killing vector, $\frac{\partial}{\partial t}$, associated with a Killing spinor have spacetime geometry [14]

$$ds_5^2 = -\mathcal{F}^2(dt + \Psi)^2 + \mathcal{F}^{-1}\,ds_N^2, \qquad (4.21)$$

where $\mathcal{F}$ is a function and $\Psi = \Psi_m\,dx^m$ is a 1-form, $ds_N^2 = h_{mn}\,dx^m\,dx^n$ is a metric on a Kähler 4-manifold N, and (t, x^m), $m = 1, 2, 3, 4$, are spacetime coordinates. The components of the metric depend only on x^n. In addition, the one-form gauge potential is

$$A = \frac{\sqrt{3}}{2}\mathcal{F}(dt + \Psi) + \frac{\ell}{2\sqrt{3}}\mathcal{P}, \qquad (4.22)$$

where ℓ is constant and $\mathcal{R}_N = d\mathcal{P}$ is the Ricci form of the Kähler manifold N. The function $\mathcal{F}$ is determined in terms of the Ricci scalar R_N of N via

$$\mathcal{F} = -\frac{24}{\ell^2 R_N}. \qquad (4.23)$$

Setting

$$\mathcal{F}\,d\Psi = G^+ + G^-, \qquad (4.24)$$

where G^+, G^- are self- and anti-self-dual 2-forms on N, the Ricci form $\mathcal{R}_N$ is constrained by

$$\mathcal{R}_N = -\frac{2}{\ell} G^+ - \frac{6}{\mathcal{F}\ell^2} J_N, \tag{4.25}$$

where J_N is the Kähler form of N.

The uplifted metric is given by [15]

$$\begin{aligned} ds_{10}^2 = {} & ds_5^2 + \ell^2\big(d\alpha^2 + \cos^2\alpha\, d\beta^2 \sin^2\alpha \cos^2\alpha \big(d\xi_1 - \sin^2\beta\, d\xi_2 - \cos^2\beta\, d\xi_3\big)^2 \\ & + \cos^2\alpha \sin^2\beta \cos^2\beta (d\xi_2 - d\xi_3)^2\big) \\ & + \left(-\frac{2}{\sqrt{3}} A - \ell \sin^2\alpha\, d\xi_1 - \ell\cos^2\alpha\big(\sin^2\beta\, d\xi_2 + \cos^2\beta\, d\xi_3\big)\right)^2. \end{aligned} \tag{4.26}$$

It is convenient to define $\chi_1 = \xi_1 - \frac{1}{2}(\xi_2 + \xi_3)$, $\chi_2 = \xi_1 + \frac{1}{2}(\xi_2 + \xi_3)$, $\phi = \frac{1}{2}(\xi_2 - \xi_3)$ and rewrite the metric as

$$\begin{aligned} ds^2 = {} & \frac{2\ell\mathcal{F}}{3}\left(dt + \Psi + \frac{\ell}{4\mathcal{F}} d\chi_2 + \frac{\ell}{6\mathcal{F}}(\mathcal{P} + \mathcal{Q})\right)\left(\frac{3}{2} d\chi_2 + \mathcal{P} + \mathcal{Q}\right) \\ & + \mathcal{F}^{-1} h_{mn}\, dx^m\, dx^n + \ell^2\, ds^2_{CP^2}, \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} ds^2_{CP^2} = {} & d\alpha^2 + \cos^2\alpha\, d\beta^2 + \sin^2\alpha \cos^2\alpha \big(d\chi_1 + \big(\cos^2\beta - \sin^2\beta\big)\, d\phi\big)^2 \\ & + 4\cos^2\alpha \sin^2\beta \cos^2\beta\, d\phi^2, \end{aligned} \tag{4.28}$$

is the Kähler–Einstein metric on CP^2 which has constant holomorphic sectional curvature. $\mathcal{Q}$ is the potential for the Ricci form of the metric (4.28), where

$$\mathcal{Q} = 3\cos^2\alpha\big(\sin^2\beta - \cos^2\beta\big)\, d\phi + \frac{3}{2}\big(\sin^2\alpha - \cos^2\alpha\big)\, d\chi_1, \tag{4.29}$$

and the Kähler form of CP^2 is $J_{CP^2} = \frac{1}{6} d\mathcal{Q}$. The Ricci scalar of (4.28) is $R_{CP^2} = 24$. Note that in the uplifted solution, the Killing vector $\frac{\partial}{\partial t}$ is null.

In order to write the uplifted solution as a pure spinor $SU(4) \ltimes \mathbb{R}^8$ background, we set $ds^2 = 2e^- e^+ + ds_8^2$, where

$$\begin{aligned} e^+ &= \left(dt + \Psi + \frac{\ell}{4\mathcal{F}} d\chi_2 + \frac{\ell}{6\mathcal{F}}(\mathcal{P} + \mathcal{Q})\right), \\ e^- &= \frac{\ell\mathcal{F}}{3}\left(\frac{3}{2} d\chi_2 + \mathcal{P} + \mathcal{Q}\right), \\ ds_8^2 &= \mathcal{F}^{-1}\, ds_N^2 + \ell^2\, ds^2_{CP^2}. \end{aligned} \tag{4.30}$$

The calculation we present below can also be carried out if CP^2 is replaced with any other four-dimensional Kähler–Einstein manifold E such that $R_E = 24$. However, we shall continue the analysis using CP^2.

The complex structure along the transverse directions to the lightcone is identified as the direct sum of the complex structure on CP^2 together with the complex structure of the Kähler base manifold N of the 5-dimensional solution. It is clear that the null Killing vector field $\partial/\partial t$ of the ten-dimensional solution has non-trivial twist.

It will be convenient to split the $SU(4)$ indices α into $\alpha = (a, \mu)$ for $a, b = 1, 2$ and $\mu, \nu = 3, 4$ and choose a Hermitian frame along the transverse directions as

$$e^a = e^{ik_1\chi_2}\mathcal{F}^{-\frac{1}{2}}\hat{e}^a,$$
$$e^\mu = e^{ik_2\chi_2}\ell\hat{e}^\mu \tag{4.31}$$

for real constants k_1, k_2 to be fixed, where $\hat{e}^a$ is a Hermitian frame of the Kähler base manifold N, $ds^2_N = 2\delta_{a\bar{b}}\hat{e}^a\hat{e}^{\bar{b}}$, $J_N = -i\delta_{a\bar{b}}\hat{e}^a \wedge \hat{e}^{\bar{b}}$, such that

$$\mathcal{L}_{\frac{\partial}{\partial t}}\hat{e}^a = \mathcal{L}_{\frac{\partial}{\partial \chi_1}}\hat{e}^a = \mathcal{L}_{\frac{\partial}{\partial \chi_2}}\hat{e}^a = \mathcal{L}_{\frac{\partial}{\partial \alpha}}\hat{e}^a = \mathcal{L}_{\frac{\partial}{\partial \beta}}\hat{e}^a = \mathcal{L}_{\frac{\partial}{\partial \phi}}\hat{e}^a = 0. \tag{4.32}$$

We also set

$$\hat{e}^3 = \frac{1}{\sqrt{2}}\big(d\alpha - i\sin\alpha\cos\alpha\big(d\chi_1 + \big(\cos^2\beta - \sin^2\beta\big)\,d\phi\big)\big),$$
$$\hat{e}^4 = \frac{1}{\sqrt{2}}(\cos\alpha\, d\beta - 2i\cos\alpha\sin\beta\cos\beta\, d\phi). \tag{4.33}$$

In addition, we have $ds^2_{CP^2} = 2\delta_{\mu\bar{\nu}}\hat{e}^\mu\hat{e}^{\bar{\nu}}$ and $J_{CP^2} = -i\delta_{\mu\bar{\nu}}\hat{e}^\mu \wedge \hat{e}^{\bar{\nu}}$. It is also convenient to write $\omega = \omega_N + \omega_{CP^2}$, where

$$\omega_N = \mathcal{F}^{-1}J_N, \qquad \omega_{CP^2} = \ell^2\tilde{J}_{CP^2}. \tag{4.34}$$

Using the expression for the spin connection which we give in Appendix A.2.2, it is straightforward to show that the constraints on the geometry imposed by supersymmetry are satisfied provided that

$$k_1 + k_2 = -\frac{3}{4}. \tag{4.35}$$

Next, note that the formula for the uplifted five-form is [15]

$$F = \frac{1}{4}(1+*)\left(-\frac{4}{\ell}d\,\mathrm{vol}_5 + \frac{\ell^2}{\sqrt{3}}\sum_{i=1}^{3} d\big(\mu_i^2\big) \wedge d\xi_i \wedge *_{(5)}\, dA\right), \tag{4.36}$$

where $\mu_1 = \sin\alpha$, $\mu_2 = \cos\alpha\sin\beta$, $\mu_3 = \cos\alpha\cos\beta$ and

$$d\,\mathrm{vol}_5 = -\frac{1}{2}\mathcal{F}^{-2}e^0 \wedge J_N \wedge J_N, \tag{4.37}$$

with $e^0 = \mathcal{F}(dt + \Psi)$. In addition, $*_{(5)}$ denotes the 5-dimensional Hodge dual taken with respect to $d\,\mathrm{vol}_5$ and $*$ denotes the standard Hodge star operation of M whose volume form can be rewritten as $d\,\mathrm{vol}_M = \frac{1}{4}e^+ \wedge e^- \wedge \omega_N \wedge \omega_N \wedge \omega_{CP^2} \wedge \omega_{CP^2}$. Note the change in normalization for the 5-form when comparing with the Killing spinor equation in [7]. In order to simplify this expression, note that

$$\sum_{i=1}^{3} d\big(\mu_i^2\big) \wedge d\xi_i = 2J_{CP^2}. \tag{4.38}$$

We then find the uplifted five-form is given by

$$\begin{aligned}F ={}& \frac{1}{4}\ell^2\mathcal{F}^{-1}e^+ \wedge e^- \wedge J_{CP^2} \wedge d\mathcal{F} + {}^*\left(\frac{1}{4}\ell^2\mathcal{F}^{-1}e^+ \wedge e^- \wedge J_{CP^2} \wedge d\mathcal{F}\right) \\ &+ e^+ \wedge \left(\frac{1}{2}\mathcal{F}^{-1}\ell^{-1}J_N \wedge J_N - \frac{1}{2}\ell^3\mathcal{F}J_{CP^2} \wedge J_{CP^2} + \frac{1}{6}\mathcal{F}\ell^2 J_{CP^2} \wedge G^+\right) \\ &+ e^- \wedge \left(-\frac{1}{4}\ell^{-1}\mathcal{F}^{-1}\omega \wedge \omega + \frac{1}{4}\ell^2\mathcal{F}^{-1}J_{CP^2} \wedge G^-\right).\end{aligned} \tag{4.39}$$

In comparing this expression with the 5-form obtained from the pure spinor classification, it is straightforward to see that

$$-\frac{1}{4}e^+\wedge d\big(e^-\wedge\omega\big)=e^+\wedge\bigg(\frac{1}{2}\mathcal{F}^{-1}\ell^{-1}J_N\wedge J_N-\frac{1}{2}\ell^3\mathcal{F}J_{CP^2}\wedge J_{CP^2}+\frac{1}{6}\mathcal{F}\ell^2 J_{CP^2}\wedge G^+\bigg)+\frac{1}{4}\ell^2\mathcal{F}^{-1}e^+\wedge e^-\wedge J_{CP^2}\wedge d\mathcal{F}. \tag{4.40}$$

Also note that

$$\begin{aligned}&\frac{i}{4}\nabla_-\omega^{2,0}\wedge\omega-\frac{i}{4}\nabla_-\omega^{0,2}\wedge\omega-\frac{1}{4\cdot 4!}(\nabla_-\operatorname{Re}\chi)\cdot\operatorname{Im}\chi\,\omega\wedge\omega\\&\quad=-\frac{1}{4}\ell^{-1}\mathcal{F}^{-1}\omega\wedge\omega+\frac{1}{4}\ell^2\mathcal{F}^{-1}J_{CP^2}\wedge G^-\\&\qquad+\frac{i}{24}\mathcal{F}^{-1}\big(G^-\big)_a{}^a(\omega_N\wedge\omega_N-\omega_N\wedge\omega_{CP^2}+\omega_{CP^2}\wedge\omega_{CP^2}).\end{aligned} \tag{4.41}$$

However, $\omega_N\wedge\omega_N-\omega_N\wedge\omega_{CP^2}+\omega_{CP^2}\wedge\omega_{CP^2}$ is a traceless $(2,2)$ form. We therefore set

$$\hat{\Psi}^{2,2}=-\frac{i}{24}\mathcal{F}^{-1}\big(G^-\big)_a{}^a(\omega_N\wedge\omega_N-\omega_N\wedge\omega_{CP^2}+\omega_{CP^2}\wedge\omega_{CP^2}). \tag{4.42}$$

Hence, the formula for the uplifted five-form matches the expression for the five-form given in (4.5). It is remarkable that in establishing the equality of the uplifted five-form with that in (4.5), we have seen that all components of the latter give a non-trivial contribution.

It remains to consider the Bianchi identity; note that one can rewrite F as

$$\begin{aligned}F&=\frac{1}{4}d\big(e^+\wedge e^-\wedge\omega\big)-\frac{\ell^2}{4}J_{CP^2}\wedge\star_N d\mathcal{F}^{-1}\\&\quad+\mathcal{F}^{-1}e^-\wedge\bigg(\frac{\ell^3}{12}J_{CP^2}\wedge\mathcal{R}_N-\frac{1}{4}J_N\wedge d\Psi-\frac{\ell^3}{2}J_{CP^2}\wedge J_{CP^2}\bigg),\end{aligned} \tag{4.43}$$

where $\star_N$ denotes the Hodge dual taken on the Kähler base with metric h and volume form $-\frac{1}{2}J_N\wedge J_N$. Hence

$$dF=J_{CP^2}\wedge\bigg(-\frac{\ell^2}{4}d\star_N\,d\mathcal{F}^{-1}+\frac{\ell^4}{36}\mathcal{R}_N\wedge\mathcal{R}_N-\frac{\ell}{2}J_N\wedge d\Psi\bigg). \tag{4.44}$$

So the Bianchi identity implies that

$$-d\star_N\,d\mathcal{F}^{-1}+\frac{4}{9}G^+\wedge G^++4\ell^{-2}\mathcal{F}^{-2}J_N\wedge J_N-2\ell^{-1}\mathcal{F}^{-1}J_N\wedge G^-=0. \tag{4.45}$$

As expected, this constraint is equivalent to that implied by the five-dimensional gauge field equations obtained in [14]. Observe that as $\mathcal{F}$ is inversely proportional to the Ricci scalar R_N, this constraint is a highly non-linear constraint on the geometry of the Kähler base space N.

The AdS_5 black holes found in [6] belong to the class of five-dimensional backgrounds which we have uplifted to ten dimensions. In fact, it was shown in [7] that the uplifted black hole solutions preserve exactly $1/16$ of the supersymmetry in IIB supergravity. Hence it follows that the Killing spinors of the uplifted black hole solution are $(1,i1)$ of the pure spinor $SU(4)\ltimes\mathbb{R}^8$ backgrounds. This can also be seen by an inspection of the conditions given in [7].

5. $Spin(7) \ltimes \mathbb{R}^8$ backgrounds

As in previous cases, it is convenient to carry out the analysis without loss of generality in the gauge $f^2 + g^2 = 1$. A direct inspection of the geometric conditions in Appendix A.3 reveals that the holonomy of the Levi-Civita connection, ∇, of the spacetime is contained in $Spin(7) \ltimes \mathbb{R}^8$,

$$\mathrm{hol}(\nabla) \subseteq Spin(7) \ltimes \mathbb{R}^8. \tag{5.1}$$

This is equivalent to requiring that the forms

$$e^-, \quad e^- \wedge \phi, \tag{5.2}$$

are ∇-parallel, where ϕ is the fundamental $Spin(7)$ self-dual four-form given by $\phi = \mathrm{Re}\,\chi - \frac{1}{2}\omega \wedge \omega$. In particular, the null vector field X is Killing and twist free. Adapting coordinates along the parallel vector field, $X = e_+ = \partial/\partial u$, one finds that the spacetime metric can be written as

$$ds^2 = 2\,dv\left(du + V\,dv + n_I\,dy^I\right) + \gamma_{IJ}\,dy^I\,dy^J, \tag{5.3}$$

where the components depend on the coordinates v, y^I. It remains to specify the fluxes. One can show using the results of Appendix A.3 that

$$F = \frac{i}{14}\partial_- \log\left(\frac{f+ig}{f-ig}\right)e^- \wedge \phi + e^- \wedge \Psi_{\mathbf{27}}. \tag{5.4}$$

To derive this, we first remark that $F = e^- \wedge \Psi$, where Ψ is a self-dual four-form in the eight-transverse directions to the light-cone. Then we use the decomposition of the four-forms in $\mathbb{R}^8$ under $Spin(7)$ representations as $\Lambda^4(\mathbb{R}^8) = \Lambda^{4+}(\mathbb{R}^8) \oplus \Lambda^{4-}(\mathbb{R}^8)$, $\Lambda^{4+}(\mathbb{R}^8) = \Lambda_{\mathbf{1}} \oplus \Lambda_{\mathbf{7}} \oplus \Lambda_{\mathbf{27}}$ and $\Lambda^{4-}(\mathbb{R}^8) = \Lambda_{\mathbf{35}}$. It turns out that the Killing spinor equations imply that the $\Psi_{\mathbf{7}}$ component of Ψ vanishes, $\Psi_{\mathbf{7}} = 0$. The component $\Psi_{\mathbf{27}}$ is not restricted by the Killing spinor equations. The Killing spinors are

$$\epsilon = (f + ig)(v)(1 + e_{1234}), \tag{5.5}$$

i.e., the functions f, g depend only on v. It is clear from the above that the spacetime is a pp-wave with rotation propagating on an eight-dimensional manifold which has holonomy $Spin(7)$. The metric of the transverse space depends on the wave-profile coordinate v.

Imposing the Bianchi identity for F one finds that F is closed iff $d\Psi_{\mathbf{27}} = 0$ up to forms of the type $e^- \wedge \mu$. The only equation that remains to be imposed to find solutions is the vanishing of the E_{--} component of the Einstein equations, $E_{--} = 0$. This equation can be easily recovered from that in [16] by setting the one-form and three-form field strengths to zero.

6. G_2 backgrounds

6.1. Geometry

We have presented the linear system for G_2 backgrounds and its solution in Appendix A.4. In particular, we have given the solution in $SU(3) \subset G_2$ representations. Here, we shall investigate the consequences that the supersymmetry conditions have on the geometry and fluxes of the theory. For this, we introduce a frame[7] (e^+, e^-, e^1, e^i), $i = 2, 3, 4, 6, 7, 8, 9$, where (e^+, e^-, e^1)

[7] This is a real frame and different from the pseudo-Hermitian frame of Appendix A.4 which we have used to solve the linear system.

span the trivial subbundle of TM and (e^i) the transverse directions of the spacetime. The metric and fluxes written in this frame are

$$\begin{aligned} &ds^2 = 2e^- e^+ + \left(e^1\right)^2 + \delta_{ij} e^i e^j, \\ &F = e^+ \wedge \Phi + e^- \wedge \Psi + e^+ \wedge e^- \wedge \mathcal{X} + {}^*\left[e^+ \wedge e^- \wedge \mathcal{X}\right], \end{aligned} \tag{6.1}$$

where Φ is an anti-self dual, and Ψ is a self-dual, four-form in the eight directions transverse to the light-cone directions, and $\mathcal{X}$ is a three-form. The expression for the flux is similar to that which we have given for F in (3.1) for the $SU(4) \ltimes \mathbb{R}^8$ backgrounds.

Choosing a basis $(1 + e_{1234}, e_{15} + e_{2345})$ in the space of G_2-invariant spinors, one can show that a basis in the space of spinor bilinears is

$$e^0, \quad e^1, \quad e^5, \quad \varphi, \tag{6.2}$$

where

$$\varphi = \mathrm{Re}\left[\left(e^2 + ie^7\right) \wedge \left(e^3 + ie^8\right) \wedge \left(e^4 + ie^9\right)\right] - e^6 \wedge \left(e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9\right) \tag{6.3}$$

is the fundamental G_2 three-form, and $e^\pm = (1/\sqrt{2})(\pm e^0 + e^5)$. It is clear that $T^*M = I^3 \oplus \mathcal{T}$ where I^3 is a trivial bundle of rank 3 and $\mathcal{T}$ are the remaining "transverse" directions of the co-tangent bundle of the spacetime.

The geometric conditions of Appendix A.4 can be expressed in G_2 representations as

$$\begin{aligned} &\mathcal{L}_V g = 0, \qquad d\left(f^2 e^1\right) = d\left(f^2 e^5\right) = 0, \qquad (Z_+)_i = -\frac{1}{12}\left[(Y_+)_i - (Y_-)_i\right], \\ &(Y_1)_i = -4(\nabla_i e_-)_+, \qquad (Y_+)_i = -4(\nabla_i e_+)_1, \qquad (Y_-)_i = 4(\nabla_i e_-)_1, \\ &\delta^{ij}(\nabla_i e_+)_j = (\nabla_1 e_+)_1, \qquad \delta^{ij}(\nabla_i e_1)_j = -\frac{1}{2}\left[(\nabla_- e_-)_1 + (\nabla_+ e_+)_1\right], \\ &X_1 = 2\left[(\nabla_+ e_1)_+ - (\nabla_- e_1)_-\right], \qquad X_2 = 0, \qquad X_4 = \theta_\varphi = 0, \end{aligned} \tag{6.4}$$

where the one-form associated with V is $\kappa = f^2 e^0$,

$$(Y_r)_i = \frac{1}{6}\nabla_r \varphi_{jkl} \star \varphi^{jkl}{}_i, \quad (Z_r)_i = \frac{1}{6}(de_r)_{kl}\varphi^{kl}{}_i, \quad r = -, +, 1, \tag{6.5}$$

and X_1, X_2, X_3 and X_4 have been defined in Appendix B.3. The $\star$ Hodge duality operation is with respect to the "transverse" volume form $d\,\mathrm{vol} = e^2 \wedge e^3 \wedge e^4 \wedge e^6 \wedge \cdots \wedge e^9$. Clearly V is a time-like Killing vector field as may have been expected from the results of [9,10]. The class X_3, which is associated with the traceless symmetric representation of G_2, is not restricted by the supersymmetry conditions.

To show that $X_2 = 0$, one has to compute

$$\Pi_{\mathbf{14}}(\star\tilde{d} \star \varphi), \tag{6.6}$$

and demonstrate that it vanishes using the conditions, where $\tilde{d}$ denotes the exterior derivative restricted to the transverse directions, and $\Pi_{\mathbf{14}}$ is the projection to the 14-dimensional representation in the decomposition of $\Lambda^2(\mathbb{R}^8) = \Lambda_{\mathbf{7}} \oplus \Lambda_{\mathbf{14}}$ in G_2 irreducible representations, $\Lambda_{\mathbf{14}} = \mathfrak{g}_2$. In particular,

$$(\Pi_{\mathbf{14}}\gamma)_{ij} = \frac{4}{6}\left(\frac{1}{4} \star \varphi^{kl}{}_{ij}\gamma_{kl} + \gamma_{ij}\right), \quad \gamma \in \Lambda^2\left(\mathbb{R}^8\right). \tag{6.7}$$

Moreover, one can immediately see using (C.3) that

$$\tilde{d} \star \varphi = 0. \tag{6.8}$$

Thus the transverse directions are *co-symplectic* or *co-calibrated*. Moreover, it turns out that using the conditions (6.4), one can show that $\mathcal{L}_V \varphi = 0$.

It is sometimes more convenient to re-express the conditions on the geometry as

$$\begin{aligned}
&\mathcal{L}_V g = 0, \qquad d\big(f^2 e^1\big) = d\big(f^2 e^5\big) = 0, \qquad \mathcal{L}_V \varphi = 0, \\
&(Y_1)_i = -4(\nabla_i e_-)_+, \qquad (Y_+)_i = -4(\nabla_i e_+)_1, \qquad (Y_-)_i = 4(\nabla_i e_-)_1, \\
&\delta^{ij}(\nabla_i e_+)_j = (\nabla_1 e_+)_1, \qquad \delta^{ij}(\nabla_i e_1)_j = -\frac{1}{2}\big[(\nabla_- e_-)_1 + (\nabla_+ e_+)_1\big], \\
&X_1 = 2\big[(\nabla_+ e_1)_+ - (\nabla_- e_1)_-\big], \qquad \tilde{d} \star \varphi = 0.
\end{aligned} \tag{6.9}$$

Observe that the above constraints imply that f, e^+, e^-, e^1 and φ are all invariant under the action of V. Therefore V preserves the G_2 structure of spacetime.

One can introduce some special coordinates on the spacetime. Solving the closure conditions, we write $e^1 = f^{-2} dx^2$ and $e^5 = f^{-2} dx^1$, and adapt a coordinate t along the Killing vector field V, $V = \partial/\partial t$, where the functions x^1 and x^2 can be thought of as spacetime coordinates. The metric of the spacetime M can be written as

$$ds^2 = -f^4(dt + m)^2 + f^{-4} \sum_{S=1}^{2} \big(dx^S\big)^2 + ds_7^2, \quad ds_7^2 = \delta_{ij} e^i e^j, \tag{6.10}$$

where ds_7^2 is the metric along the "transverse" directions, $e^i = e^i_S\, dx^S + e^i_I\, dy^I$, $m = V_S\, dx^S + m_I\, dy^I$, and y^I are the remaining coordinates of the spacetime.

The spacetime is foliated by eight-dimensional Lorentzian manifolds N given by $x^S = \text{const}$. In turn N is a real line bundle over a seven-dimensional base space B. A special case arises whenever $m_I = 0$. In such a case M is foliated by seven-dimensional manifolds B given by $t, x^S = \text{const}$. Moreover the conditions (6.9) imply that B is a co-symplectic G_2 manifold. This is perhaps the most significant geometric property of this class of IIB supersymmetric backgrounds. As we shall see the D3-brane and the LLM solutions are of this type.

6.2. Fluxes

To find the conditions that supersymmetry imposes on the flux F in (6.1), it is convenient to write

$$\Phi = e^1 \wedge \alpha - \star\alpha, \qquad \Psi = e^1 \wedge \beta + \star\beta, \qquad \mathcal{X} = e^1 \wedge \gamma + \delta, \tag{6.11}$$

where α, β, δ are three-forms and γ is a two-form. Then using the decomposition $\Lambda^3(\mathbb{R}^7) = \Lambda^3_{\mathbf{1}} \oplus \Lambda^3_{\mathbf{7}} \oplus \Lambda^3_{\mathbf{27}}$ in G_2 irreducible representations, α, β, δ can be written as

$$\begin{aligned}
&\alpha = p_1\varphi + v_1 \bar{\wedge} \star \varphi + s_1 \bar{\wedge} \varphi, \qquad \beta = p_2\varphi + v_2 \bar{\wedge} \star \varphi + s_2 \bar{\wedge} \varphi, \\
&\delta = p_3\varphi + v_3 \bar{\wedge} \star \varphi + s_3 \bar{\wedge} \varphi,
\end{aligned} \tag{6.12}$$

respectively, where s_1, s_2 and s_3 are symmetric traceless 2-tensors associated with the **27**-dimensional representation of G_2. In particular, one has

$$\begin{aligned}
&p_1 = \frac{1}{42}\varphi^{ijk}\alpha_{ijk}, \qquad (v_1)_i = -\frac{1}{24}\alpha_{jkl} \star \varphi^{jkl}{}_i, \\
&(s_1)_{ij} = \frac{1}{4}\alpha_{kl(i}\varphi^{kl}{}_{j)} - \frac{1}{28}\delta_{ij}\alpha_{klm}\varphi^{klm}
\end{aligned} \tag{6.13}$$

and similarly for the rest. Using $\Lambda^2(\mathbb{R}^7) = \Lambda_{\mathbf{7}} \oplus \mathfrak{g}_2$, one has

$$\gamma = u\bar{\wedge}\varphi + \hat{\gamma}, \tag{6.14}$$

where $\hat{\gamma} \in \mathfrak{g}_2$ and

$$u_i = \frac{1}{6}\gamma_{jk}\varphi^{jk}{}_i, \qquad \hat{\gamma}_{ij} = \gamma_{ij}^{\mathfrak{g}_2} = (\Pi_{\mathbf{14}}\gamma)_{ij}. \tag{6.15}$$

The IIB Killing spinor equations imply that all the five-form flux is determined in terms of the geometry. In particular a calculation reveals that

$$\begin{aligned}
&p_1 = \pm\frac{\sqrt{2}}{14}(\nabla_+ e_+)_1, \qquad (v_1)_i = \pm\frac{\sqrt{2}}{8}(\nabla_+ e_+)_i,\\
&(s_1)_{ij} = \mp\frac{\sqrt{2}}{8}\left[\frac{1}{2}(s_\Gamma)_{ij} - \left((\nabla_{(i} e_1)_{j)} - \frac{\delta_{ij}}{7}(\nabla_k e_1)^k\right)\right],\\
&p_2 = \mp\frac{\sqrt{2}}{14}(\nabla_- e_-)_1, \qquad (v_2)_i = \pm\frac{\sqrt{2}}{8}(\nabla_- e_-)_i,\\
&(s_2)_{ij} = \mp\frac{\sqrt{2}}{8}\left[\frac{1}{2}(s_\Gamma)_{ij} + \left((\nabla_{(i} e_1)_{j)} - \frac{\delta_{ij}}{7}(\nabla_k e_1)^k\right)\right],\\
&p_3 = \pm\frac{\sqrt{2}}{7}(\nabla_- e_-)_+, \qquad (v_3)_i = \mp\frac{\sqrt{2}}{8}(\nabla_+ e_1 - \nabla_- e_1)_i,\\
&(s_3)_{ij} = \pm\frac{\sqrt{2}}{4}\left[(\nabla_{(i} e_+)_{j)} - \frac{\delta_{ij}}{7}(\nabla_k e_+)^k\right],
\end{aligned} \tag{6.16}$$

where s_Γ is defined in Appendix B.3. Similarly, we get

$$\begin{aligned}
&u_i = \pm\frac{\sqrt{2}}{12}\big((\nabla_j e_+)_k - (\nabla_j e_-)_k\big)\varphi^{jk}{}_i,\\
&\hat{\gamma}_{ij} = \mp\frac{1}{2\sqrt{2}}\big[(\nabla_{[i} e_+)_{j]} - (\nabla_{[i} e_-)_{j]}\big]^{\mathfrak{g}_2}.
\end{aligned} \tag{6.17}$$

Substituting these expressions back into the flux and after some computation, one finds that

$$F = \Theta + {}^*\Theta, \tag{6.18}$$

where

$$\Theta = \mp\frac{1}{4}\big\{f^{-4}\kappa \wedge d\big(f^2\varphi\big) + f^{-2}e^+ \wedge e^- \wedge e^1 \wedge d\kappa\big\}, \tag{6.19}$$

and $\kappa = f^2 e^0$. This is a remarkably simple expression for the flux. To summarize the solution of the Killing spinor equations is given by (6.10) and (6.19) subject to the conditions (6.9).

7. Co-calibrated G_2 manifolds and supergravity backgrounds

7.1. Co-calibrated G_2 manifolds

Co-calibrated or co-symplectic G_2 manifolds are seven-dimensional manifolds with a G_2 structure which have the property that

$$\tilde{d} \star \varphi = 0, \tag{7.1}$$

i.e., the dual of the fundamental G_2 form is closed. If one in addition requires that $X_3 = 0$, then from the results of Appendix B.3 it follows that $\tilde{d}\varphi = \lambda \star \varphi$, $\lambda \neq 0$. These are the *weak holonomy* or *nearly parallel* G_2 manifolds. So the co-symplectic G_2 manifolds include the nearly parallel G_2 manifolds, which in turn include the tri-Sasakian ones.

Nearly parallel G_2 manifolds have appeared before in the context of eleven-dimensional supergravity compactifications, see, e.g., [34,35]. Compact homogeneous examples are the squashed 7-sphere, $Sp(2) \times Sp(1)/Sp(1) \times Sp(1)$, the Aloff–Wallach spaces $N(k,\ell) = SU(3)/U(1)$ [36] and $SO(5)/SO(3)$ [35], where the embedding of $U(1)$ in $SU(3)$ is given as $\mathrm{diag}(e^{ik\theta_1}, e^{i\ell\theta_2}, e^{-i(\theta_1+\theta_2)})$ and $SO(3)$ is maximal in $SO(5)$. These are the only strictly nearly parallel G_2 compact homogeneous manifolds [24]. However, there are additional homogeneous tri-Sasakian and Einstein–Sasakian manifolds, see, e.g., [24,37,38]. More examples of nearly parallel G_2 manifolds can be constructed by squashing the tri-Sasakian ones along the orbits of $\mathfrak{so}(3)$ [24,40]. Using the tri-Sasakian examples of [39], one can construct infinite families of such manifolds which are neither homogeneous nor tri-Sasakian.

There are co-symplectic G_2 manifolds which are *not* nearly parallel. This can be easily seen from the results of [19]. Unlike the nearly parallel case, there is no classification of compact homogeneous co-symplectic G_2 manifolds. It is expected that the class of compact homogeneous co-symplectic G_2 manifolds is large.[8]

The co-homogeneity one co-symplectic G_2 manifolds are more tractable. A classification has been given in [19]. It has been found that the principal orbits, which are six-dimensional homogeneous manifolds, are one of the following spaces

$$\begin{aligned} &S^6 = G_2/SU(3), \qquad \mathbb{C}P^3 = Sp(2)/SU(2)U(1), \qquad F_{1,2} = SU(3)/T^2, \\ &S^3 \times S^3 = SU(2)^3/SU(2) = SU(2)^2 T^1/T^1 = SU(2)^2, \\ &S^5 \times S^1 = SU(3)T^1/SU(2), \qquad S^3 \times \left(S^1\right)^3 = SU(2)T^3, \qquad \left(S^1\right)^6 = T^6, \end{aligned} \tag{7.2}$$

up to discrete identifications. The metric and fundamental form of co-homogeneity one manifolds can be written as

$$ds_7^2 = h^2(y)\,dy^2 + ds_6^2\big(\mu(y)\big), \qquad \varphi = \varphi\big(h(y), \mu(y), \theta(y)\big), \tag{7.3}$$

where μ are the homogeneous moduli of the G_2 structure which have been promoted to functions of the inhomogeneous coordinate y. We shall explain the θ dependence later. Using a y coordinate transformation, one can set $h = 1$ as was done in [19]. However, we shall not do so because in the associated supergravity backgrounds h may depend on another two coordinates. Setting $e^6 = h\,dy$, it is known that the stability subgroup of a vector in the seven-dimensional representation of G_2 is $SU(3)$. There is a unique fundamental $SU(3)$ two-form and the space of fundamental $SU(3)$ 3-forms is two-dimensional. Using this, the G_2 invariant three form can be written as

$$\varphi = \mathrm{Re}\big(e^{i\theta}\hat{\chi}\big) + e^6 \wedge \hat{\omega}, \tag{7.4}$$

where $\hat{\omega}$ is the Hermitian form and $\hat{\chi}$ is the $(3,0)$-form. The isotropy groups of the homogeneous spaces may leave more forms invariant, and so there may be many ways to construct a G_2 invariant 3-form φ. As we shall demonstrate below, the bubbling AdS solutions are examples of backgrounds based on co-symplectic co-homogeneity one G_2 manifolds whose principal orbit is

[8] We thank R. Cleyton, S. Ivanov and A. Swann for discussions on this point.

$S^3 \times S^3$. In fact one uses a very special family of such manifolds with isometry $SO(4) \times SO(4)$. Solutions based on various families with principal orbits given in (7.2) will be presented elsewhere [41].

7.2. D3-brane

The D3-brane solution can be viewed as a special case of solutions with G_2-invariant Killing spinors. For this write the metric as

$$ds^2 = -h^{-\frac{1}{2}}\,dt^2 + h^{\frac{1}{2}}\big(dx^2 + dy^2\big) + h^{\frac{1}{2}}\,dw^2 + h^{-\frac{1}{2}}\,d\mathbf{z}^2 + h^{\frac{1}{2}}\,d\mathbf{q}^2, \tag{7.5}$$

where $(t, \mathbf{z})$ are the worldvolume and $(x, y, w, \mathbf{q})$ are the transverse coordinates, respectively and h is the usual harmonic function $h = h(w, x, y, \mathbf{q})$. The time-like Killing vector field is $\kappa = \partial/\partial t$ and so $f^4 = h^{-\frac{1}{2}}$. The fundamental G_2 three-form is

$$\varphi = \mathrm{Re}\prod_{r=1}^{3}\big[\wedge\big(h^{-\frac{1}{4}}\,d\mathbf{z}^r + i h^{\frac{1}{4}}\,d\mathbf{q}^r\big)\big] - h^{\frac{1}{4}}dw \wedge \sum_{r=1}^{3} d\mathbf{z}^r \wedge d\mathbf{q}^r, \tag{7.6}$$

and $e^1 = h^{\frac{1}{4}}\,dx$ and $e^5 = h^{\frac{1}{4}}\,dy$. It is straightforward to verify that $\tilde{d} \star \varphi = 0$. Moreover $Y_r = X_1 = 0$. The remaining conditions are also satisfied. A consequence of this is that the seven-dimensional “transverse” space B admits a G_2 structure for which the only non-vanishing class is X_3. Asymptotically, B is flat space and at the origin is $H_4 \times S^3$.

7.3. Bubbling AdS solutions

The solution found by Lin, Lunin and Maldacena in [5] is an example of a G_2 IIB supergravity background and so it is associated with a co-symplectic G_2 geometry. The co-symplectic geometry is of co-homogeneity one and has principal orbit $S^3 \times S^3$. To show this, we write the metric in the form given in (6.10) by introducing a frame

$$\begin{aligned}
&e^0 = h^{-1}\big(dt + V_S\,dx^S\big), \qquad e^5 = h\,dx^1, \qquad e^1 = h\,dx^2,\\
&e^6 = h\,dy, \qquad e^{1+a} = \frac{\sqrt{y}}{2}e^{\frac{G}{2}}\sigma^a, \qquad e^{6+a} = \frac{\sqrt{y}}{2}e^{-\frac{G}{2}}\hat{\sigma}^a,
\end{aligned} \tag{7.7}$$

where $\sigma^a, \hat{\sigma}^a$, $a = 1, 2, 3$, are left-invariant 1-forms on $SU(2) \times SU(2)$ given by

$$\begin{aligned}
&\sigma^1 = -\sin\psi\,d\theta + \cos\psi\,\sin\theta\,d\phi, \qquad \sigma^2 = \cos\psi\,d\theta + \sin\psi\,\sin\theta\,d\phi,\\
&\sigma^3 = d\psi + \cos\theta\,d\phi,
\end{aligned} \tag{7.8}$$

and $\hat{\sigma}^a$ are defined in exactly the same way, but with θ, ϕ, ψ replaced with $\hat{\theta}$, $\hat{\phi}$, $\hat{\psi}$ throughout, and the rest of the components of the metric depend on y, x^S. Moreover, the functions h and G are related by

$$h^{-2} = 2y\cosh G, \tag{7.9}$$

and

$$\frac{\partial V_1}{\partial y} = \frac{1}{2y\cosh^2 G}\frac{\partial G}{\partial x^2}, \qquad \frac{\partial V_2}{\partial y} = -\frac{1}{2y\cosh^2 G}\frac{\partial G}{\partial x^1},$$
$$\frac{\partial V_2}{\partial x^1} - \frac{\partial V_1}{\partial x^2} = \frac{1}{2y\cosh^2 G}\frac{\partial G}{\partial y}. \tag{7.10}$$

Writing $z = \frac{1}{2}\tanh G$, z is further constrained by

$$\partial_S\partial_S z + y\partial_y\left(\frac{\partial_y z}{y}\right) = 0. \tag{7.11}$$

The five-form is given by

$$F = \frac{1}{8}F_{(1)} \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 - \frac{1}{8}F_{(2)} \wedge \hat{\sigma}^1 \wedge \hat{\sigma}^2 \wedge \hat{\sigma}^3, \tag{7.12}$$

where $F_{(1)}$, $F_{(2)}$ are two 2-forms. In turn these are given by

$$F_{(1)} = dB_t \wedge \left(dt + V_S\, dx^S\right) + B_t\, dV + d\tilde{B}, \tag{7.13}$$

with

$$B_t = -\frac{1}{4}y^2 e^{2G}, \qquad d\tilde{B} = -\frac{1}{4}y^3 \star_3 d\left(\frac{z+\frac{1}{2}}{y^2}\right), \tag{7.14}$$

and

$$F_{(2)} = dC_t \wedge \left(dt + V_S\, dx^S\right) + C_t\, dV + d\tilde{C}, \tag{7.15}$$

with

$$C_t = -\frac{1}{4}y^2 e^{-2G}, \qquad d\tilde{C} = -\frac{1}{4}y^3 \star_3 d\left(\frac{z-\frac{1}{2}}{y^2}\right), \tag{7.16}$$

where $\star_3$ denotes Hodge duality on $\mathbb{R}^3$ with metric $dy^2 + (dx^1)^2 + (dx^2)^2$ with positive orientation given by $dy \wedge dx^1 \wedge dx^2$.

In this basis the transverse metric is

$$ds_7^2 = \left(e^2\right) + \left(e^3\right)^2 + \left(e^4\right)^2 + \left(e^6\right)^2 + \left(e^7\right)^2 + \left(e^8\right)^2 + \left(e^9\right)^2, \tag{7.17}$$

equipped with the fundamental G_2 form

$$\varphi = \mathrm{Re}\left(e^{iH}\left(e^2 + ie^7\right) \wedge \left(e^3 + ie^8\right) \wedge \left(e^4 + ie^9\right)\right)$$
$$- e^6 \wedge \left(e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9\right), \tag{7.18}$$

where

$$e^{iH} = \frac{1}{\sqrt{e^G + e^{-G}}}\left(-e^{\frac{G}{2}} + ie^{-\frac{G}{2}}\right). \tag{7.19}$$

Observe that the expression for the fundamental form is as expected in the context of cohomogeneity one G_2 structures (7.4). The frame we have chosen is not adapted in the G_2 structure because the components of φ depend on H. However, we can adapt a frame by rotating the frame as

$$e^2 + ie^7 \to e^{\frac{iH}{3}}\left(e^2 + ie^7\right), \tag{7.20}$$

and similarly for the other two pairs while leaving e^6 as it is. Observe that the metric remains invariant under this rotation. However, in what follows in this section we shall continue to work with the frame we have originally introduced.

It is straightforward to verify that φ satisfies

$$\tilde{d} \star_7 \varphi = 0. \tag{7.21}$$

This establishes that the LLM solution is associated with a co-symplectic G_2 structure. Moreover, it is straightforward to see by setting $y, x^S = \text{const}$ that the principal orbit is $S^3 \times S^3$. Observe that this principal orbit is one of the possibilities that appear in the classification of co-homogeneity one co-symplectic manifolds in [19] and are listed in (7.2). In fact, it is associated with the most symmetric family of such co-symplectic structures which has $SO(4) \times SO(4)$ symmetry. There are families of the same orbit that possess fewer isometries. Although φ is co-closed on the transverse space, $\tilde{d}\varphi \neq 0$, and we find that

$$X_1 = \frac{4\sqrt{y}}{\sqrt{e^G + e^{-G}}} \frac{\partial G}{\partial y}. \tag{7.22}$$

The function f appearing in (6.10) is given by

$$f^4 = 2y \cosh G. \tag{7.23}$$

It remains to find the forms α, β, δ which appear in the decomposition of F, (6.11). These are

$$\begin{aligned}
\alpha &= \frac{1}{2\sqrt{2y}} e^{-\frac{G}{2}} \left(y e^G \frac{\partial G}{\partial x^2} + 1 - y \frac{\partial G}{\partial y} \right) e^2 \wedge e^3 \wedge e^4 \\
&\quad + \frac{1}{2\sqrt{2y}} e^{\frac{G}{2}} \left(y e^{-G} \frac{\partial G}{\partial x^2} + 1 + y \frac{\partial G}{\partial y} \right) e^7 \wedge e^8 \wedge e^9, \\
\beta &= \frac{1}{2\sqrt{2y}} e^{-\frac{G}{2}} \left(-y e^G \frac{\partial G}{\partial x^2} + 1 - y \frac{\partial G}{\partial y} \right) e^2 \wedge e^3 \wedge e^4 \\
&\quad + \frac{1}{2\sqrt{2y}} e^{\frac{G}{2}} \left(-y e^{-G} \frac{\partial G}{\partial x^2} + 1 + y \frac{\partial G}{\partial y} \right) e^7 \wedge e^8 \wedge e^9,
\end{aligned} \tag{7.24}$$

and

$$\delta = \frac{1}{2}\sqrt{y} e^{\frac{G}{2}} \frac{\partial G}{\partial x^1} e^2 \wedge e^3 \wedge x^4 + \frac{1}{2}\sqrt{y} e^{-\frac{G}{2}} \frac{\partial G}{\partial x^1} e^7 \wedge e^8 \wedge x^9. \tag{7.25}$$

Moreover, γ vanishes. It is then straightforward to verify, using (7.10), that all of the remaining geometric constraints are satisfied, and the components of F are determined in terms of the spin connection as set out in the previous section.

There are other solutions which are extensions of the LLM solution and so they are associated with a co-symplectic G_2 structure. In particular observe that (6.10) is compatible with some recent results in [28] where bubbling solutions have been investigated that preserve 4, 8 and 16 supersymmetries.

8. Concluding remarks

IIB backgrounds with $N \geqslant 2$ supersymmetry and active five-form flux have four distinct types of geometries. Killing spinors with isotropy group $SU(4) \ltimes \mathbb{R}^8$ give rise to two types of geometry

Table 1
The columns contain the stability subgroups of the Killing spinors in $Spin(9,1)$, the type of transverse geometry, and the conditions on the fundamental forms, respectively

Stab	Transverse structure	Conditions
$SU(4)\ltimes\mathbb{R}^8$ pure	relatively balanced $SU(4)$ Hermitian	$\tilde{d}\omega^3 = W_4\wedge\omega^3$, $W_1 = W_2 = 0$, $W_4 = W_5$
$Spin(7)\ltimes\mathbb{R}^8$	holonomy $Spin(7)$	$\tilde{d}\phi = 0$
G_2	co-symplectic G_2	$\tilde{d}\star\varphi = 0$

depending on whether the Killing spinors are generic or pure. In both cases, the spacetime admits a null Killing vector field and an almost complex structure on the transverse directions to the lightcone. For generic backgrounds, the twist of the Killing vector field is trivial but the almost complex structure non-integrable, while in the pure case the twist is non-trivial but the almost complex structure is integrable. In the latter case, if one assumes that the twist of the Killing vector field is trivial, then all the geometric conditions take a very simple form. In particular, the cubic power of a certain rescaled Hermitian form and a certain rescaled fundamental (4, 0)-form must be closed. Examples of backgrounds that admit pure Killing spinors are the D3-brane and the lifts of supersymmetric backgrounds of gauged five-dimensional supergravity which include the AdS_5 black holes. The null Killing vector field of the D3-brane background has a trivial twist but that of the AdS_5 black hole does not. $N\geqslant 2$ backgrounds with Killing spinors which have isotropy group $Spin(7)\ltimes\mathbb{R}^8$ are pp-waves propagating on a manifold with holonomy contained in $Spin(7)$.

The remaining type of geometry is associated with IIB backgrounds that admit G_2-invariant Killing spinors. The tangent bundle of such spacetimes decomposes with respect to the spacetime metric into a trivial bundle of rank three and a bundle of rank seven which are the transverse directions of the spacetime. One direction in the trivial bundle is associated with a time-like Killing vector field and the other two directions are associated with closed one forms. The transverse geometry is that of a co-symplectic G_2 manifold. We give a description of how one can construct a supersymmetric IIB background from a family of co-symplectic G_2 manifolds. In addition, we show that the co-symplectic G_2 manifold associated with the bubbling AdS solution of [5] has principal orbit $S^3\times S^3$. The transverse structures of the supersymmetric backgrounds are summarized in Table 1.

The holonomy of the supercovariant connection of IIB supergravity with only F flux is contained in $SL(16,\mathbb{C})$. This can be easily seen from the computation of the supercovariant curvature for this case in [23]. This is a subgroup of $SL(32,\mathbb{R})$ which is the holonomy group of the IIB supercovariant connection [25]. Arguments based on the reduction of holonomy in the presence of parallel spinors similar to those of [26,27] suggest that there may exist supersymmetric IIB backgrounds with F flux with any even number of supersymmetries. We have identified the geometries of $N=2$ backgrounds and the $N=32$, maximally supersymmetric, backgrounds have been classified in [23]. It is also tractable to investigate the existence of $N=30$ backgrounds by applying the technique developed in [29] to this case. If backgrounds exist with strictly $N=30$ supersymmetry, they are expected to be homogeneous spaces [31]. We hope to report the outcome of this investigation in the future.

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Appendix A. Linear systems

A.1. Solution of the linear system of generic $SU(4) \ltimes \mathbb{R}^8$ backgrounds

The Killing spinor is given in (2.2). For generic backgrounds, we take $f, g_1, g_2 \neq 0$ and $f \neq \pm g_2$. The conditions that arise from the Killing spinor equations for supersymmetric backgrounds with F flux, $P = G = 0$, and $SU(4) \ltimes \mathbb{R}^8$-invariant Killing spinors can be easily read from those of $N = 1$ backgrounds in [9]. First consider the conditions that arise from the gravitino Killing spinor equation $\mathcal{D}_+\epsilon = 0$. These can be written as

$$\begin{aligned}
&\Omega_{+,\alpha\beta} = 0, \qquad \Omega_{+,\beta}{}^{\beta} = 0, \qquad \partial_+ f + \frac{1}{2}\Omega_{+,-+} f = 0,\\
&\partial_+ g_1 + \frac{1}{2}\Omega_{+,-+} g_1 = 0, \qquad \partial_+ g_2 + \frac{1}{2}\Omega_{+,-+} g_2 = 0,\\
&\Omega_{+,+\alpha} = \Omega_{+,+\bar\alpha} = 0.
\end{aligned} \tag{A.1}$$

Some of the conditions associated with the gravitino Killing spinor equations $\mathcal{D}_\alpha\epsilon = \mathcal{D}_{\bar\alpha}\epsilon = 0$ give

$$\begin{aligned}
&\Omega_{\alpha,+\beta} = 0, \qquad F_{+\bar\alpha\beta_1\beta_2\beta_3} = 0, \qquad \Omega_{\bar\alpha,+\beta} = 0, \qquad F_{+\bar\alpha\beta\gamma}{}^{\gamma} = 0,\\
&2\Omega_{\bar\alpha,\beta_1\beta_2}\big(f^2 - (g_2 + ig_1)^2\big) - \big(f^2 + g_1^2 + g_2^2\big)\Omega_{\bar\alpha,\bar\gamma_1\bar\gamma_2}\epsilon^{\bar\gamma_1\bar\gamma_2}{}_{\beta_1\beta_2} = 0,\\
&4iF_{-+\bar\alpha\beta_1\beta_2} + 4ig_{\bar\alpha[\beta_1}F_{\beta_2]-+\delta}{}^{\delta} + 2\Omega_{\bar\alpha,\bar\gamma_1\bar\gamma_2}\epsilon^{\bar\gamma_1\bar\gamma_2}{}_{\beta_1\beta_2}\frac{fg_2}{f^2 - (g_2 + ig_1)^2} = 0.
\end{aligned} \tag{A.2}$$

Next some of the conditions of $\mathcal{D}_-\epsilon = 0$, their complex conjugate and dual give

$$\begin{aligned}
&-4ifg_2F_{-\bar\alpha_1\bar\alpha_2\beta}{}^{\beta} + 2\big[f^2 + g_1^2 + g_2^2\big]\Omega_{-,\bar\alpha_1\bar\alpha_2} - \big[f^2 - (g_2 + ig_1)^2\big]\Omega_{-,\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar\alpha_1\bar\alpha_2} = 0,\\
&\big[f^2 + g_1^2 + g_2^2\big]\Omega_{-,+\bar\alpha} - 2ifg_2F_{-+\bar\alpha\beta}{}^{\beta} = 0,\\
&\big[f^2 - (g_2 - ig_1)^2\big]\Omega_{-,+\bar\alpha} - \frac{2i}{3}fg_2F_{-+\gamma_1\gamma_2\gamma_3}\epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar\alpha} = 0.
\end{aligned} \tag{A.3}$$

Some of the $\mathcal{D}_-\epsilon = 0$ and some of the remaining $\mathcal{D}_\alpha\epsilon = \mathcal{D}_{\bar\alpha}\epsilon = 0$ conditions give

$$\begin{aligned}
&\left[D_{\bar\alpha} + \frac{1}{2}\Omega_{\bar\alpha,\beta}{}^{\beta} + \frac{1}{2}\Omega_{\bar\alpha,-+} + \Omega_{-,+\bar\alpha} + iF_{-+\bar\alpha\beta}{}^{\beta}\right](f - g_2 + ig_1) = 0,\\
&\left[D_{\bar\alpha} + \frac{1}{2}\Omega_{\bar\alpha,\beta}{}^{\beta} + \frac{1}{2}\Omega_{\bar\alpha,-+} + \Omega_{-,+\bar\alpha} - iF_{-+\bar\alpha\beta}{}^{\beta}\right](f + g_2 - ig_1) = 0,\\
&D_{\bar\alpha}\big(f^2 + g_1^2 + g_2^2\big) + [\Omega_{\bar\alpha,-+} + \Omega_{-,\bar\alpha+}]\big(f^2 + g_1^2 + g_2^2\big) = 0.
\end{aligned} \tag{A.4}$$

In addition, one finds that

$$\begin{aligned}
&D_{\bar\alpha}\log\big[f^2 - (g_2 + ig_1)^2\big] - \Omega_{\bar\alpha,\beta}{}^{\beta} + \Omega_{\bar\alpha,-+} = 0,\\
&D_{\bar\alpha}\log\left[\frac{f + g_2 + ig_1}{f - g_2 - ig_1}\right] - 2iF_{-+\bar\alpha\beta}{}^{\beta} = 0.
\end{aligned} \tag{A.5}$$

From (A.4), one gets

$$D_{\bar\alpha}\log\big[f^2-(g_2-ig_1)^2\big]+\Omega_{\bar\alpha,\beta}{}^{\beta}+\Omega_{\bar\alpha,-+}+2\Omega_{-,+\bar\alpha}=0,$$
$$D_{\bar\alpha}\log\left[\frac{f-g_2+ig_1}{f+g_2-ig_1}\right]+2iF_{-+\bar\alpha\beta}{}^{\beta}=0. \tag{A.6}$$

Thus

$$D_{\bar\alpha}\log\left[\frac{(f+ig_1)^2-g_2^2}{(f-ig_1)^2-g_2^2}\right]=0, \qquad 4iF_{-+\bar\alpha\beta}{}^{\beta}+D_{\bar\alpha}\log\frac{(f-g_2)^2+g_1^2}{(f+g_2)^2+g_1^2}=0. \tag{A.7}$$

Comparing with (A.3), one finds that

$$\Omega_{-,+\bar\alpha}-\frac{fg_2}{2\big(f^2+g_1^2+g_2^2\big)}D_{\bar\alpha}\log\left[\frac{(f+g_2)^2+g_1^2}{(f-g_2)^2+g_1^2}\right]=0. \tag{A.8}$$

Furthermore

$$D_{\bar\alpha}\log\big[\big((f-g_2)^2+g_1^2\big)\big((f+g_2)^2+g_1^2\big)\big]+2\Omega_{\bar\alpha,-+}+2\Omega_{-,+\bar\alpha}=0,$$
$$D_{\bar\alpha}\log\left[\frac{f^2-(g_2-ig_1)^2}{f^2-(g_2+ig_1)^2}\right]+2\Omega_{\bar\alpha,\beta}{}^{\beta}+2\Omega_{-,+\bar\alpha}=0. \tag{A.9}$$

Taking the trace of the last two equations in (A.2), one gets the additional geometric conditions

$$-\Omega_{\bar\beta,\alpha}{}^{\bar\beta}+\frac{\big(f^2+g_1^2+g_2^2\big)^2}{4f^2g_2^2}\Omega_{-,+\alpha}=0,$$
$$\frac{\big(f^2+g_1^2+g_2^2\big)}{2fg_2}\Omega_{-,+\alpha}+\frac{fg_2}{f^2-(g_2+ig_1)^2}\Omega_{\bar\gamma_1,\bar\gamma_2\bar\gamma_3}\epsilon^{\bar\gamma_1\bar\gamma_2\bar\gamma_3}{}_{\alpha}=0. \tag{A.10}$$

The remaining $\mathcal{D}_-\epsilon=0$ equations give

$$\partial_-\big(f^2-(g_2-ig_1)^2\big)+\frac{i}{3}fg_2F_{-\gamma_1\ldots\gamma_4}\epsilon^{\gamma_1\ldots\gamma_4}$$
$$+\big[\Omega_{-,\gamma}{}^{\gamma}+\Omega_{-,-+}\big]\big(f^2-(g_2-ig_1)^2\big)=0,$$
$$-2ig_2\partial_-g_1+2ig_1\partial_-g_2+\big(f^2+g_1^2+g_2^2\big)\Omega_{-,\alpha}{}^{\alpha}-ifg_2F_{-\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}=0,$$
$$\partial_-\big[f^2+g_1^2+g_2^2\big]+\Omega_{-,-+}\big[f^2+g_1^2+g_2^2\big]=0. \tag{A.11}$$

We have not been able to simplify the solution of the linear system further. These conditions have been expressed in terms of the fundamental $SU(4)$ forms in Section 3.

A.2. Pure $SU(4)\ltimes\mathbb{R}^8$ Killing spinors

A.2.1. Solution of the linear system

We choose the Killing spinor to be $\epsilon=h1$. Then after some straightforward computation using the results of [9], the linear system arising from the Killing spinor equations can be solved. The conditions on the geometry are

$$\partial_+h+\frac{1}{2}\Omega_{+,-+}h=0, \qquad \partial_-h+\frac{1}{2}\Omega_{-,-+}h=0, \qquad \partial_\alpha h+\frac{1}{2}(\Omega_{\alpha,-+}+\Omega_{-,\alpha+})h=0,$$
$$\Omega_{+,\beta}{}^{\beta}=0, \qquad \Omega_{+,\alpha\beta}=0, \qquad \Omega_{\alpha,+}{}^{\alpha}=0,$$
$$\Omega_{+,+\alpha}=0, \qquad \Omega_{\alpha,+\beta}=0, \qquad \Omega_{\alpha,+\bar\beta}+\Omega_{\bar\beta,+\alpha}=0,$$
$$\Omega_{\alpha,\beta\gamma}=0, \qquad \Omega_{\bar\alpha,\beta}{}^{\beta}+\Omega_{\beta,\bar\alpha}{}^{\beta}=0, \qquad \Omega_{\bar\alpha,\beta}{}^{\beta}=-\Omega_{-,+\bar\alpha}. \tag{A.12}$$

In addition, one finds the following restrictions on the fluxes

$$
\begin{aligned}
&F_{+\alpha\bar\beta_1\bar\beta_2\bar\beta_3}=0, \qquad iF_{\alpha+\bar\beta\gamma}{}^{\gamma}+\Omega_{\alpha,+\bar\beta}=0, \qquad F_{\alpha\bar\beta_1\bar\beta_2\bar\beta_3\bar\beta_4}=F_{-+\bar\beta_1\bar\beta_2\bar\beta_3}=0,\\
&iF_{\bar\alpha-+\beta}{}^{\beta}=\Omega_{\bar\alpha,\beta}{}^{\beta}, \qquad 2iF_{\alpha-+\bar\beta_1\bar\beta_2}-2ig_{\alpha[\bar\beta_1}F_{\bar\beta_2]-+\gamma}{}^{\gamma}+\Omega_{\alpha,\bar\beta_1\bar\beta_2}=0,\\
&F_{-\alpha_1\alpha_2\alpha_3\alpha_4}=0, \qquad \Omega_{-,\bar\beta\bar\gamma}+iF_{-\bar\beta\bar\gamma\delta}{}^{\delta}=0,\\
&\Omega_{-,\beta}{}^{\beta}+\frac{i}{2}F_{-\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}=0, \qquad \Omega_{-,+\alpha}+iF_{-+\alpha\beta}{}^{\beta}=0.
\end{aligned}
\tag{A.13}
$$

The above conditions on the geometry and the fluxes imposed by supersymmetry have been re-expressed in terms of the fundamental $SU(4)$ forms in Section 4.

A.2.2. *Spin connection*

The non-vanishing components of the spin connection along the transverse directions of the spacetime associated with the uplift of the five-dimensional supersymmetric solution can be expressed as

$$
\begin{aligned}
&\Omega_{a,\nu\bar\rho}=-\frac{2i}{3}k_2\mathcal{P}_a\delta_{\nu\bar\rho}, \qquad \Omega_{\mu,b\bar c}=-\frac{2i}{3}k_1\mathcal{Q}_\mu\delta_{b\bar c},\\
&\Omega_{a,bc}=e^{-3ik_1\chi_2}\mathcal{F}^{\frac12}\hat\Omega_{\hat a,\hat b\hat c},\\
&\Omega_{a,b\bar c}=-\frac12\mathcal{F}^{-1}\nabla_b\mathcal{F}\delta_{a\bar c}+\frac12\mathcal{F}^{-1}\nabla_a\mathcal{F}\delta_{b\bar c}-\frac{2i}{3}k_1\mathcal{P}_a\delta_{b\bar c}+\mathcal{F}^{\frac12}e^{-ik_1\chi_2}\hat\Omega_{\hat a,\hat b\hat{\bar c}},\\
&\Omega_{a,\bar b\bar c}=\frac12\mathcal{F}^{-1}\nabla_{\bar b}\mathcal{F}\delta_{a\bar c}-\frac12\mathcal{F}^{-1}\nabla_{\bar c}\mathcal{F}\delta_{a\bar b}+e^{ik_1\chi_2}\mathcal{F}^{\frac12}\hat\Omega_{\hat a,\hat{\bar b}\hat{\bar c}}.
\end{aligned}
\tag{A.14}
$$

Here hatted and unhatted frame indices are with respect to the bases e^a and $\hat e^a$, and $\hat\Omega_{\hat a,\hat b\hat c}$, $\hat\Omega_{\hat a,\hat b\hat{\bar c}}$, $\hat\Omega_{\hat a,\hat{\bar b}\hat{\bar c}}$ are the components of the spin connection of N with respect to the basis $\hat e^a$. As N is Kähler with Kähler form J_N, we have $\hat\Omega_{\hat a,\hat b\hat c}=0$ and $\hat\Omega_{\hat a,\hat{\bar b}\hat{\bar c}}=0$.

Similarly, we have

$$
\begin{aligned}
&\Omega_{\mu,\nu\rho}=e^{-3ik_2\chi_2}\ell^{-1}\hat\Omega_{\hat\mu,\hat\nu\hat\rho},\\
&\Omega_{\mu,\nu\bar\rho}=-\frac{2i}{3}\mathcal{Q}_\mu\delta_{\nu\bar\rho}+e^{-ik_2\chi_2}\ell^{-1}\hat\Omega_{\hat\mu,\hat\nu\hat{\bar\rho}},\\
&\Omega_{\mu,\bar\nu\bar\rho}=e^{ik_2\chi_2}\ell^{-1}\hat\Omega_{\hat\mu,\hat{\bar\nu}\hat{\bar\rho}},
\end{aligned}
\tag{A.15}
$$

for the directions along CP^2.

A.3. *Solution of the linear system of Spin(7)* $\ltimes\mathbb{R}^8$ *backgrounds*

The Killing spinor is $\epsilon=(f+ig)(1+e_{1234})$, i.e., it is given as in (2.2) by setting $g_2=0$ and $g_1=g$. A straightforward computation using the results of [9] reveals that the conditions on the geometry are

$$
\begin{aligned}
&\partial_+(f+ig)+\frac12\Omega_{+,-+}(f+ig)=0,\\
&\partial_-\big(f^2+g^2\big)+\Omega_{-,-+}\big(f^2+g^2\big)=0,\\
&\partial_{\bar\alpha}f+\frac12\Omega_{\bar\alpha,-+}f=0, \qquad \partial_{\bar\alpha}g+\frac12\Omega_{\bar\alpha,-+}g=0,
\end{aligned}
$$

$$
\begin{aligned}
&\Omega_{+,\bar\beta_1\bar\beta_2}=\frac{1}{2}\epsilon_{\bar\beta_1\bar\beta_2}{}^{\gamma_1\gamma_2}\Omega_{+,\gamma_1\gamma_2}, \qquad \Omega_{+,\gamma}{}^{\gamma}=0, \qquad \Omega_{+,+\alpha}=0,\\
&\Omega_{\alpha,+\beta}=0, \qquad \Omega_{\alpha,+\bar\beta}=0,\\
&\Omega_{-,\beta_1\beta_2}-\frac{1}{2}\epsilon_{\beta_1\beta_2}{}^{\bar\gamma_1\bar\gamma_2}\Omega_{-,\bar\gamma_1\bar\gamma_2}=0, \qquad \Omega_{-,\beta}{}^{\beta}=0, \qquad \Omega_{-,+\alpha}=0,\\
&\Omega_{\bar\alpha,\beta_1\beta_2}-\frac{1}{2}\epsilon_{\beta_1\beta_2}{}^{\bar\gamma_1\bar\gamma_2}\Omega_{\bar\alpha,\bar\gamma_1\bar\gamma_2}=0, \qquad \Omega_{\bar\alpha,\beta}{}^{\beta}=0,
\end{aligned}
\tag{A.16}
$$

and the conditions on the fluxes are

$$
\begin{aligned}
&F_{+\bar\alpha\beta_1\beta_2\beta_3}=0, \qquad F_{+\bar\alpha\beta\gamma}{}^{\gamma}=0, \qquad F_{-+\gamma_1\gamma_2\gamma_3}=0,\\
&F_{-\bar\beta_1\bar\beta_2\delta}{}^{\delta}=-\frac{1}{2}\epsilon_{\bar\beta_1\bar\beta_2}{}^{\gamma_1\gamma_2}F_{-\gamma_1\gamma_2\delta}{}^{\delta}, \qquad F_{-+\bar\alpha\beta_1\beta_2}=0,\\
&\partial_-\log\left(\frac{f+ig}{f-ig}\right)+\frac{i}{2}F_{-\gamma}{}^{\gamma}{}_{\delta}{}^{\delta}+\frac{i}{6}F_{-\gamma_1\ldots\gamma_4}\epsilon^{\gamma_1\cdots\gamma_4}.
\end{aligned}
\tag{A.17}
$$

As a special case suppose that a background admits a Majorana–Weyl Killing spinor. One can use the gauge symmetry to write $\epsilon=f(1+e_{1234})$. The conditions are then easily derived and they can be read from the ones above by setting $g=0$. The conditions above on the geometry and fluxes can be re-expressed in terms of the fundamental $Spin(7)$ form in Section 5.

A.4. Solution of the linear system of G_2 backgrounds

The Killing spinor is $\epsilon=f(1+e_{1234})+ig(e_{51}+e_{5234})$, where f and g are real spacetime functions. The equations that arise from the linear system are simplified considerably by making a gauge transformation of the form $e^{b\Gamma_{+-}}$ to set $g^2=f^2$. We therefore take $g=\pm f$. Unlike the previous cases, the computation for the G_2 case is more involved and it is convenient to organize the conditions in terms of $SU(3)\subset G_2$ representations. For this, we choose a pseudo-Hermitian frame $(e^+,e^-,e^1,e^{\bar 1},e^p,e^{\bar p})$, $p=2,3,4$, on the spacetime which is adapted to the choice of the Killing spinors. The linear system can be easily derived from that of [10] by setting the one-form and three-form field strengths to zero. The conditions that arise from the linear system are as follows:

The conditions on the geometry and fluxes which transform as $SU(3)$ singlets are

$$
\begin{aligned}
&F_{+-1p}{}^{p}=\mp\frac{1}{2}\big(\Omega_{1,-1}+\Omega_{\bar 1,-\bar 1}+\Omega_{+,p}{}^{p}-\Omega_{-,p}{}^{p}\big),\\
&F_{-1\bar 1p}{}^{p}=\pm\frac{1}{4}\big(\Omega_{1,p}{}^{p}-\Omega_{\bar 1,p}{}^{p}+\Omega_{1,1\bar 1}-\Omega_{\bar 1,1\bar 1}+\Omega_{-,-1}+\Omega_{-,-\bar 1}\big),\\
&F_{+1\bar 1p}{}^{p}=\pm\frac{1}{4}\big(\Omega_{1,p}{}^{p}-\Omega_{\bar 1,p}{}^{p}-\Omega_{1,1\bar 1}+\Omega_{\bar 1,1\bar 1}-\Omega_{+,+1}-\Omega_{+,+\bar 1}\big),\\
&F_{-1234}=\pm\frac{1}{8}\big(-\Omega_{1,p}{}^{p}+\Omega_{\bar 1,p}{}^{p}-\Omega_{1,1\bar 1}+\Omega_{\bar 1,1\bar 1}+3\Omega_{-,-1}-\Omega_{-,-\bar 1}\big),\\
&F_{+-234}=\pm\frac{1}{4}\big(\Omega_{1,-\bar 1}+\Omega_{\bar 1,-1}+\Omega_{+,p}{}^{p}+\Omega_{-,p}{}^{p}\big),\\
&F_{+1\bar 2\bar 3\bar 4}=\pm\frac{1}{8}\big(-\Omega_{1,p}{}^{p}+\Omega_{\bar 1,p}{}^{p}+\Omega_{1,1\bar 1}-\Omega_{\bar 1,1\bar 1}-3\Omega_{+,+1}+\Omega_{+,+\bar 1}\big),\\
&\Omega_{1,p}{}^{p}+\Omega_{\bar 1,p}{}^{p}=\frac{1}{2}(\Omega_{+,+1}-\Omega_{+,+\bar 1}-\Omega_{-,-1}+\Omega_{-,-\bar 1}),
\end{aligned}
$$

$$\begin{aligned}
&\Omega_{1,1\bar{1}}+\Omega_{\bar{1},1\bar{1}}=-\frac{1}{2}(\Omega_{+,+1}-\Omega_{+,+\bar{1}}+\Omega_{-,-1}-\Omega_{-,-\bar{1}}),\\
&\Omega_{1,-+}=\frac{1}{2}(\Omega_{+,+1}-\Omega_{-,-1}),\\
&\Omega_{1,-\bar{1}}-\Omega_{\bar{1},-1}=-\Omega_{+,p}{}^{p}-\Omega_{-,p}{}^{p},\\
&\Omega_{1,+1}=\frac{1}{2}\big(\Omega_{1,-1}+\Omega_{\bar{1},-\bar{1}}+\Omega_{+,p}{}^{p}-\Omega_{-,p}{}^{p}\big),\\
&\Omega_{1,+\bar{1}}=\frac{1}{2}\big(\Omega_{1,-\bar{1}}+\Omega_{\bar{1},-1}+\Omega_{+,p}{}^{p}+\Omega_{-,p}{}^{p}\big),\\
&\Omega_{p,}{}^{p}{}_{\bar{1}}+\Omega_{\bar{p},}{}^{\bar{p}}{}_{1}=\frac{1}{2}\big(\Omega_{1,p}{}^{p}-\Omega_{\bar{1},p}{}^{p}+\Omega_{1,1\bar{1}}-\Omega_{\bar{1},1\bar{1}},\\
&\qquad\qquad-\Omega_{p_1,p_2p_3}\epsilon^{p_1p_2p_3}-\Omega_{\bar{p}_1,\bar{p}_2\bar{p}_3}\epsilon^{\bar{p}_1\bar{p}_2\bar{p}_3}+\Omega_{-,-1}+\Omega_{-,-\bar{1}}\big),\\
&\Omega_{p,}{}^{p}{}_{+}=-\frac{1}{2}\big(\Omega_{1,-1}+\Omega_{\bar{1},-\bar{1}}+\Omega_{+,p}{}^{p}-\Omega_{-,p}{}^{p}\big),\\
&\Omega_{p,}{}^{p}{}_{-}=-\frac{1}{2}\big(\Omega_{1,-1}+\Omega_{\bar{1},-\bar{1}}-\Omega_{+,p}{}^{p}+\Omega_{-,p}{}^{p}\big),\\
&\Omega_{p_1,p_2p_3}\epsilon^{p_1p_2p_3}-\Omega_{\bar{p}_1,\bar{p}_2\bar{p}_3}\epsilon^{\bar{p}_1\bar{p}_2\bar{p}_3}=-2\big(\Omega_{p,}{}^{p}{}_{\bar{1}}-\Omega_{\bar{p},}{}^{\bar{p}}{}_{1}\big),\\
&\Omega_{p,}{}^{p}{}_{1}=\frac{1}{4}\big(\Omega_{p_1,p_2p_3}\epsilon^{p_1p_2p_3}+\Omega_{\bar{p}_1,\bar{p}_2\bar{p}_3}\epsilon^{\bar{p}_1\bar{p}_2\bar{p}_3}-\Omega_{1,p}{}^{p}+\Omega_{\bar{1},p}{}^{p},\\
&\qquad+\Omega_{+,+1}+\Omega_{+,+\bar{1}}+\Omega_{1,1\bar{1}}-\Omega_{\bar{1},1\bar{1}}\big)-\frac{1}{2}\big(\Omega_{p,}{}^{p}{}_{\bar{1}}-\Omega_{\bar{p},}{}^{\bar{p}}{}_{1}\big),\\
&\Omega_{-,-+}=-\frac{1}{2}(\Omega_{1,-1}+\Omega_{\bar{1},-\bar{1}}+\Omega_{\bar{1},-1}+\Omega_{1,-\bar{1}}),\\
&\Omega_{\bar{1},-\bar{1}}-\Omega_{1,-1}=\Omega_{-,p}{}^{p}-\Omega_{+,p}{}^{p},\\
&\Omega_{-,1\bar{1}}=-\Omega_{-,p}{}^{p},\\
&\Omega_{+,1\bar{1}}=\Omega_{+,p}{}^{p},\\
&\Omega_{+,-+}=\frac{1}{2}(\Omega_{1,-1}+\Omega_{\bar{1},-\bar{1}}+\Omega_{\bar{1},-1}+\Omega_{1,-\bar{1}}),\\
&\Omega_{+,-1}=0,\\
&\Omega_{-,+1}=0,
\end{aligned}\tag{A.18}$$

and

$$\begin{aligned}
&f^{-1}\partial_+f=-\frac{1}{2}\Omega_{-,-+},\\
&f^{-1}\partial_-f=-\frac{1}{2}\Omega_{-,-+},\\
&f^{-1}\partial_1f=-\frac{1}{4}(\Omega_{+,+1}+\Omega_{-,-1}),
\end{aligned}\tag{A.19}$$

and their complex conjugates.

The conditions on the geometry and fluxes that transform as $(1,1)$ tensors under $SU(3)$ are

$$\begin{aligned}
&F_{+-1p\bar{q}}=\pm\frac{1}{2}\big(\Omega_{p,+\bar{q}}+\delta_{p\bar{q}}\Omega_{r,}{}^{r}{}_{+}\big),\\
&F_{-1\bar{1}p\bar{q}}=\mp\frac{1}{4}\big(2\Omega_{p,\bar{q}\bar{1}}+\Omega_{p,q_1q_2}\epsilon^{q_1q_2}{}_{\bar{q}}\big)\pm\frac{1}{4}\delta_{p\bar{q}}\big(2\Omega_{r,}{}^{r}{}_{\bar{1}}+\Omega_{r_1,r_2r_3}\epsilon^{r_1r_2r_3}\big),
\end{aligned}$$

$$
\begin{aligned}
&F_{+1\bar{1}p\bar{q}} = \pm\frac{1}{4}\big(2\Omega_{p,\bar{q}1} - \Omega_{p,q_1q_2}\epsilon^{q_1q_2}{}_{\bar{q}}\big) \mp \frac{1}{4}\delta_{p\bar{q}}\big(2\Omega_{r,}{}^{r}{}_{1} - \Omega_{r_1,r_2r_3}\epsilon^{r_1r_2r_3}\big),\\
&\Omega_{p,\bar{q}+} = \Omega_{\bar{q},p-},\\
&2\Omega_{p,\bar{q}\bar{1}} + \Omega_{p,q_1q_2}\epsilon^{q_1q_2}{}_{\bar{q}} = 2\Omega_{\bar{q},p1} + \Omega_{\bar{q},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p},\\
&2\Omega_{p,\bar{q}1} - \Omega_{p,q_1q_2}\epsilon^{q_1q_2}{}_{\bar{q}} = 2\Omega_{\bar{q},p\bar{1}} - \Omega_{\bar{q},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p},
\end{aligned}
\tag{A.20}
$$

and their complex conjugates.

The conditions on the geometry and fluxes that transform under the fundamental representation of $SU(3)$ are

$$
\begin{aligned}
&F_{+-1\bar{1}p} = \pm\frac{1}{2}\left(\Omega_{p,+1} - \Omega_{p,-1} + \Omega_{+,1p} - \frac{1}{2}\Omega_{+,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p}\right),\\
&F_{+-pq}{}^{q} = \pm\frac{1}{2}\left(-\Omega_{p,-1} - \Omega_{p,+1} - \Omega_{+,1p} + \frac{1}{2}\Omega_{+,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p}\right),\\
&F_{-1pq}{}^{q} = \pm\left(-\frac{1}{2}\Omega_{\bar{1},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} + \Omega_{\bar{1},1p} - \Omega_{p,1\bar{1}} - \Omega_{p,q}{}^{q}\right),\\
&F_{+\bar{1}pq}{}^{q} = \pm\left(-\frac{1}{2}\Omega_{\bar{1},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} + \Omega_{p,-+} + \Omega_{\bar{1},1p} - \Omega_{p,q}{}^{q}\right),\\
&F_{+-1\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} = \mp\Omega_{p,+1},\\
&F_{+-\bar{1}\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} = \mp\Omega_{\bar{1},+p},\\
&F_{-1\bar{1}\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} = \pm\left(-\frac{1}{2}\Omega_{\bar{1},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} + \Omega_{\bar{1},1p}\right),\\
&F_{+1\bar{1}\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} = \pm\left(\Omega_{p,-+} - \Omega_{p,1\bar{1}} + \Omega_{\bar{1},1p} - \frac{1}{2}\Omega_{\bar{1},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p}\right),\\
&\Omega_{p,-\bar{1}} = \Omega_{\bar{1},+p},\\
&\Omega_{p,+\bar{1}} = -\Omega_{+,1p} + \frac{1}{2}\Omega_{+,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} - \Omega_{p,+1},\\
&\Omega_{1,-p} = \Omega_{p,+1},\\
&\Omega_{1,+p} = \Omega_{p,-1},\\
&\Omega_{1,\bar{1}p} = \Omega_{p,-+} - \Omega_{p,1\bar{1}} + \Omega_{\bar{1},1p} - \frac{1}{2}\Omega_{1,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} - \frac{1}{2}\Omega_{\bar{1},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p},\\
&\Omega_{1,1p} = -\frac{1}{2}\Omega_{\bar{1},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} + \Omega_{\bar{1},1p} + \frac{1}{2}\Omega_{1,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} - \Omega_{p,1\bar{1}} - \Omega_{p,q}{}^{q},\\
&\Omega_{\bar{q},}{}^{\bar{q}}{}_{p} = \Omega_{\bar{q}_1,\bar{q}_2\bar{1}}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p},\\
&\Omega_{\bar{q}_1,\bar{q}_2-}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} = -\Omega_{p,+1} + \frac{1}{2}\Omega_{+,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} + \Omega_{p,-1} - \Omega_{+,1p},\\
&\Omega_{\bar{q}_1,\bar{q}_2+}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} = \Omega_{p,+1} - \frac{1}{2}\Omega_{+,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} - \Omega_{p,-1} + \Omega_{+,1p},\\
&\Omega_{\bar{q}_1,\bar{q}_21}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} = -\Omega_{\bar{q}_1,\bar{q}_2\bar{1}}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p},\\
&\Omega_{\bar{1},-p} = -\Omega_{+,1p} + \frac{1}{2}\Omega_{+,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} - \Omega_{p,+1},\\
&\Omega_{\bar{1},\bar{1}p} = \Omega_{p,-+} + \Omega_{\bar{1},1p} - \Omega_{\bar{1},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_{p} - \Omega_{p,q}{}^{q},
\end{aligned}
$$

$$
\begin{aligned}
&\Omega_{-,1p} = \frac{1}{2}\Omega_{-,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_p, \\
&\Omega_{-,-p} = -\Omega_{\bar{1},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_p - \Omega_{p,1\bar{1}} + 2\Omega_{\bar{1},1p} - \Omega_{p,q}{}^q, \\
&\Omega_{-,+p} = 0, \\
&\Omega_{-,\bar{1}p} = -\Omega_{\bar{1},+p} - \Omega_{p,-1} - \frac{1}{2}\Omega_{-,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_p, \\
&\Omega_{+,-p} = 0, \\
&\Omega_{+,+p} = 2\Omega_{\bar{1},1p} + 2\Omega_{p,-+} - \Omega_{p,1\bar{1}} - \Omega_{\bar{1},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_p - \Omega_{p,q}{}^q, \\
&\Omega_{+,\bar{1}p} = -\frac{1}{2}\Omega_{+,\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_p,
\end{aligned}
\tag{A.21}
$$

together with

$$
f^{-1}\partial_p f = \frac{1}{2}\big(-\Omega_{p,-+} + \Omega_{p,1\bar{1}} - 2\Omega_{\bar{1},1p} + \Omega_{\bar{1},\bar{q}_1\bar{q}_2}\epsilon^{\bar{q}_1\bar{q}_2}{}_p + \Omega_{p,q}{}^q\big),
\tag{A.22}
$$

and their complex conjugates.

Lastly, the conditions on the geometry and fluxes that transform as (2, 0) tensors under $SU(3)$ are

$$
\begin{aligned}
&F_{+-\bar{q}_1\bar{q}_2(p}\epsilon^{\bar{q}_1\bar{q}_2}{}_{q)} = \mp\Omega_{(p,|-|q)}, \\
&F_{-1\bar{q}_1\bar{q}_2(p}\epsilon^{\bar{q}_1\bar{q}_2}{}_{q)} = \pm\left(\Omega_{(p,|1|q)} - \frac{1}{2}\Omega_{(p,|\bar{q}_1\bar{q}_2|}\epsilon^{\bar{q}_1\bar{q}_2}{}_{q)}\right), \\
&F_{+\bar{1}\bar{q}_1\bar{q}_2(p}\epsilon^{\bar{q}_1\bar{q}_2}{}_{q)} = \mp\left(\Omega_{(p,|\bar{1}|q)} + \frac{1}{2}\Omega_{(p,|\bar{q}_1\bar{q}_2|}\epsilon^{\bar{q}_1\bar{q}_2}{}_{q)}\right), \\
&\Omega_{(p,|-|q)} = \Omega_{(p,|+|q)},
\end{aligned}
\tag{A.23}
$$

and their complex conjugates. As we have seen the above conditions considerably simplify when they are written in terms of the fundamental forms of G_2.

Appendix B. Null structures

B.1. Null vectors and $SO(n) \ltimes \mathbb{R}^n$ structures

The stability subgroup in the special Lorentz group $SO(n+1,1)$ of a nowhere vanishing null vector X is $SO(n) \ltimes \mathbb{R}^n$. Therefore, geometrically the structure of a Lorentzian manifold that admits a non-vanishing null vector reduces to $SO(n) \ltimes \mathbb{R}^n$. Topologically, the structure reduces further to the maximal compact subgroup $SO(n)$. Let X be a non-vanishing null vector field on the spacetime. It is always possible to introduce a frame e^+, e^-, e^i such that

$$
ds^2 = 2e^+e^- + \delta_{ij}e^ie^j, \qquad X = e_+,
\tag{B.1}
$$

where e_A is the co-frame, $e^A(e_B) = \delta^A{}_B$. It is convenient to use the Lorentzian metric to construct the associated null one-form $\kappa = e^-$ to X. Next consider the covariant derivative of κ, $\nabla\kappa$ with respect to the Levi-Civita connection of the Lorentzian metric. One way to determine the $SO(n) \ltimes \mathbb{R}^n$ structures is to decompose $\nabla\kappa$ under the irreducible representations of either the geometric structure group $SO(n) \ltimes \mathbb{R}^n$ or the topological structure group $SO(n)$. Since the representation of $SO(n) \ltimes \mathbb{R}^n$ on the space of one-forms is reducible but indecomposable, a more refined characterization of the geometry can be achieved by decomposing $\nabla\kappa$ under the topological structure group $SO(n)$. In particular, one finds that

$$\nabla\kappa = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Z_1 + Z_2 + Z_3, \tag{B.2}$$

where

$$\begin{aligned} &Y_1 = \nabla_+\kappa_-, \qquad Y_2 = \nabla_-\kappa_-, \qquad (Y_3)_i = \nabla_i\kappa_-, \qquad (Y_4)_i = \nabla_-\kappa_i, \qquad (Y_5)_i = \nabla_+\kappa_i, \\ &(Z_1)_{ij} = 2\nabla_{[i}\kappa_{j]}, \qquad Z_2 = \nabla^i\kappa_i, \qquad (Z_3)_{ij} = \nabla_{(i}\kappa_{j)} - \frac{1}{n}\delta_{ij}\nabla^l\kappa_l. \end{aligned} \tag{B.3}$$

Clearly, if κ is parallel, then all the classes vanish. If X is Killing, then $Y_1 = Y_2 = Y_3 + Y_4 = Y_5 = Z_2 = Z_3 = 0$, and similarly if X is self-parallel, $Y_1 = Y_5 = 0$. In all, there are 2^8 possible structures.

B.2. Null $U(n) \ltimes \mathbb{C}^n$, Cauchy–Riemann and $SU(n) \ltimes \mathbb{C}^n$ structures

A $(2n+2)$-dimensional Lorentzian manifold with a $U(n) \ltimes \mathbb{C}^n$-structure admits a nowhere vanishing null one-form κ and a three-form $\sigma = \kappa \wedge \omega$, where ω is a Hermitian form. In an adapted basis, one has

$$\kappa = e^-, \qquad \sigma = -ie^- \wedge \delta_{\alpha\bar{\beta}} e^\alpha \wedge e^{\bar{\beta}}, \qquad ds^2 = 2e^-e^+ + 2\delta_{\alpha\bar{\beta}} e^\alpha e^{\bar{\beta}}. \tag{B.4}$$

It is worth pointing out that the stability subgroup of e^- and ω in the special Lorentz group $SO(2n+1,1)$ is not $U(n) \ltimes \mathbb{C}^n$ and so the introduction of the null three-form σ is necessary.

To determine the different $U(n) \ltimes \mathbb{C}^n$ structures on the spacetime, we decompose $\nabla\kappa$ and $\nabla\sigma$ under representations of $U(n)$, the maximal compact subgroup of $U(n) \ltimes \mathbb{C}^n$, which is the topological structure group of the spacetime.[9] It turns out that all independent structures can be found by considering the covariant derivative of κ, $\nabla_A\kappa_B$, and the component, $\nabla_A\omega_{ij} = \nabla_A\sigma_{-ij} - \nabla_A\kappa_-\omega_{ij}$, of $\nabla\sigma$. In particular the $SO(2n) \ltimes \mathbb{R}^{2n}$ classes which we have previously investigated are further decomposed as

$$\begin{aligned} &Y_3 = Y_3^{1,0} + Y_3^{0,1}, \qquad Y_4 = Y_4^{1,0} + Y_4^{0,1}, \qquad Y_5 = Y_5^{1,0} + Y_5^{0,1}, \\ &Z_1 = Z_1^{2,0} + Z_1^{0,2} + \hat{Z}_1^{1,1} - \frac{i}{n}\omega \wedge (\mathcal{Z}_2 - \bar{\mathcal{Z}}_2), \qquad Z_2 = \mathcal{Z}_2 + \bar{\mathcal{Z}}_2, \qquad \mathcal{Z}_2 = \nabla^\alpha\kappa_\alpha, \\ &Z_3 = Z_3^{2,0} + Z_3^{0,2} + \hat{Z}_3^{1,1}. \end{aligned} \tag{B.5}$$

In addition, there are classes associated with $\nabla\sigma$. The independent ones are

$$V_1^{2,0} = \nabla_+\omega_{\alpha\beta}, \qquad V_2^{2,0} = \nabla_-\omega_{\alpha\beta} \tag{B.6}$$

and W_1, W_2, W_3, W_4 which are associated with $\nabla_i\omega_{jk}$. The latter are related to those of Gray–Hervella for almost Hermitian manifolds, see [17].

A Cauchy–Riemann structure is a null $U(n) \ltimes \mathbb{C}^n$ structure. A Cauchy–Riemann structure determines an integrable distribution spanned by $e^-, e^{\bar{\alpha}}$. Using the torsion free conditions, we observe that this requires that

$$\Omega_{[\alpha,\beta]+} = \Omega_{+,+\alpha} = \Omega_{[\alpha,\beta]\gamma} = \Omega_{[+,\beta]\gamma} = 0. \tag{B.7}$$

[9] One may also consider decompositions under the $U(n) \ltimes \mathbb{C}^n$ group. However, the decomposition under $U(n)$ is more convenient since the irreducible representations are easy to identify.

In turn these give

$$\Omega_{\alpha,\beta\gamma} = \Omega_{\alpha,\beta+} = \Omega_{+,+\alpha} = \Omega_{+,\alpha\beta} = 0. \tag{B.8}$$

Observe that in this setting κ is not self-parallel. This in addition will require that $\Omega_{+,+-} = 0$. Therefore a Cauchy–Riemann structure is specified by the vanishing of the classes

$$W_1 = W_2 = 0, \qquad V_1^{2,0} = V_2^{0,2} = 0, \qquad Y_5 = Z_1^{2,0} = Z_1^{0,2} = Z_3^{2,0} = Z_3^{0,2} = 0. \tag{B.9}$$

If κ is self-parallel, then in addition $Y_1 = 0$.

The null $SU(n) \ltimes \mathbb{C}^n$ structures can be investigated in a similar way. The associated nowhere vanishing forms in the basis introduced for the $U(n) \ltimes \mathbb{C}^n$ case are

$$\kappa = e^-, \qquad \sigma = e^- \wedge \omega, \qquad \rho = e^- \wedge \chi, \tag{B.10}$$

where χ is the $SU(n)$-invariant holomorphic $(n, 0)$-form. It turns out that the decomposition of $\nabla\kappa$ is as in the $U(n) \ltimes \mathbb{C}^n$ case above. In addition, one can also define the classes V_1 and V_2. There are two new additional classes

$$V_3^{n-1,1} = \nabla_+(\mathrm{Re}\,\chi)_{\alpha_1\ldots\alpha_{n-1}\bar\beta}, \qquad V_4^{n-1,1} = \nabla_-(\mathrm{Re}\,\chi)_{\alpha_1\ldots\alpha_{n-1}\bar\beta}. \tag{B.11}$$

The remaining classes are determined by $\nabla_i\omega_{jk}$ and $\nabla_i\chi_{j_1\ldots j_n}$. These can be expressed in terms of the W_1, W_2, W_3, W_4 and W_5 classes of the $SU(n)$ structures, see, e.g., [18]. The normalization which we use for these classes is that of Appendix C in [32]. In particular for the $SU(4)$ case which is relevant to the results of this paper, we have that

$$\begin{aligned}
\tilde d\omega &= W_1 + W_3 + \frac{1}{3} W_4 \wedge \omega, \\
\tilde d\,\mathrm{Re}\,\chi &= W_5 \wedge \mathrm{Re}\,\chi + \left(-\frac{1}{3}W_1 + \frac{1}{2}W_2\right) \bar\wedge\, \mathrm{Im}\,\chi, \\
\tilde d\,\mathrm{Im}\,\chi &= W_5 \wedge \mathrm{Im}\,\chi - \left(-\frac{1}{3}W_1 + \frac{1}{2}W_2\right) \bar\wedge\, \mathrm{Re}\,\chi,
\end{aligned} \tag{B.12}$$

where $\tilde d$ denotes the exterior derivative evaluated along the transverse directions, W_4 is the Lee form of ω, W_5 is the Lee form of $\mathrm{Re}\,\chi$. It is clear that if $W_1 = W_2 = 0$, then one has that

$$\begin{aligned}
\tilde d\omega^3 &= W_4 \wedge \omega^3, \\
\tilde d\chi &= W_5 \wedge \chi.
\end{aligned} \tag{B.13}$$

Observe that any Hermitian eight-dimensional manifold with an $SU(4)$ structure satisfies these conditions. As we have seen supersymmetry imposes in addition that $W_4 = W_5$. We refer to these as *relatively balanced Hermitian* $SU(4)$ *structures* because the difference of the two Lee forms vanishes. One consequence of (B.13) is that if B admits such relatively balanced Hermitian structure, then $\tilde d W_4 = \tilde d W_5$, is a trace-less $(1, 1)$-form.

B.3. Null $Spin(7) \ltimes \mathbb{R}^8$ *structures*

The null $Spin(7) \ltimes \mathbb{R}^8$ structure is associated with the forms

$$\kappa = e^-, \qquad \sigma = e^- \wedge \phi, \tag{B.14}$$

where ϕ is the self-dual $Spin(7)$-invariant four-form.

The different $Spin(7) \ltimes \mathbb{R}^8$ null structures can be determined by decomposing $\nabla\kappa$ and $\nabla\sigma$ in $Spin(7)$ representations. It turns out that

$$\nabla\kappa = Y_1 + \cdots + Y_5 + Z_1 + \cdots + Z_4, \tag{B.15}$$

where

$$\begin{aligned}
&Y_1 = \nabla_+\kappa_-, \qquad Y_2 = \nabla_-\kappa_-, \qquad (Y_3)_i = \nabla_i\kappa_-, \qquad (Y_4)_i = \nabla_-\kappa_i, \qquad (Y_5)_i = \nabla_+\kappa_i, \\
&(Z_1)_{ij} = d\kappa_{ij}|_{\mathbf{7}}, \qquad (Z_2)_{ij} = d\kappa_{ij}|_{\mathbf{21}}, \qquad Z_3 = \nabla^i\kappa_i, \\
&(Z_4)_{ij} = \nabla_{(i}\kappa_{j)} - \frac{1}{8}\delta_{ij}\nabla^l\kappa_l,
\end{aligned} \tag{B.16}$$

where we have used the decomposition $\Lambda^2(\mathbb{R}^8) = \Lambda_{\mathbf{7}} \oplus \Lambda_{\mathbf{21}}$ under $Spin(7)$. It remains to investigate $\nabla\sigma$. It turns out that the remaining independent structures are given by $\nabla_+\phi$, $\nabla_-\phi$ and $\nabla_i\phi_{j_1\ldots j_4}$. So we define

$$V_1 = \nabla_+\phi_{j_1\ldots j_4}, V_2 = \nabla_-\phi_{j_1\ldots j_4}. \tag{B.17}$$

It is easy to see that both V_1 and V_2 lie in the fundamental seven-dimensional representation of $Spin(7)$. Furthermore, $\nabla_i\phi_{j_1\ldots j_4}$ determines two classes W_1, W_2 which are precisely those expected for eight-dimensional manifolds with a $Spin(7)$ structure.

Appendix C. G_2-structures in ten dimensions

The G_2-structure we consider is characterized by the existence of three one-forms e^+, e^-, e^1 and a three-form φ. The metric can be written as

$$ds^2 = 2e^+e^- + \left(e^1\right)^2 + \delta_{ij}e^ie^j. \tag{C.1}$$

The three one-forms span a trivial bundle in the decomposition $T^*M = I^3 \oplus \mathcal{T}^*$ we have mentioned in Section 2. The form φ is the fundamental G_2 three-form in the transverse directions $\mathcal{T}^*$. Throughout, we use the notation of [22]. Following the analysis in [17] and using a unified notation for the three one-forms as κ_r, $r = +, -1$, the different G_2 structures are determined by decomposing the covariant derivatives $\nabla\kappa_r$ and $\nabla\varphi$ in G_2 representations. It turns out that it suffices to consider the classes

$$\begin{aligned}
&\nabla_r(\kappa_s)_t \leftrightarrow T_{rst}, \\
&\nabla_i(\kappa_s)_t \leftrightarrow (V_1)_{st}, \\
&\nabla_r(\kappa_s)_i \leftrightarrow (V_2)_{rs}, \\
&\nabla_i(\kappa_r)_j = \frac{1}{2}(\tilde{d}\kappa_r)_{ij} + \nabla_{(i}(\kappa_r)_{j)} \leftrightarrow Z_r + F_r + T'_r + S_r, \\
&\nabla_r\varphi_{ijk} \leftrightarrow Y_r, \\
&\nabla_i\varphi_{jkl} \leftrightarrow X_1 + X_2 + X_3 + X_4,
\end{aligned} \tag{C.2}$$

where X_1, X_2, X_3 and X_4 are the usual G_2 classes in seven dimensions. The representations T, T' and X_1 are singlets, V_1, V_2, Z, Y and X_4 are 7-dimensional, F and X_2 are 14-dimensional, and S and X_3 are 27-dimensional.

One can write

$$\tilde{d}\varphi \leftrightarrow X_1 + X_3 + X_4,$$

$$
\begin{aligned}
&\tilde{d}\varphi \wedge \varphi \leftrightarrow X_1, \\
&\tilde{d} \star \varphi \leftrightarrow X_2 + X_4, \\
&\star(\tilde{d} \star \varphi) \wedge \star\varphi \leftrightarrow X_4,
\end{aligned}
\tag{C.3}
$$

where $\tilde{d}$ is the restriction of the exterior derivative along the transverse directions. In particular, one finds

$$
\begin{aligned}
&\tilde{d}\varphi = \frac{1}{7} X_1 \star \varphi + \frac{3}{4}\theta \wedge \varphi - \frac{1}{2} s_\Gamma \bar{\wedge} \star \varphi, \\
&\tilde{d} \star \varphi = X_2 \wedge \varphi + \theta \wedge \star\varphi,
\end{aligned}
\tag{C.4}
$$

where $X_2 = \Pi_{\mathbf{14}} \star \tilde{d} \star \varphi$ and $X_4 = \theta_\varphi = -\frac{1}{3} \star (\star\tilde{d}\varphi \wedge \varphi)$ is the Lee form,

$$
\begin{aligned}
&X_1 = \frac{1}{4!} (\tilde{d}\varphi)_{i_1 \ldots i_4} \star \varphi^{i_1 \ldots i_4}, \\
&(s_\Gamma)_{ij} = \frac{1}{3!} (\tilde{d}\varphi)_{k_1 k_2 k_3 (i} \star \varphi_{j)}{}^{k_1 k_2 k_3} + \frac{1}{42} \delta_{ij} (\tilde{d}\varphi)_{k_1 k_2 k_3 k_4} \star \varphi^{k_1 k_2 k_3 k_4}.
\end{aligned}
\tag{C.5}
$$

Moreover we choose s_Γ to represent X_3. The classes $X_1, \ldots, X_4$ are analogous to the Fernandez–Gray classes of seven-dimensional manifolds with a G_2 structure [33].

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