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Inverse systems and I-favorable spaces

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ABSTRACT

We show that a compact space is I-favorable if, and only if it can be represented as the limit of a σ-complete inverse system of compact metrizable spaces with skeletal bonding maps. We also show that any completely regular I-favorable space can be embedded as a dense subset of the limit of a σ-complete inverse system of separable metrizable spaces with skeletal bonding maps.

1. Introduction

We investigate the class of all limits of σ-complete inverse systems of compact metrizable spaces with skeletal bonding maps. Notations are used the same as in the monograph [5]. For example, a compact space is Hausdorff, and a regular space is T_1. A directed set Σ is said to be σ-complete if any countable chain of its elements has least upper bound in Σ. An inverse system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is said to be a σ-complete, whenever Σ is σ-complete and for every chain $\{\sigma_n\colon n \in \omega\} \subseteq \Sigma$, such that $\sigma = \sup\{\sigma_n\colon n \in \omega\} \in \Sigma$, there holds

$$X_\sigma = \varprojlim\{X_{\sigma_n}, \pi^{\sigma_{n+1}}_{\sigma_n}\},$$

compare [15]. However, we will consider inverse systems where bonding maps are surjections. Another details about inverse systems one can find in [5, pp. 135–144]. For basic facts about I-favorable spaces we refer to [4], compare also [10].

Through the course of this note we modify quotient topologies and quotient maps, introducing $\mathcal{Q}_\mathcal{P}$-topologies and $\mathcal{Q}_\mathcal{P}$-maps, where $\mathcal{P}$ is a family of subsets of X. Next, we assign the family $\mathcal{P}_{seq}$ (of all sets with some properties of cozero sets) to a given family $\mathcal{P}$. Frink's theorem is used to show that the $\mathcal{Q}_\mathcal{P}$-topology is completely regular, whenever $\mathcal{P} \subseteq \mathcal{P}_{seq}$ is a ring of subsets of X, see Theorem 5. Afterwards, some special club filters are described as systems of countable skeletal families. This yields that each family which belongs to a such club filter is a countable skeletal family, which produces a skeletal map onto a compact metrizable space. Theorem 12 is the main result: I-favorable compact spaces coincides with limits of σ-complete inverse systems of compact metrizable spaces with skeletal bonding maps.

E.V. Shchepin has considered several classes of compact spaces in a few papers, for example [13,14] and [15]. He introduced the class of compact openly generated spaces. A compact space X is called *openly generated*, whenever X is the limit of a σ-complete inverse system of compact metrizable spaces with open bonding maps. Originally, Shchepin used another

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name: open-generated spaces; see [15]. A.V. Ivanov showed that a compact space X is openly generated if, and only if its superextension is a Dugundji space, see [9]. Then Shchepin established that the classes of openly generated compact spaces and of κ-metrizable spaces are the same, see Theorem 21 in [15]. Something likewise is established for compact I-favorable spaces in Theorem 12.

A Boolean algebra $\mathbb{B}$ is semi-Cohen (regularly filtered) if, and only if $[\mathbb{B}]^\omega$ has a closed unbounded set of countable regular subalgebras, in other words $[\mathbb{B}]^\omega$ contains a club filter. Hence, the Stone space of a semi-Cohen algebras is I-favorable. Translating Corollary 5.5.5 of L. Heindorf and L.B. Shapiro [7] on topological notions, one can obtain our's main result in zero-dimensional cases, compare also Theorem 4.3 of B. Balcar, T. Jech and J. Zapletal [2]. We get Theorem 11 which says that each completely regular I-favorable space is homeomorphic to a dense subspace of the limit of an inverse system $\{X/\mathcal{R}, q^{\mathcal{R}}_{\mathcal{P}}, \mathcal{C}\}$, where spaces $X/\mathcal{R}$ are metrizable and separable, bonding maps $q^{\mathcal{R}}_{\mathcal{P}}$ are skeletal and the directed set $\mathcal{C}$ is σ-complete.

2. $\mathcal{Q}_{\mathcal{P}}$-topologies

Let $\mathcal{P}$ be a family of subsets of X. We say that $y \in [x]_{\mathcal{P}}$, whenever $x \in V$ if, and only if $y \in V$, for each $V \in \mathcal{P}$. The family of all classes $[x]_{\mathcal{P}}$ is denoted $X/\mathcal{P}$. Note that $[x]_{\mathcal{P}} \subseteq V$ if, and only if $[x]_{\mathcal{P}} \cap V \neq \emptyset$, for each $V \in \mathcal{P}$. Put $q(x) = [x]_{\mathcal{P}}$. The function $q : X \to X/\mathcal{P}$ is called a *$\mathcal{Q}_{\mathcal{P}}$-map*. The coarser topology on $X/\mathcal{P}$ which contains all images $q[V] = \{[x]_{\mathcal{P}}\colon x \in V\}$, where $V \in \mathcal{P}$, is called a *$\mathcal{Q}_{\mathcal{P}}$-topology*. If $V \in \mathcal{P}$, then $q^{-1}(q[V]) = V$. Indeed, we have $V \subseteq q^{-1}(q[V])$, since $q : X \to X/\mathcal{P}$ is a surjection. Suppose $x \in q^{-1}(q[V])$. Then $q(x) \in q[V]$, and $[x]_{\mathcal{P}} \cap V \neq \emptyset$. We get $[x]_{\mathcal{P}} \subseteq V$, since $V \in \mathcal{P}$. Therefore $x \in V$.

Lemma 1. *Let $\mathcal{P}$ be a family of open subsets of a topological space X. If $\mathcal{P}$ is a closed under finite intersections, then the $\mathcal{Q}_{\mathcal{P}}$-map $q : X \to X/\mathcal{P}$ is continuous. Moreover, if $X = \bigcup \mathcal{P}$, then the family $\{q[V]\colon V \in \mathcal{P}\}$ is a base for the $\mathcal{Q}_{\mathcal{P}}$-topology.*

Proof. We have $q[V \cap U] = q[V] \cap q[U]$, for every $U, V \in \mathcal{P}$. Hence, the family $\{q[V]\colon V \in \mathcal{P}\}$ is closed under finite intersections. This family is a base for the $\mathcal{Q}_{\mathcal{P}}$-topology, since $X = \bigcup \mathcal{P}$ implies that $X/\mathcal{P}$ is a union of basic sets. Obviously, the $\mathcal{Q}_{\mathcal{P}}$-map q is continuous. □

Additionally, if X is a compact space and $X/\mathcal{P}$ is Hausdorff, then the $\mathcal{Q}_{\mathcal{P}}$-map $q : X \to X/\mathcal{P}$ is a quotient map. Also, the $\mathcal{Q}_{\mathcal{P}}$-topology coincides with the quotient topology, compare [5, p. 124].

Let $\mathcal{R}$ be a family of subsets of X. Denote by $\mathcal{R}_{seq}$ the family of all sets W which satisfy the following condition: *There exist sequences $\{U_n\colon n \in \omega\} \subseteq \mathcal{R}$ and $\{V_n\colon n \in \omega\} \subseteq \mathcal{R}$ such that $U_k \subseteq (X \setminus V_k) \subseteq U_{k+1}$, for any $k \in \omega$, and $\bigcup\{U_n\colon n \in \omega\} = W$.*

If $\mathcal{R}_{seq} \neq \emptyset$, then $\bigcup \mathcal{R} = X$. Indeed, take $W \in \mathcal{R}_{seq}$. Whenever U_n and V_n are elements of sequences witnessing $W \in \mathcal{R}_{seq}$, then $X \setminus V_k \subseteq U_{k+1} \subseteq W$ implies $U_{k+1} \cup V_k = X$.

If X is a completely regular space and $\mathcal{T}$ consists of all cozero sets of X, then $\mathcal{T} = \mathcal{T}_{seq}$. Indeed, for each $W \in \mathcal{T}$, fix a continuous function $f : X \to [0,1]$ such that $W = f^{-1}((0,1])$. Put $U_n = f^{-1}((\frac{1}{n},1])$ and $X \setminus V_n = f^{-1}([\frac{1}{n},1])$.

Recall that, a family of sets is called a *ring of sets* whenever it is closed under finite intersections and finite unions.

Lemma 2. *If a ring of sets $\mathcal{R}$ is contained in $\mathcal{R}_{seq}$, then any countable union $\bigcup\{U_n \in \mathcal{R}\colon n \in \omega\}$ belongs to $\mathcal{R}_{seq}$.*

Proof. Suppose that sequences $\{U^n_k\colon k \in \omega\} \subseteq \mathcal{R}$ and $\{V^n_k\colon k \in \omega\} \subseteq \mathcal{R}$ witnessing $U_n \in \mathcal{R}_{seq}$, respectively. Then sets $U^0_n \cup U^1_n \cup \cdots \cup U^n_n$ and $V^0_n \cap V^1_n \cap \cdots \cap V^n_n$ are successive elements of sequences which witnessing $\bigcup\{U_n \in \mathcal{R}\colon n \in \omega\} \in \mathcal{R}_{seq}$. □

Lemma 3. *If a family of sets $\mathcal{P}$ is contained in $\mathcal{P}_{seq}$, then the $\mathcal{Q}_{\mathcal{P}}$-topology is Hausdorff.*

Proof. Take $[x]_{\mathcal{P}} \neq [y]_{\mathcal{P}}$ and $W \in \mathcal{P}$ such that $x \in W$ and $y \notin W$. Fix sequences $\{U_n\colon n \in \omega\}$ and $\{V_n\colon n \in \omega\}$ witnessing $W \in \mathcal{P}_{seq}$. Choose $k \in \omega$ such that $x \in U_k$ and $y \in V_k$. Hence $[x]_{\mathcal{P}} \subseteq U_k$ and $[y]_{\mathcal{P}} \subseteq V_k$. Therefore, sets $q[U_k]$ and $q[V_k]$ are disjoint neighbourhoods of $[x]_{\mathcal{P}}$ and $[y]_{\mathcal{P}}$, respectively. □

Lemma 4. *If a non-empty family of sets $\mathcal{P} \subseteq \mathcal{P}_{seq}$ is closed under finite intersections, then $\mathcal{Q}_{\mathcal{P}}$-topology is regular.*

Proof. We have $q[A] \cap q[B] = q[A \cap B]$ for each $A, B \in \mathcal{P}$. The family $\{q[A]\colon A \in \mathcal{P}\}$ is a base of open sets for the $\mathcal{Q}_{\mathcal{P}}$-topology. Fix $x \in W \in \mathcal{P}$ and sequences $\{U_n\colon n \in \omega\} \subseteq \mathcal{P}$ and $\{V_n\colon n \in \omega\} \subseteq \mathcal{P}$ witnessing $W \in \mathcal{P}_{seq}$. Take any $U_k \subseteq W$ such that $[x]_{\mathcal{P}} \subseteq U_k \in \mathcal{P}$. We get $q(x) \in q[U_k] \subseteq \operatorname{cl} q[U_k] \subseteq q[X \setminus V_k] = X/\mathcal{P} \setminus q[V_k] \subseteq q[W]$, where $\bigcup \mathcal{P} = X$. □

To show which $\mathcal{Q}_{\mathcal{P}}$-topologies are completely regular, we apply the Frink's theorem, compare [6] or [5, p. 72].

Theorem *(O. Frink (1964)). A T_1-space X is completely regular if, and only if there exists a base $\mathcal{B}$ satisfying*:

(1) *If $x \in U \in \mathcal{B}$, then there exists $V \in \mathcal{B}$ such that $x \notin V$ and $U \cup V = X$.*
(2) *If $U, V \in \mathcal{B}$ and $U \cup V = X$, then there exist disjoint sets $M, N \in \mathcal{B}$ such that $X \setminus U \subseteq M$ and $X \setminus V \subseteq N$.*

Theorem 5. *If $\mathcal{P}$ is a ring of subsets of X and $\mathcal{P} \subseteq \mathcal{P}_{seq}$, then the $\mathcal{Q}_{\mathcal{P}}$-topology is completely regular.*

Proof. The $\mathcal{Q}_{\mathcal{P}}$-topology is Hausdorff by Lemma 3. Let $\mathcal{B}$ be the minimal family which contains $\{q[V]\colon V \in \mathcal{P}\}$ and is closed under countable unions. This family is a base for the $\mathcal{Q}_{\mathcal{P}}$-topology, by Lemma 1. We should show that $\mathcal{B}$ fulfills conditions (1) and (2) in Frink's theorem.

Let $[x]_{\mathcal{P}} \in q[W] \in \mathcal{B}$. Fix sequences $\{U_k\colon k \in \omega\}$ and $\{V_k\colon k \in \omega\}$ witnessing $W \in \mathcal{P}_{seq}$ and $k \in \omega$ such that $x \in X \setminus V_k \subseteq W$. We have $W \cup V_k = X$. Therefore $[x]_{\mathcal{P}} \notin q[V_k]$ and $q[W] \cup q[V_k] = X/\mathcal{P}$. Thus $\mathcal{B}$ fulfills (1).

Fix sets $\bigcup\{U_n\colon n \in \omega\} \in \mathcal{B}$ and $\bigcup\{V_n\colon n \in \omega\} \in \mathcal{B}$ such that

$$X/\mathcal{P} = \bigcup\bigl\{q[U_n]\colon n \in \omega\bigr\} \cup \bigcup\bigl\{q[V_n]\colon n \in \omega\bigr\},$$

where U_n and V_n belong to $\mathcal{P}$. Thus, $U = \bigcup\{U_n\colon n \in \omega\} \in \mathcal{P}_{seq}$ and $V = \bigcup\{V_n\colon n \in \omega\} \in \mathcal{P}_{seq}$ by Lemma 2. Next, fix sequences $\{A_n\colon n \in \omega\}$, $\{B_n\colon n \in \omega\}$, $\{C_n\colon n \in \omega\}$ and $\{D_n\colon n \in \omega\}$ witnessing $U \in \mathcal{P}_{seq}$ and $V \in \mathcal{P}_{seq}$, respectively. Therefore

$$A_k \subseteq (X \setminus B_k) \subseteq A_{k+1} \subseteq U \quad \text{and} \quad C_k \subseteq (X \setminus D_k) \subseteq C_{k+1} \subseteq V$$

for every $k \in \omega$. Put $N_n = A_n \cap D_n$ and $M_n = C_n \cap B_n$. Let

$$M = \bigcup\{M_n\colon n \in \omega\} \quad \text{and} \quad N = \bigcup\{N_n\colon n \in \omega\}.$$

Sets $q[M]$ and $q[N]$ fulfill (2) in Frink's theorem. Indeed, if $k \leqslant n$, then

$$A_k \cap D_k \cap C_n \cap B_n \subseteq A_n \cap B_n = \emptyset$$

and

$$A_n \cap D_n \cap C_k \cap B_k \subseteq C_n \cap D_n = \emptyset.$$

Consequently $M_k \cap N_n = \emptyset$, for any $k, n \in \omega$. Hence sets $q[M]$ and $q[N]$ are disjoint. Also, it is $q[V] \cup q[N] = X/\mathcal{P}$. Indeed, suppose that $x \notin V$, then $x \in U$ and there is k such that $x \in A_k$. Since $x \notin V$, then $x \in D_k$ for all $k \in \omega$. We have $x \in A_k \cap D_k = N_k \subseteq N$. Therefore $[x]_{\mathcal{P}} \in q[N]$. Similarly, one gets $q[U] \cup q[M] = X/\mathcal{P}$. Thus $\mathcal{B}$ fulfills (2). □

If $\mathcal{P} \subseteq \mathcal{P}_{seq}$ is finite, then $X/\mathcal{P}$ is discrete, being a finite Hausdorff space. Whenever $\mathcal{P} \subseteq \mathcal{P}_{seq}$ is countable and closed under finite intersections, then $X/\mathcal{P}$ is a regular space with a countable base. Therefore, $X/\mathcal{P}$ is metrizable and separable.

3. Skeletal families and skeletal functions

A continuous surjection is called *skeletal* whenever for any non-empty open sets $U \subseteq X$ the closure of $f[U]$ has non-empty interior. If X is a compact space and Y Hausdorff, then a continuous surjection $f : X \to Y$ is skeletal if, and only if $\operatorname{Int} f[U] \neq \emptyset$, for every non-empty and open $U \subseteq X$. One can find equivalent notions *almost-open* or *semi-open* in the literature, see [1] and [8]. Following J. Mioduszewski and L. Rudolf [11] we call such maps skeletal, compare [14, p. 413]. In a fact, one can use the next proposition as a definition for skeletal functions.

Proposition 6. *Let $f : X \to Y$ be a skeletal function. If an open set $V \subseteq Y$ is dense, then the preimage $f^{-1}(V) \subseteq X$ is dense, too.*

Proof. Suppose that a non-empty open set $W \subseteq X$ is disjoint with $f^{-1}(V)$. Then the image $\operatorname{cl} f[W]$ has non-empty interior and $\operatorname{cl} f[W] \cap V = \emptyset$, a contradiction. □

There are topological spaces with no skeletal map onto a dense in itself metrizable space. For example, the remainder of the Čech–Stone compactification βN. Also, if I is a compact segment of connected Souslin line and X is metrizable, then each skeletal map $f : I \to X$ is constant. Indeed, let Q be a countable and dense subset of $f[I] \subseteq X$. Suppose a skeletal map $f : I \to X$ is non-constant. Then the preimage $f^{-1}(Q)$ is nowhere dense in I as countable union of nowhere dense subset of a Souslin line. So, for each open set $V \subseteq I \setminus f^{-1}(Q)$ there holds $\operatorname{Int} f[V] = \emptyset$, a contradiction. Regular Baire space X with a category measure μ, for a definition of this space see [12, pp. 86–91], gives an another example of a space with no skeletal map onto a dense in itself, separable and metrizable space. In [3] A. Błaszczyk and S. Shelah are considered separable extremally disconnected spaces with no skeletal map onto a dense in itself, separable and metrizable space. They formulated the result in terms of Boolean algebra: *There is a nowhere dense ultrafilter on ω if, and only if there is a complete, atomless, σ-centered Boolean algebra which contains no regular, atomless, countable subalgebra.*

A family $\mathcal{P}$ of open subsets of a space X is called a *skeletal family*, whenever for every non-empty open set $V \subseteq X$ there exists $W \in \mathcal{P}$ such that $U \subseteq W$ and $\emptyset \neq U \in \mathcal{P}$ implies $U \cap V \neq \emptyset$. The following proposition explains connection between skeletal maps and skeletal families.

Proposition 7. *Let $f : X \to Y$ be a continuous function and let $\mathcal{B}$ be a π-base for Y. The family $\{f^{-1}(V)\colon V \in \mathcal{B}\}$ is skeletal if, and only if f is a skeletal map.*

Proof. Assume, that f is a skeletal map. Fix a non-empty open set $V \subseteq X$. There exists $W \in \mathcal{B}$ such that $W \neq \emptyset$ and $W \subseteq \operatorname{Int}\operatorname{cl} f[V]$. Also, for any $U \in \mathcal{B}$ such that $\emptyset \neq U \subseteq W$ there holds $f^{-1}(U) \cap V \neq \emptyset$. Indeed, if $f^{-1}(U) \cap V = \emptyset$, then $U \cap \operatorname{cl} f[V] = \emptyset$, a contradiction. Thus the family $\{f^{-1}(V)\colon V \in \mathcal{B}\}$ is skeletal.

Assume, that function $f : X \to Y$ is not skeletal. Then there exists a non-empty open set $U \subseteq X$ such that $\operatorname{Int}\operatorname{cl} f[U] = \emptyset$. Since $\mathcal{B}$ is a π-base for Y, then for each $W \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $V \subseteq W$ and $V \cap f[U] = \emptyset$. The family $\{f^{-1}(V)\colon V \in \mathcal{B}\}$ is not skeletal. □

It is well know—compare a comment following the definition of compact open-generated spaces in [15]—that all limit projections are open in any inverse system with open bonding maps. And conversely, if all limit projections of an inverse system are open, then so are all bonding maps. Similar fact holds for skeletal maps.

Proposition 8. *If $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is an inverse system such that all bonding maps π^σ_ϱ are skeletal and all projections π_σ are onto, then any projection π_σ is skeletal.*

Proof. Fix $\sigma \in \Sigma$. Consider a non-empty basic set $\pi_\zeta^{-1}(V)$ for the limit $\varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$. Take $\tau \in \Sigma$ such that $\zeta \leqslant \tau$ and $\sigma \leqslant \tau$. We get $\pi_\zeta^{-1}(V) = \pi_\tau^{-1}((\pi^\tau_\zeta)^{-1}(V))$. Hence

$$\pi_\tau\big[\pi_\zeta^{-1}(V)\big] = \pi_\tau\big[\pi_\tau^{-1}\big(\big(\pi^\tau_\zeta\big)^{-1}(V)\big)\big] = \big(\pi^\tau_\zeta\big)^{-1}(V),$$

the set $\pi_\tau[\pi_\zeta^{-1}(V)]$ is open and non-empty. We have

$$\pi_\sigma\big[\pi_\zeta^{-1}(V)\big] = \pi^\tau_\sigma\big[\pi_\tau\big[\pi_\zeta^{-1}(V)\big]\big],$$

since $\pi^\tau_\sigma \circ \pi_\tau = \pi_\sigma$. The bonding map π^τ_σ is skeletal, hence the closure of $\pi_\sigma[\pi_\zeta^{-1}(V)]$ has non-empty interior. □

4. The open–open game

Players are playing at a topological space X in the open–open game. Player I chooses a non-empty open subset $A_0 \subseteq X$ at the beginning. Then Player II chooses a non-empty open subsets $B_0 \subseteq A_0$. Player I chooses a non-empty open subset $A_n \subseteq X$ at the nth inning, and then Player II chooses a non-empty open subset $B_n \subseteq A_n$. Player I wins, whenever the union $B_0 \cup B_1 \cup \cdots \subseteq X$ is dense. One can assume that Player II wins for other cases. The space X is called I-*favorable* whenever Player I can be insured that he wins no matter how Player II plays. In other words, Player I has a winning strategy. A strategy for Player I could be defined as a function

$$\sigma : \bigcup\big\{\mathcal{T}^n\colon n \geqslant 0\big\} \to \mathcal{T},$$

where $\mathcal{T}$ is a family of non-empty and open subsets of X. Player I has a winning strategy, whenever he knows how to define $A_0 = \sigma(\emptyset)$ and succeeding $A_{n+1} = \sigma(B_0, B_1, \ldots, B_n)$ such that for each game

$$\big(\sigma(\emptyset), B_0, \sigma(B_0), B_1, \sigma(B_0, B_1), B_2, \ldots, B_n, \sigma(B_0, B_1, \ldots, B_n), B_{n+1}, \ldots\big)$$

the union $B_0 \cup B_1 \cup B_2 \cup \cdots \subseteq X$ is dense. For more details about the open–open game see P. Daniels, K. Kunen and H. Zhou [4].

Consider a countable sequence $\sigma_0, \sigma_1, \ldots$ of strategies for Player I. For a family $\mathcal{Q} \subseteq \mathcal{T}$ let $\mathcal{P}(\mathcal{Q})$ be the minimal family such that $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{Q}) \subseteq \mathcal{T}$, and if $\{B_0, B_1, \ldots, B_n\} \subseteq \mathcal{P}(\mathcal{Q})$, then $\sigma_k(B_0, B_1, \ldots, B_n) \in \mathcal{P}(\mathcal{Q})$, and $\sigma_k(\emptyset) \in \mathcal{P}(\mathcal{Q})$, for all σ_k. We say that $\mathcal{P}(\mathcal{Q})$ is the *closure of $\mathcal{Q}$ under strategies σ_k*. In particular, if σ is a winning strategy and the closure of $\mathcal{Q}$ under σ equals $\mathcal{Q}$, then $\mathcal{Q}$ is closed under a winning strategy.

Lemma 9. *If $\mathcal{P}$ is closed under a winning strategy for Player I, then for any open set $V \neq \emptyset$ there is $W \in \mathcal{P}$ such that whenever $U \in \mathcal{P}$ and $U \subseteq W$ then $U \cap V \neq \emptyset$.*

Proof. Let σ be a winning strategy for Player I. Consider an open set $V \neq \emptyset$. Suppose that for any $W \in \mathcal{P}$ there is $U_W \in \mathcal{P}$ such that $U_W \subseteq W$ and $U_W \cap V = \emptyset$. Then Player II wins any game whenever he always chooses sets $U_W \in \mathcal{P}$, only. In particular, the game

$$\sigma(\emptyset), U_{\sigma(\emptyset)}, \sigma(U_{\sigma(\emptyset)}), U_{\sigma(U_{\sigma(\emptyset)})}, \sigma(U_{\sigma(\emptyset)}, U_{\sigma(U_{\sigma(\emptyset)})}), U_{\sigma(U_{\sigma(\emptyset)}, U_{\sigma(U_{\sigma(\emptyset)})})}, \ldots$$

would be winning for him, since all sets chosen by Player II:

$$U_{\sigma(\emptyset)}, U_{\sigma(U_{\sigma(\emptyset)})}, U_{\sigma(U_{\sigma(\emptyset)}, U_{\sigma(U_{\sigma(\emptyset)})})}, \ldots;$$

are disjoint with V, a contradiction. □

Theorem 10. *If a ring $\mathcal{P}$ of open subsets of X is closed under a winning strategy and $\mathcal{P} \subseteq \mathcal{P}_{seq}$, then $X/\mathcal{P}$ is a completely regular space and the $\mathcal{Q}_{\mathcal{P}}$-map $q : X \to X/\mathcal{P}$ is skeletal.*

Proof. Take a non-empty open subset $V \subseteq X$. Since $\mathcal{P}$ is closed under a winning strategy, there exists $W \in \mathcal{P}$ such that if $U \in \mathcal{P}$ and $U \subseteq W$, then $U \cap V \neq \emptyset$, by Lemma 9. This follows $q[U] \cap q[V] \neq \emptyset$, for any basic set $q[U]$ such that $U \subseteq W$ and $U \in \mathcal{P}$. Therefore $q[W] \subseteq \operatorname{cl} q[V]$, since $\{q[U]:\ U \in \mathcal{P}\}$ is a base for the $\mathcal{Q}_{\mathcal{P}}$-topology. The $\mathcal{Q}_{\mathcal{P}}$-map $q : X \to X/\mathcal{P}$ is continuous by Lemma 1. By Theorem 5, the space $X/\mathcal{P}$ is completely regular. □

Fix a π-base $\mathcal{Q}$ for a space X. Following [4], compare [10], any family $\mathcal{C} \subset [\mathcal{Q}]^{\omega}$ is called *a club filter* whenever:

The family $\mathcal{C}$ is closed under ω-chains with respect to inclusion, i.e. if $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots$ is an ω-chain which consists of elements of $\mathcal{C}$, then $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \in \mathcal{C}$; For any countable subfamily $\mathcal{A} \subseteq \mathcal{Q}$, where $\mathcal{Q}$ is the π-base fixed above, there exists $\mathcal{P} \in \mathcal{C}$ such that $\mathcal{A} \subseteq \mathcal{P}$; and

($\mathcal{S}$) *For any non-empty open set V and each $\mathcal{P} \in \mathcal{C}$ there is $W \in \mathcal{P}$ such that if $U \in \mathcal{P}$ and $U \subseteq W$, then U meets V, i.e. $U \cap V \neq \emptyset$.*

In fact, the condition ($\mathcal{S}$) gives reasons to look into I-favorable spaces with respect to skeletal families. Any $\mathcal{P}$ closed under a winning strategy for Player I fulfills ($\mathcal{S}$), by Lemma 9. There holds, see [4, Theorem 1.6], compare [10, Lemmas 3 and 4]: *A topological space has a club filter if, and only if it is I-favorable*. In the next part we modify a little the definition of club filters. We introduce $\mathcal{T}$-clubs, i.e. club filters with some additional properties.

Suppose a completely regular space X is I-favorable. Let $\mathcal{T}$ be the family of all cozero subsets of X. For each $W \in \mathcal{T}$ fix sequences $\{U_n^W:\ n \in \omega\}$ and $\{V_n^W:\ n \in \omega\}$ witnessing $W \in \mathcal{T}_{seq}$. First, for each k choose $\sigma_k^*(\emptyset) \in \mathcal{T}$. Next, put $\sigma_{2n}^*(W) = U_n^W$ and $\sigma_{2n+1}^*(W) = V_n^W$, and $\sigma_k^*(\mathcal{S}) = \sigma_k^*(\emptyset)$ for other $\mathcal{S} \in \bigcup\{\mathcal{T}^n:\ n \geqslant 0\}$. Then, a family $\mathcal{P} \subseteq \mathcal{T}$ is closed under strategies σ_k^*, whenever $\mathcal{P} \subseteq \mathcal{P}_{seq}$. Also, $\mathcal{P}$ is closed under finite unions, whenever it is closed under the strategy which assigns the union $A_0 \cup A_1 \cup \cdots \cup A_n$ to each sequence $(A_0, A_1, \ldots, A_n)$. And also, $\mathcal{P}$ is closed under finite intersections, whenever it is closed under the strategy which assigns the intersection $A_0 \cap A_1 \cap \cdots \cap A_n$ to each $(A_0, A_1, \ldots, A_n)$.

Consider a collection $\mathcal{C} = \{\mathcal{P}(\mathcal{Q}):\ \mathcal{Q} \in [\mathcal{T}]^{\omega}\}$. Assume that each $\mathcal{P} \in \mathcal{C}$ is countable and closed under a winning strategy for Player I and all strategies σ_k^*, and closed under finite intersections and finite unions. Then, the family $\mathcal{C}$ is called *$\mathcal{T}$-club*. By the definitions, any *$\mathcal{T}$-club* $\mathcal{C}$ is closed under ω-chains with respect to the inclusion. Each $\mathcal{P} \in \mathcal{C}$ is a countable ring of sets and $\mathcal{P} \subseteq \mathcal{P}_{seq}$ and it is closed under a winning strategy for Player I. By Theorem 10, the $\mathcal{Q}_{\mathcal{P}}$-map $q : X \to X/\mathcal{P}$ is skeletal and onto a metrizable separable space, for every $\mathcal{P} \in \mathcal{C}$.

Thus, we are ready to build an inverse system with skeletal bonding maps onto metrizable separable spaces. Any $\mathcal{T}$-club $\mathcal{C}$ is directed by the inclusion. For each $\mathcal{P} \in \mathcal{C}$ it is assigned the space $X/\mathcal{P}$ and the skeletal function $q_{\mathcal{P}} : X \to X/\mathcal{P}$. If $\mathcal{P}, \mathcal{R} \in \mathcal{C}$ and $\mathcal{P} \subseteq \mathcal{R}$, then put $q_{\mathcal{P}}^{\mathcal{R}}([x]_{\mathcal{R}}) = [x]_{\mathcal{P}}$. Thus, we have defined the inverse system $\{X/\mathcal{R}, q_{\mathcal{P}}^{\mathcal{R}}, \mathcal{C}\}$. Spaces $X/\mathcal{R}$ are metrizable and separable, bonding maps $q_{\mathcal{P}}^{\mathcal{R}}$ are skeletal and the directed set $\mathcal{C}$ is σ-complete.

Theorem 11. *Let X be an I-favorable completely regular space. If $\mathcal{C}$ is a $\mathcal{T}$-club, then the limit $Y = \varprojlim\{X/\mathcal{R}, q_{\mathcal{P}}^{\mathcal{R}}, \mathcal{C}\}$ contains a dense subspace which is homeomorphic to X.*

Proof. For any $\mathcal{P} \in \mathcal{C}$, put $f(x)_{\mathcal{P}} = q_{\mathcal{P}}(x)$. We have defined the function $f : X \to Y$ such that $f(x) = \{f(x)_{\mathcal{P}}\}$. If $\mathcal{R}, \mathcal{P} \in \mathcal{C}$ and $\mathcal{P} \subseteq \mathcal{R}$, then $q_{\mathcal{P}}^{\mathcal{R}}(f(x)_{\mathcal{R}}) = f(x)_{\mathcal{P}}$. Thus $f(x)$ is a thread, i.e. $f(x) \in Y$.

The function f is continuous. Indeed, let $\pi_{\mathcal{P}}$ be the projection of Y to $X/\mathcal{P}$. By [5, Proposition 2.5.5], the family $\{\pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[U]):\ U \in \mathcal{P} \in \mathcal{C}\}$ is a base for Y. Also,

$$f^{-1}\big(\pi_{\mathcal{P}}^{-1}\big(q_{\mathcal{P}}[U]\big)\big) = q_{\mathcal{P}}^{-1}\big(q_{\mathcal{P}}[U]\big) = U$$

holds for any $U \in \mathcal{P} \in \mathcal{C}$.

Verify that f is injection. Let $x, y \in X$ and $x \neq y$. Take $\mathcal{P} \in \mathcal{C}$ such that $x \in U$ and $y \in V$ for some disjoint sets $U, V \in \mathcal{P}$. Sets $q_{\mathcal{P}}[U]$ and $q_{\mathcal{P}}[V]$ are disjoint, hence $\pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[U])$ and $\pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[V])$ are disjoint neighbourhoods of $f(x)$ and $f(y)$, respectively.

There holds $f[U] = f[X] \cap \pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[U])$, whenever $U \in \mathcal{P} \in \mathcal{C}$. Indeed, $f[U] \subseteq \pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[U])$ implies $f[U] \subseteq f[X] \cap \pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[U])$. Suppose, there exists $y \in \pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[U]) \cap f[X]$ such that $y \notin f[U])$. Take $x \in X$ such that $f(x) = y$ and $x \notin U$. We get $\pi_{\mathcal{P}}(f(x)) = q_{\mathcal{P}}(x) \notin q_{\mathcal{P}}[U]$, but this follows $f(x) \notin \pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[U])$, a contradiction.

Thus, f is open, since $\mathcal{T} = \bigcup \mathcal{C}$ is a base for X. But $f[X] \subseteq Y$ is dense, since the family $\{\pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[U]):\ U \in \mathcal{P} \in \mathcal{C}\}$ is a base for Y. □

5. Reconstruction of I-favorable spaces

Now, we are ready to prove the announce analog of Shchepin's openly generated spaces.

Theorem 12. *If X is an I-favorable compact space, then*

$$X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\},$$

where $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is a σ-complete inverse system, all spaces X_σ are compact and metrizable, and all bonding maps π^σ_ϱ are skeletal and onto.

Proof. Let $\mathcal{C}$ be a $\mathcal{T}$-club. Put

$$\{X_\sigma, \pi^\sigma_\varrho, \Sigma\} = \{X/\mathcal{R}, q^{\mathcal{R}}_{\mathcal{P}}, \mathcal{C}\}.$$

Each space $X_\sigma = X/\mathcal{R}$ has countable base, by the definition of $\mathcal{T}$-club. Also, each $\mathcal{Q}_\mathcal{R}$-map $q_\mathcal{R} : X \to X/\mathcal{R}$ is continuous, by Lemma 1. Hence, any space X_σ is compact and metrizable, by Lemma 4. Each $\mathcal{Q}_\mathcal{R}$-map $q_\mathcal{R} : X \to X_\sigma$ is skeletal, by Theorem 10. Thus, all bonding maps π^σ_ϱ are skeletal, too. The space X is homeomorphic to a dense subspace of $\varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$, by Theorem 11. We get $X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$, since X is compact.

The inverse system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is σ-complete. Indeed, suppose that $\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \cdots$ and all $\mathcal{P}_n \in \mathcal{C}$. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n\colon n \in \omega\} \in \mathcal{C}$. Put

$$\big(h([x]_\mathcal{P})\big)_{\mathcal{P}_n} = q^{\mathcal{P}}_{\mathcal{P}_n}\big([x]_\mathcal{P}\big) = [x]_{\mathcal{P}_n}.$$

Since maps $q^{\mathcal{P}}_{\mathcal{P}_n}$ are continuous, we have defined a continuous function $h : X/\mathcal{P} \to \varprojlim\{X/\mathcal{P}_n, q^{\mathcal{P}_{n+1}}_{\mathcal{P}_n}\}$. Whenever $\{[x_n]_{\mathcal{P}_n}\}$ is a thread in the inverse system $\{X/\mathcal{P}_n, q^{\mathcal{P}_{n+1}}_{\mathcal{P}_n}\}$, then there exists $x \in \bigcap\{[x_n]_{\mathcal{P}_n}\colon n \in \omega\}$, since sets $[x_n]_{\mathcal{P}_n}$ consist of a centered family of non-empty closed sets in a compact space X. Thus $h^{-1}(\{[x_n]_{\mathcal{P}_n}\}) = [x]_\mathcal{P} \in X/\mathcal{P}$, hence h is a bijection. □

To obtain the converse of Theorem 12 one should consider an inverse system of compact metrizable spaces with all bonding maps skeletal. Such assumptions are unnecessary. So, we assume that spaces X_σ have countable π-bases, only.

Theorem 13. *Let $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ be a σ-complete inverse system such that all bonding maps π^σ_ϱ are skeletal and all projections π_σ are onto. If all spaces X_σ have countable π-base, then the limit $\varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is I-favorable.*

Proof. Let $\leqslant$ denotes the relation which directs Σ. Describe the following strategy for a match playing at the limit $X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$. Assume that Players play with basic sets of the form $\pi^{-1}_\sigma(V)$, where V is non-empty and open in X_σ and $\sigma \in \Sigma$.

Player I chooses an open non-empty set $A_0 \subseteq X$ at the beginning. Let $\mathcal{B}_0 = \{B_0\}$ be a respond of Player II. Take $\sigma_0 \in \Sigma$ such that $B_0 = \pi^{-1}_{\sigma_0}(V^0_0) \subseteq A_0$. Fix a countable π-base $\{V^0_0, V^0_1, \ldots\}$ for X_{σ_0}.

Assume, that we have just settled indexes $\sigma_0 \leqslant \sigma_1 \leqslant \cdots \leqslant \sigma_n$ and π-bases $\{V^k_0, V^k_1, \ldots\}$ for X_{σ_k}, where $0 \leqslant k \leqslant n$. Additionally assume, that for any V^k_m there exists V^{k+1}_j such that $\pi^{-1}_{\sigma_{k+1}}(V^{k+1}_j) = \pi^{-1}_{\sigma_k}(V^k_m)$. Now, Player I plays each set from

$$\mathcal{A}_{n+1} = \big\{\pi^{-1}_{\sigma_k}\big(V^k_m\big)\colon\ k \leqslant n \text{ and } m \leqslant n\big\}$$

one after the other. Let $\mathcal{B}_{n+1}$ denote the family of all responds of Player II, for innings from $\mathcal{A}_{n+1}$. Choose $\sigma_{n+1} \geqslant \sigma_n$ and a countable π-base $\{V^{n+1}_0, V^{n+1}_1, \ldots\}$ for $X_{\sigma_{n+1}}$ which contains the family

$$\big\{\big(\pi^{\sigma_{n+1}}_{\sigma_k}\big)^{-1}\big(V^k_m\big)\colon\ k \leqslant n \text{ and } m \in \omega\big\}$$

and such that for any $V \in \mathcal{B}_{n+1}$ there exists V^{n+1}_j such that $\pi^{-1}_{\sigma_{n+1}}(V^{k+1}_j) = V$.

Let $\sigma = \sup\{\sigma_n\colon n \in \omega\} \in \Sigma$. Any set $\pi_{\sigma_n}[\bigcup\{\bigcup\mathcal{B}_n\colon n \in \omega\}]$ is dense in X_{σ_n}, since it intersects any π-basic set $V^n_j \subseteq X_{\sigma_n}$. The inverse system is σ-complete, hence the set $\pi_\sigma[\bigcup\{\bigcup\mathcal{B}_n\colon n \in \omega\}]$ is dense in X_σ. The projection π_σ is skeletal by Proposition 8. So, the set $\bigcup\{\bigcup\mathcal{B}_n\colon n \in \omega\}$ is dense in X by Proposition 6. □

A continuous and open map is skeletal, hence every compact openly generated space is I-favorable.

Corollary 14. *Any compact openly generated space is I-favorable.*

The converse is not true. For instance, the Čech–Stone compactification βN of positive integers with the discrete topology is I-favorable and extremally disconnected. But βN is not openly generated, since a compact extremally disconnected and openly generated space has to be discrete, see Theorem 11 in [13].

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