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Equivariant Index and the Moment Map for Completely Integrable Torus Actions

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Consider a completely integrable torus action on a compact Spin^c manifold. The equivariant index of the Dirac operator is a virtual representation of the torus and is determined by the multiplicities of the weights which occur it in. We prove that these multiplicities are equal to values of the density function for the Duistermaat–Heckman measure, once this is defined appropriately. (The two-form that we take is half of the curvature of the line bundle which is associated to the Spin^c structure. It is closed and invariant but not necessarily symplectic.) We deduce that these multiplicities are equal to the topological degree of the "descended moment map" $\bar{\Phi}: M/T \to \mathfrak{t}^*$, which in nice cases can be described as sums of certain winding numbers. © 1998 Academic Press

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1. INTRODUCTION

This paper is a response to the following question of Raoul Bott. Take a Spin^c manifold M with an action of a torus T of half the dimension of the manifold. The equivariant index of the Dirac operator is then a virtual representation of T. In this representation are all the nonzero multiplicities equal to ± 1?

The answer is no. We show that the multiplicity of a weight $\alpha \in \mathfrak{t}^*$ is equal to the winding number around α of a certain map into $\mathfrak{t}^*$. This number can be any integer. We remark that Bott's question was motivated by the *Bott–Samelson manifolds* which are relevant to representation theory and for which the multiplicities are ± 1 [GK].

The answer "no" was already discovered in [KT]; Sue Tolman found a complex line bundle on a toric variety for which the multiplicities are -1 and -2. We generalize the results of [KT] to manifolds other than toric varieties and to Spin^c structures instead of holomorphic line bundles.

This paper can be viewed in the context of geometric quantization; more specifically, Kähler quantization. In Kähler quantization one usually starts from a symplectic manifold (M, ω). If there is a complex structure on M such that ω is Kähler, and a holomorphic Hermitian line bundle $L \to M$ whose curvature is $i\omega$, then the holomorphic sections of $L^{\otimes k}$ form a (finite dimensional) Hilbert space which for large k can be recovered from the symplectic data through the Riemann–Roch formula. The only place where this uses the nondegeneracy of ω is that if ω is Kähler then by the Kodaira vanishing theorem [We] the higher cohomology groups vanish for large k, and so the space of holomorphic section coincides with the index. In this paper we remove the demand that ω be Kähler and we replace the space of holomorphic sections by the index $\sum(-1)^i H^i(M, \mathcal{O}_L)$. The two-form ω need not even be symplectic, and we do not require it to be compatible with the complex structure (or the almost complex structure, or the Spin^c structure). Only the cohomology class of ω is important.

The paper consists of two parts; the "symplectic story" (Sections 4–7) and the "index story" (Sections 8–13). Sections 2 and 3 are more detailed introductions to these two parts. (The word "symplectic" is not quite in place; we allow closed degenerate two-forms.) A special case of a Spin^c structure on a manifold, which we discuss in detail, is a complex line bundle over an almost complex manifold. In Sections 6 and 13 we construct *equivariant connected sums* of toric varieties to provide examples of spaces which are not toric varieties and to which our theorems apply. Such creatures also occur as natural submanifolds of coadjoint orbits. The rest of the paper contains definitions, precise statements, proofs of the theorems, and examples.

From Bott we learned to look at the index instead of at the space of holomorphic sections (as in the Borel–Weil–Bott theorem [B]). Boutet de Monvel and Guillemin have used the twisted Dolbeault operator for almost complex structures [BG], and Vergne has used the Dirac spin operator [V2] for quantizing symplectic manifolds. We learned from Guillemin and Sternberg that the multiplicities of a representation is related to symplectic geometry and in particular to the Duistermaat–Heckman measure [GS2, GLS1, GLS2]. Indeed, our computations of multiplicities go through symplectic machinery. This is different from the method used in [KT], where the multiplicities were computed directly.

This work evolved from earlier work of the authors and of Tolman [G, KT]; see the end of Section 3. Earlier versions of this paper appeared as notes that the authors distributed in a sequence of talks at Tel-Aviv University in August 1992, and in the second author's thesis [K, Chaps. 2, 3]. The second author was supported by the Weizmann Institute of Science in the summer of 1992, by the Alfred P. Sloan dissertation fellowship in the academic year 1992/93, and partially by NSF Grant 9404404-DMS in 1994.

The second author thanks Susan Tolman for explaining the Atiyah–Bott–Lefschetz fixed point formula for holomorphic torus actions in terms of multiplicities; this description (and its analogue for Spin^c structures) was the key to our result. We wish to thank Chris Woodward for the wonderful examples on coadjoint orbits in Section 6.

2. THE (PRE-)SYMPLECTIC STORY

Let M be a C^∞-smooth oriented manifold and T a torus which acts on M. This means that we are given a homomorphism from T to the group of diffeomorphisms of M such that the associated map $T \times M \to M$ is smooth. We assume that the action is *effective*, i.e., that the only element of T which acts trivially is the identity. If the dimension n of the torus T is one half of the dimension of the manifold M then we say that the action is *completely integrable*.

A *symplectic* form on M is a differential two-form which is closed and nondegenerate. If we drop the nondegeneracy requirement then we can still define the Liouville measure and the moment map (see below). Traditionally, these have been associated to symplectic forms, therefore we use the term *presymplectic* in this context to mean simply a *closed two-form*.

Let ω be a T-invariant closed two-form on M. Let $\mathfrak{t}$ be the Lie algebra of T and let $\mathfrak{t}^*$ be the dual vector space. As is usual we define a moment map $\Phi: M \to \mathfrak{t}^*$ by the equation

$$\langle d\Phi, \xi \rangle = -\iota(\xi_M)\, \omega \qquad \text{for all} \quad \xi \in \mathfrak{t}, \tag{2.1}$$

where ξ_M is the vector field on M which generates the action of the one parameter subgroup $\exp(t\xi)$, $t \in \mathbb{R}$. The following properties are proved exactly as in the symplectic case: if the moment map Φ exists then it is unique up to a translation in $\mathfrak{t}^*$, the obstruction for the existence of Φ lies in the first De Rham cohomology of the manifold, and the moment map Φ is constant on the orbits of the torus T (see [Au]).

From now on until the end of this section we assume that M is compact. We define *Liouville measure* v by

$$v(U) = \int_U \frac{\omega^n}{(-2\pi)^n \, n!},$$

where U is an open subset of the manifold M. If the form ω is symplectic and the orientation is chosen appropriately then this is a positive measure. Otherwise, this defines a *signed measure* on M. A typical situation is that ω is degenerate along a hypersurface, the measure v is positive on one side of the hypersurface and negative on the other side; this can be seen by expressing ω^n as the product of an arbitrary volume form on M with a real valued function and looking at a neighborhood of a point where the function vanishes but its differential is nonzero.

The push-forward of Liouville measure by the moment map is defined by

$$m_{DH}(W) = v(\Phi^{-1}(W)), \tag{2.2}$$

where W is an open subset of $\mathfrak{t}^*$. It is also called the *Duistermaat–Heckman measures*. It is a signed measure which is supported on (a subset of) the image of Φ in $\mathfrak{t}^*$.

If ω is symplectic then the convexity theorem of Atiyah, Guillemin, and Sternberg [GS1, A] states that the image of the moment map is a convex polytope in the vector space $\mathfrak{t}^*$; it is called the *moment polytope*. If the torus action is completely integrable then the Duistermaat–Heckman measure is Lebesgue measure on the moment polytope. We note that all the compact symplectic manifolds with completely integrable Hamiltonian torus actions are Kähler toric varieties [De, G1, G2].

If ω is not symplectic then the image of the moment map can be almost anything, but the push-forward measure still behaves rigidly; we can then write

$$m_{DH} = \rho(\alpha) \cdot |d\alpha|,$$

where $|d\alpha|$ denotes Lebesgue measure on $\mathfrak{t}^*$ and where the density function $\rho(\alpha)$ is piecewise polynomial on $\mathfrak{t}^*$. This is known for symplectic forms. It is perhaps less widely known for closed two-forms, so we will say a few words about the proof. Duistermaat and Heckman [DH] showed, for symplectic forms, that the density function $\rho(\alpha)$ is piecewise polynomial, and deduced an *exact stationary phase formula* for the Fourier transform of the push-forward measure. This formula was later proved by Berline, Vergne, Atiyah, and Bott [BV, AB3] as a special case of a localization formula in equivariant cohomology, which also applies when the two-form is degenerate. Guillemin, Lerman, and Sternberg [GLS1, GLS2] have shown how the localization formula implies that the density function is piecewise polynomial; this also works for degenerate two-forms. In particular this implies that the boundaries of the regions in which $\rho(\alpha)$ is polynomial are contained in a finite union of hyperplanes. The above proof involves a global computation on M. The local result, that the push-forward measure is polynomial on every region of regular values of Φ, can also be derived as a special case of a recent formula of Vergne [V1, Theorem 7].

The polynomials which form the density function $\rho(\alpha)$ are of degree at most $\frac{1}{2}\dim M - \dim T$. In particular, if the action is completely integrable then the density function is piecewise constant.

Since the moment map is constant on T-orbits, it descends to a map $\bar{\Phi}$ defined on the quotient space M/T. If M is locally toric, i.e., is locally isomorphic to a smooth toric variety, then this quotient is topologically a manifold with boundary. ("Locally isomorphic" means equivariantly diffeomorphic on small *invariant* open sets.) If $\alpha \in \mathfrak{t}^* \setminus \bar{\Phi}(\partial(M/T))$ then we can define the winding number around α of every boundary component; see Section 5. We prove:

The density function $\rho(\alpha)$ is equal to the sum of these winding numbers.

EXAMPLE 2.3. Take the unit sphere, $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, with the circle action generated by the vector field $\partial/\partial\theta$, where r, θ, and z are cylindrical coordinates on $\mathbb{R}^3$. The standard area form is $d\theta \wedge dz$. An invariant two-form can be written as $\omega = f(z)\, dz \wedge d\theta$. The corresponding Liouville measure is positive wherever $f(z)$ is positive and is negative wherever $f(z)$ is negative. The function $\Phi(r, \theta, z) = F(z)$ is a moment map if $F(z)$ satisfies $\partial F/\partial z = f$. The quotient M/T is a closed interval, parametrized by $z \in [-1, 1]$. The function $\bar{\Phi}(z) = F(z)$ is increasing or decreasing near z according to the sign of $f(z)$. In particular, if we take $\omega = (2z+1)\, dz \wedge d\theta$ then $\bar{\Phi}(z) = z^2 + z$. The endpoints of the interval $[-1, 1] = M/T$ map to the points 0 and 2 in $\mathbb{R}$ and there is a "fold" at $\bar{\Phi}(-1/2) = -1/4$; see Fig. 1. The image of Φ is the interval $[-1/4, 2]$, but the contributions of the overlapping pieces cancel each other and the push-forward measure M_{DH} is Lebesgue measure on the interval $[0, 2]$. Indeed, if $g(\alpha)$ is a continuous function on $\mathbb{R}$ then $\int_{\mathbb{R}} g(\alpha)\, dm_{DH} = \int_{S^2} (\Phi^* g)(-w/2\pi) = 1/2\pi \int_{S^2} g(F(z))\, f(z)\, d\theta \wedge dz = \int_0^2 g(u)\, du$ using the change of variable $u = F(z)$.

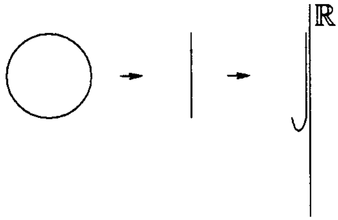

Fig. 1. A moment map for a presymplectic from on S^2.

3. THE STORY OF THE INDEX

EXAMPLE 3.1. The circle group acts on the vector space of holomorphic functions on $\mathbb{C}$ by $(\lambda f)(z) = f(\lambda^{-1} z)$. In particular, if $f(z) = z^k$ then $(\lambda f)(z) = \lambda^{-k} f(z)$, i.e., the monomial z^k is a weight vector with weight $-k$. Since any holomorphic function can be expanded into a power series,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots;$$

the weights which occur are precisely the nonpositive integers, each of which occurs with multiplicity 1. Figure 2 shows the *multiplicity diagram* for this representation. We note that our representation space is the kernel of the differential operator $\bar{\partial}$ acting on smooth functions on $\mathbb{C}$. On the other hand, if we take the moment map $\Phi(z) = -\frac{1}{2}|z|^2$ for the standard symplectic form $r\, d\theta \wedge dr$ then the density function for the Duistermaat–Heckman measure is equal to 1 on negative numbers and 0 on positive numbers; this gives a picture very similar to Fig. 2.

In this paper we describe and generalize a relationship between symplectic data and holomorphic data as was illustrated in the above example. We now give an outline of our story.

Take a compact smooth oriented manifold M with a completely integrable action of a torus T. We consider a certain extra (equivariant) structure on M. The simplest such structure could be a complex structure on M and a holomorphic line bundle over M. More generally, we can work with an almost complex structure on M and a smooth line bundle. Even more generally, we can take a Spin^c structure on M. This structure determines a virtual representation of the torus, the *equivariant index of the*

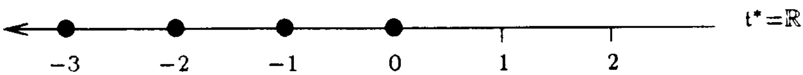

Fig. 2. Multiplicity diagram for the circle action on holomorphic functions on $\mathbb{C}$.

Dirac operator, which can be described in terms of its multiplicity function, mult: $\ell^* \to \mathbb{Z}$, where ℓ^* is the integral weight lattice in $\mathfrak{t}^*$. The extra structure (holomorphic line bundle, Spin^c structure...) also determines a closed two-form on M (essentially the curvature), so we can define the push-forward of Liouville measure Let $\rho(\alpha)$ be its density function. Our main theorem states:

$$\mathrm{mult}(\alpha) = \rho(\alpha) \qquad \text{for all} \quad \alpha \in \ell^*.$$

If we combine this with the previous description of the density function then, for locally toric spaces, we get:

There exists a map $\bar{\Phi}\colon M/T \to \mathfrak{t}^*$ such that $\bar{\Phi}(\partial(M/T)) \cap \ell^* = \emptyset$ and such that the multiplicity of the weight α in the equivariant index is equal to the sum of the winding numbers of the boundary components of M/T around α.

The motivation to look for winding numbers came from the first author's thesis [G]; he computed the twisted Dolbeault index in the special case of *Bott towers* and obtained shapes which he called *twisted cubes* (also see [GK]).

In the paper [KT], Tolman and the second author computed the twisted Dolbeault index for a holomorphic line bundle over a toric variety and identified the multiplicities as certain *winding numbers* which come from a map $\bar{\Phi}$ of M/T into $\mathfrak{t}^*$. They showed that the density function for the Duistermaat–Heckman measure is given by the same winding numbers and, hence,

$$\mathrm{mult}(\alpha) = \rho(\alpha) \tag{3.2}$$

when both are defined. There is a subset of $\mathfrak{t}^*$ which is contained in a finite union of hyperplanes and where the function $\rho(\alpha)$ is not defined; $\rho(\alpha)$ is locally constant outside this subset and is discontinuous along it. The multiplicity for α in this subset required a special treatment.

The results of [KT] generalize some classical results on toric varieties; Danilov showed that if the line bundle is positive then $\mathrm{mult}(\alpha) = 1$ for α in a certain polytope and $\mathrm{mult}(\alpha) = 0$ otherwise. Atiyah observed that this polytope coincides with the image of the moment map Φ. We note that Danilov computed the twisted Dolbeault cohomology for an arbitrary holomorphic line bundle but did not interpret the index in terms of winding numbers.

In this paper we generalize [KT] in two ways: our results apply to manifolds with torus actions which are not toric varieties, and we replace the twisted Dolbeault index by the more general Dirac Spin^c index. The statement of our theorem is cleaner for the Dirac operator than it is for the Dolbeault operator because in the case of the Dirac operator the hyperplanes in $\mathfrak{t}^*$ which contain the problematic α's do not intersect the weight lattice ℓ^* and (3.2) holds for *all* $\alpha \in \ell^*$. On the other hand, our proof relies on heavier machinery—localization formulas—whereas in [KT] Tolman computed the index directly and explicitly.

Our proof of (3.2) is quite simple. We express the density function $\rho(\alpha)$ by a localization formula of Guillemin, Lerman, and Sternberg (the "GLS formula" [GLS1] as a sum of contributions from fixed point data. We express the equivariant index as a sum of power series, again using the methods of [GLS1]. This yields an expression for multiplicities as a sum of contributions from the fixed point data. We then show that the two formulas, for the density function and for the multiplicities, coincide term by term when $\dim T = \frac{1}{2} \dim M$.

A similar proof should give a relationship between the functions $\mathrm{mult}(\alpha)$ and $\rho(\alpha)$ in the nonintegrable case, when $\dim T < \frac{1}{2} \dim M$, and where T acts with isolated fixed points: Denote by mult_k the multiplicity function associated to the line bundle $L^{\otimes k}$ and let $r = \frac{1}{2} \dim M - \dim T$, then

$$\lim_{k \to \infty} \frac{1}{k^r} \mathrm{mult}_k(k\alpha) = \rho(\alpha). \tag{3.3}$$

This generalizes results of Heckman [He] for flag manifolds, and of Guillemin and Sternberg [GS2] for arbitrary Kähler manifolds. As in Section 12 we can express both sides of (3.3) as sums of terms which correspond to the fixed points in M. For each fixed point p, the term on the right can be interpreted as $\pm$ the area of a convex polytope Δ in $\mathbb{R}^r$. The term on the left is $\pm$ the number of points in the intersection $\Delta \cap (1/k)\, \mathbb{Z}^r$. This can be viewed as a Riemann sum which approaches the integral of 1, i.e., the area of the polytope, when $k \to \infty$.

4. THE TORUS

In this section we establish the sign conventions and 2π factors used throughout the paper. We urge the reader to skip this section and refer to it as necessary. The circle group is $S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Its Lie algebra is $i\mathbb{R}$. The standard n-dimensional torus is $T^n = S^1 \times \cdots \times S^1$ (n factors). An n-dimensional torus T is a group which is isomorphic to T^n. Denote its Lie algebra by $\mathfrak{t}$. We have the exponential map, $\exp : \mathfrak{t} \to T$. Its kernel is a lattice, $\ell \subset \mathfrak{t}$. The *integral weight lattice* is defined as $\ell^* = \{\alpha \in \mathfrak{t}^* \otimes \mathbb{C} \mid \langle \alpha, \beta \rangle \in 2\pi i \mathbb{Z}\ \forall \beta \in \ell\}$. For the standard torus we have $\mathfrak{t} = i\mathbb{R}^n$, $\ell = 2\pi i \mathbb{Z}^n$, and $\ell^* = \mathbb{Z}^n$.

Every $\alpha \in \ell^*$ defines a homomorphism $\rho\colon T \to S^1$ by $\exp(\beta) \mapsto e^{\langle \alpha, \beta \rangle}$ for $\beta \in \mathfrak{t}$. Equivalently, we write $\rho(\lambda) = \lambda^\alpha$ for $\lambda \in T$. For the standard torus this makes sense as multi-index notation, so that $\alpha \in \mathbb{Z}^n$ and $\lambda^\alpha = \lambda_1^{\alpha(1)} \cdots \lambda_n^{(n)}$. The homomorphism ρ gives a one-dimensional representation of T by $\lambda\colon z \mapsto \rho(\lambda)\, z$. We denote this representation by $\mathbb{C}_\alpha$.

Suppose that we are given a linear representation of T on a real vector space W. If the dimension of W is even then there exist weights $\alpha_1, ..., \alpha_r \in \ell^*$ such that $W \cong \mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_r}$ as real representations of T. If $\dim W$ is odd then we can write $W \cong \mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_r} \oplus \mathbb{R}$ and T acts trivially on the last factor $\mathbb{R}$. The weights α_i are determined only up to sign because $\mathbb{C}_\alpha$ and $\mathbb{C}_{-\alpha}$ are isomorphic as real representations. If we are given an orientation of W and we require the isomorphism $W \cong \mathbb{C}^n$ to respect this orientation then the weights are determined up to a simultaneous sign change of an even number of weights. If $\dim T = n$ and $\dim W = 2n$ then the linear T-action on W is effective if and only if $\det(\alpha_1 \cdots \alpha_n) = \pm 1$. In this case no nonzero vector is fixed by the action. It follows, via Koszul's slice theorem [Au], that a completely integrable torus action on a smooth manifold has only isolated fixed points.

Convention 4.1. Suppose that we have an effective action of a torus T on an oriented manifold M with $\dim T = \frac{1}{2} \dim M = n$. A tubular neighborhood U of a free orbit in M is equivariantly diffeomorphic to the product $T \times D$, where D is an n-dimensional disc. Choose coordinates $x_1, ..., x_n \pmod{2\pi}$ on T and $y_1, ..., y_n$ on D such that the top-form $dy_1 \wedge dx_1 \wedge \cdots \wedge dy_n \wedge dx_n$ on U is positive relative to the given orientation. The choice of an orientation on T determines an orientation on the quotient space M/T in the following way. If $dx_1 \wedge \cdots \wedge dx_n$ is positive on T then we declare $dy_1 \wedge \cdots \wedge dy_n$ to be positive on M/T. Such a choice also determines an orientation on $\mathfrak{t}^*$; take a positive basis on $\mathfrak{t}$ and declare the dual basis to be positive on $\mathfrak{t}^*$.

The following example is helpful in determining any ambiguities with regard to plus/minus signs and factors of 2π.

EXAMPLE 4.2. Take the hyperplane bundle over $\mathbb{CP}^1$:

$$\{([z : w], f) \mid f \text{ is a linear functional on the complex line in } \mathbb{C}^2 \text{ spanned by } (z, w)\}.$$

Its total space has two coordinate charts, (z, a) and (w, b), with transition functions $w = z^{-1}$, $b = wa$. ((z, a) denotes the point $([z : 1], f)$ with $f(\lambda z, \lambda) = a\lambda$.)

Take the connection one-form

$$\beta = \frac{da}{a} - \frac{|z|^2}{1+|z|^2}\frac{dz}{z} = \frac{db}{b} - \frac{|w|^2}{1+|w|^2}\frac{dw}{w}.$$

Its curvature is

$$F = d\beta = \frac{dw \wedge d\bar{w}}{(1+|w|^2)^2};$$

it takes values in $\mathrm{Lie}(S^1) = i\mathbb{R}$. To get a real valued two-form we take $\omega = -iF$. Most people prefer the two-form $(i/2\pi)\,F$ because if F_n is the curvature of the bundle $\mathcal{O}(n)$ then $\int_{\mathbb{CP}^1}((i/2\pi)\,F_n) = n$. Our choice makes the moment map look nicer and more closely related to multiplicities, as we shall soon see, but causes a factor of $(-2\pi)^n$ to appear in the definition of Liouville measure.

For the hyperplane bundle we have $\omega = (-2r\,dr \wedge d\theta)/(1+r^2)^2$ in the polar coordinates $w = re^{i\theta}$. The moment map is determined by the equation $\iota(\partial/\partial\theta)\,\omega = -d\Phi$; we get $\Phi = 1/(1+r^2)$. The image of the moment map is the interval $[0, 1]$. Liouville measure is $(r\,dr\,d\theta)/\pi(1+r^2)^2$; its push-forward is Lebesgue measure on the interval $[0, 1]$.

Now we pass to the holomorphic picture. The space $H^0(\mathbb{CP}^1, \mathcal{O}(1))$ of global holomorphic sections is spanned by two sections, which are locally given by $b = w$ and $b = 1$. (In the other local charts these are $a = 1$ and $a = z$, respectively.) The circle group fixes the first section and acts on the second section with the weight 1. (I.e., $\lambda \in S^1$ acts as multiplication by λ.) So the weights are 0, 1 with both multiplicities $= 1$. Note that these are exactly the integral points in the image of the moment map.

Remark 4.3. To be absolutely precise, we should think of the curvature F as taking values in $\mathrm{Lie}(S^1)$. If we take the curvature F itself, instead of the two-form $\omega = iF$, then the moment map defined by $d\Phi^{\xi} = -\iota(\xi_M)\,F$ takes values in $\mathfrak{t}^* \otimes \mathrm{Lie}(S^1)$. The weights are elements of $\mathfrak{t}^* \otimes \mathrm{Lie}(S^1)$ because a weight is the differential of a homomorphism $T \to S^1$. So the weights and the moment map both take values in the same space, $\mathfrak{t}^* \otimes \mathrm{Lie}(S^1)$. The multiplicity of a weight $\alpha \in \mathfrak{t}^* \otimes \mathrm{Lie}(S^1)$ is equal to the density function at α of the push-forward measure $\Phi_* \nu$, where Liouvile measure ν is defined by integrating $((i/2\pi)\,F)^n$.

5. THE PUSH-FORWARD MEASURE AS A TOPOLOGICAL DEGREE

Throughout this section we consider an effective action of a torus T on a compact oriented manifold M with $\dim T = \frac{1}{2}\dim M$, a two-form ω on M

which is closed and invariant, and a moment map $\Phi: M \to \mathfrak{t}^*$ as defined in the introduction (2.1).

Denote by M_{free} the set of points in M with a trivial stabilizer. The image of M_{free} in M/T consists precisely of the smooth points of M/T, i.e., the points whose neighborhood is diffeomorphic to $\mathbb{R}^n$; this follows from the equivariant tubular neighborhood theorem. (The smooth structure on M/T is defined by declaring a function to be smooth if it lifts to a smooth T-invariant function on M.) Take a point p in M_{free} and let q be its image in M/T. Then p is a regular point for Φ if and only if the differential $d\bar{\Phi}_q$ is an isomorphism from $T_q(M/T)$ to $\mathfrak{t}^*$ (by a dimension count).

DEFINITION 5.1. Let q be a smooth point of M/T, where $\bar{\Phi}$ is regular; then we define $\operatorname{sign}(d\bar{\Phi}_q)$ to be 1 or -1, according to whether the map $d\bar{\Phi}_q: T_q(M/T) \to \mathfrak{t}^*$ preserves or reverses orientation. (Although these orientations depend on the orientation of T, $\operatorname{sign}(d\bar{\Phi}_q)$ is independent of this choice; see Convention 4.1.)

PROPOSITION 5.2. *Let $p \in M$ be a regular point for the moment map Φ and let q be its image in M/T, then q is a smooth point of M/T and is a regular point for the descended moment map $\bar{\Phi}$. In particular,* $\operatorname{sign}(d\bar{\Phi}_q)$ *is well defined.*

PROPOSITION 5.3. *Let $p \in M$ be a regular point for the moment map Φ and let q be its image in M/T. Then there is a T-invariant neighborhood U of p in M such that the push-forward of Liouville measure from U to* $\mathfrak{t}^*$ *is equal to Lebesgue measure on the image, $\Phi(U)$, times* $\operatorname{sign}(d\bar{\Phi}_q)$.

Propositions 5.2 and 5.3 will be proved by a series of lemmas.

LEMMA 5.4. *Suppose that $p \in M$ is a regular point for Φ. Then the stabilizer of p is trivial and the two-form ω is nondegenerate on a neighborhood of p.*

Proof of Lemma 5.4. Take any nonzero element $\xi \in \mathfrak{t}$. We have

$$\iota(\xi_M)\,\omega_p = -d\Phi^{\xi}|_p \neq 0 \tag{5.5}$$

because $d\Phi|_p$ is onto $\mathfrak{t}^*$. Therefore $\xi_M \neq 0$ and hence the stabilizer of p in T is discrete. Moreover, (5.5) implies that the tangent space to the orbit at p is transverse to the null-space of ω_p. But the orbit is isotropic (the proof of this is identical to the proof in the symplectic case; see [KKS]) and we have just shown that it has dimension n. Therefore the null-space of ω_p is trivial, i.e., ω is nondegenerate at p (and hence, on a neighborhood of p).

We have shown that the stabilizer group of p is finite and we will now show that it is trivial. By the equivariant tubular theorem there is an invariant neighborhood U of the orbit of P and a T-equivariant fibration $\pi: U \to T \cdot p$ whose fibers are transverse to the T-orbits. The moment map provides a trivialization, $\Phi: U \to \mathfrak{t}^*$; therefore the neighborhood U of the orbit looks like the product $(T \cdot p) \times B^n$ with B^n a ball in $\mathfrak{t}^*$, and the torus T acts on the first factor only. The torus action is effective on M, therefore it is effective on U, and so T must act freely on the orbit. ∎

Proof of Proposition 5.2. By Lemma 5.4 $p \in M_{\text{free}}$, therefore q is a smooth point of M/T. It is regular for $\bar{\Phi}$ because p is regular for Φ. ∎

LEMMA 5.6. *Suppose that $p \in M$ is a regular point for Φ and that Liouville measure ν, defined by integrating $(-\omega/2\pi)^n/n!$, is positive near p. Then there exists an invariant neighborhood U of the orbit $T \cdot p$ such that*

$$\Phi_*(\nu|_U) = \text{Lebesgue measure on } \Phi(U).$$

Proof. This is a special case of the Duistermaat–Heckman theorem. As in the proof of Lemma 5.4 we can write $U = (T \cdot p) \times B^n$. Since the orbit is Lagrangian we can choose a slice to the orbit which is Lagrangian and then the fibers of $U \to T \cdot p$ are Lagrangian. By a suitable choice of coordinates $x_1, ..., x_n$ (mod 2π) on T and $y_1, ..., y_n$ on $\mathfrak{t}^*$ we can write the symplectic form in a standard way,

$$\omega = \sum_{i=1}^{n} dy_i \wedge dx_i.$$

Liouville measure is then equal to the Haar measure on $\mathfrak{t}^*$ with total measure 1, times Lebesgue measure on $\mathfrak{t}^*$. Its push-forward via the moment map,

$$(x_1, y_1, ..., x_n, y_n) \mapsto (y_1, ..., y_n),$$

is Lebesgue measure $|dy_1 \wedge \cdots \wedge dy_n|$. ∎

Let $p \in M$ be a regular point for Φ. If Liouville measure is positive on a neighborhood of p then we write up $\nu_p > 0$ and sign $\nu_p = 1$; if it is negative then we write $\nu_p < 0$ and sign $\nu_p = -1$. Lemma 5.4 guarantees that one of these two cases must occur.

LEMMA 5.7. *If $p \in M$ is a regular point for Φ then*

$$\text{sign } \nu_p = \text{sign}(d\bar{\Phi}_q),$$

where q is the image of p in M/T.

Proof. We use the notation of the proof of Lemma 5.6. The y_i's denote coordinates on M/T and, also, coordinates on $\mathfrak{t}^*$. The descended moment map is then

$$\bar{\Phi}(y_1, ..., y_n) = (y_1, ..., y_n).$$

It preserves orientation if and only if the coordinates $x_1, y_1, ..., x_n, y_n$ give a positive orientation on M (where we use Convention 4.1), i.e., if and only if $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n = (-\omega)^n/n! = (2\pi)^n \nu$ is positive. ▮

Proof of Proposition 5.3. Lemma 5.6 implies that

$$\Phi_*(\nu|_U) = \text{sign}(\nu_p) \cdot (\text{Lebesgue measure on } \Phi(U)),$$

and by Lemma 5.7, sign $\nu_p = \text{sign}(d\bar{\Phi}_q)$. ▮

THEOREM 5.1. *Consider an effective action of a torus T on a compact oriented manifold M with* $\dim T = \frac{1}{2}\dim M = n$. *Let ω be a closed T-invariant two-form on M and let $\Phi: M \to \mathfrak{t}^*$ be a moment map. The topological degree of the descended moment map $\bar{\Phi}: M/T \to \mathfrak{t}^*$ at α,*

$$\rho(\alpha) = \sum_{q \in \bar{\Phi}^{-1}(\alpha)} \text{sign}(d\bar{\Phi}_q), \tag{5.8}$$

is defined for an open dense set of α's in $\mathfrak{t}^*$. It is equal to the density function for the push-forward measure with respect to Lebesgue measure on $\mathfrak{t}^*$.

Proof. The right-hand side of (5.8) is well defined whenever α is a regular value for the moment map Φ. Indeed, the summands are well defined by Proposition 5.2, and the sum is finite because Φ is proper. The set of regular values is open and dense because Φ is proper and by Sard's theorem. Let $\{q_i\}$ be the pre-images of α in M/T. There is a neighborhood V of α in $\mathfrak{t}^*$ whose pre-image in M/T consists of a finite disjoint union of open sets U'_i on which Φ is a diffeomorphism $U'_i \to V$; this follows from Proposition 5.2 and from the properness of Φ. Denote by U_i the pre-image of U'_i in M. Then the push-forward measure, restricted to V, is equal to the sum of the push-forwards from the individual U_i's. If we choose V small enough then, by Proposition 5.3, the push-forward measure from U_i to V has a density function equal to $\pm 1 = \text{sign}(d\bar{\Phi}_{q_i})$. The sum of these is what we need. ▮

Remark 5.9. If we have a torus action on a manifold and a closed invariant two-form ω whose cohomology class is integral then, even if there is no moment map into $\mathfrak{t}^*$, one can still define a moment map which takes values in a torus [MD, Wei]. Theorem 5.1 remains valid in this case.

DEFINITION 5.10. A *locally toric space* is an oriented manifold M^{2n} with a completely integrable action of a torus T such that every point has an invariant neighborhood which is equivariantly diffeomorphic to an open subset of a toric variety.

Equivalently, every point has a neighborhood which is equivariantly diffeomorphic to a neighborhood of the zero section in $T \times_H (\mathbb{C}^k \times \mathbb{R}^{n-k})$ with T acting by left translations and where the stabilizer H is a k-dimensional subtorus of T which acts effectively on $\mathbb{C}^k$ and trivially on $\mathbb{R}^{n-k}$. For a locally toric space, the quotient M/T is homeomorphic (but not diffeomorphic) to an oriented manifold with boundary $(M \backslash M_{\text{free}})/T$.

THEOREM 5.2. *Let M be a compact locally toric space of dimension $2n$ with a closed invariant two-form and a moment map $\Phi\colon M \to \mathfrak{t}^*$. Then the push-forward measure can be expressed as a sum of winding numbers as follows. Suppose that $\alpha \in \mathfrak{t}^*$ is not in the image of the boundary $\partial(M/T)$. For every boundary component $N_i \subseteq \partial(M/T)$ we consider the descended moment map $\bar{\Phi}\colon N_i \to \mathfrak{t}^* \backslash \alpha$ and the induced map*

$$H_{n-1}(N_i) \to H_{n-1}(\mathfrak{t}^* \backslash \alpha). \tag{5.11}$$

These homology groups are isomorphic to $\mathbb{Z}$; an isomorphism is determined by the choices of orientations as in Convention 4.1. *We define the* winding number *of N_i around α to be the image of the number* 1 *under the map* (5.11) *and we denote it $d_i(\alpha)$. The density function for the push-forward measure is then equal to the sum of these winding numbers*;

$$\rho(\alpha) = \sum_i d_i(\alpha).$$

Remark 5.13. In particular, if we fix the manifold M and the T-action, then the push-forward measure only depends on the restriction of the moment map to the boundary of M/T. Furthermore, it only depends on the images of the fixed points; this follows from the GLS formula (Theorem 7.1).

Proof of Theorem 5.2. Let α be a regular value of the moment map. By Proposition 5.4, $\Phi^{-1}(\alpha) \subseteq M_{\text{free}}$. Since $\partial(M/T) = (M \backslash M_{\text{free}})/T$, α is not in the image of $\partial(M/T)$, and so the right-hand side of (5.12) is well defined. The map $\bar{\Phi}\colon M/T \to \mathfrak{t}^*$ intertwines the long exact sequences for the pairs $(M/T, \partial(M/T))$ and $(\mathfrak{t}^*, \mathfrak{t}^* \backslash \alpha)$. We get

$$\begin{array}{ccc} H_n(M/T, \partial(M/T)) & \xrightarrow{\delta} & H_{n-1}(\partial(M/T)) \\ \Big\downarrow \bar{\Phi}_* & & \Big\downarrow \bar{\Phi}_* \\ H_n(\mathfrak{t}^*, \mathfrak{t}^*\backslash\alpha) & \xrightarrow{\cong} & H_{n-1}(\mathfrak{t}^*\backslash\alpha) \end{array} \tag{5.14}$$

Theorem 5.1 says that $\rho(\alpha) = \bar{\Phi}_*([M/T])$, where $[M/T]$ denotes the generator of $H_n(M/T, \partial(M/T))$. Since the diagram (5.14) commutes, we also have $\rho(\alpha) = \Phi_* \delta([M/T])$, where we identify $H_{n-1}(\mathfrak{t}^*, \mathfrak{t}^*\backslash\alpha)$ with $\mathbb{Z}$. But $\delta([M/T]) = \sum_i [N_i]$, where $[N_i]$ is the generator of $H_{n-1}(N_i)$. This shows that $\rho(\alpha)$ is equal to $\sum_i \bar{\Phi}_*([N_i]) = \sum_i d_i(\alpha)$ for every regular α. Since the set of regular αs is open and dense, we are done. ∎

6. EXAMPLES OF COMPLETELY INTEGRABLE SPACES

In this section we give examples of compact manifolds with completely integrable torus actions.

EXAMPLE 6.1. Take $\mathbb{CP}^1$ with the circle action $\lambda \cdot [z, w] = [z, \lambda w]$. This space is isomorphic to the two-sphere being rotated as in Example 2.3.

EXAMPLE 6.2. The standard two-torus acts on the complex projective plane $\mathbb{CP}^2$ by $(\lambda_1, \lambda_2) \cdot [z_0, z_1, z_2] = [z_0, \lambda_1 z_1, \lambda_2 z_2]$. The Fubini–Study symplectic form on $\mathbb{CP}^2$ is induced from the standard symplectic form on $\mathbb{C}^3$ restricted to the unit sphere. The moment map is $[z_0, z_1, z_2] \mapsto (-\frac{1}{2}|z_1|^2, -\frac{1}{2}|z_2|^2)$ when $|z_0|^2 + |z_1|^2 + |z_2|^2 = 1$. This map identifies the quotient M/T with the triangle $\{(x, y) \mid x \leqslant 0, y \leqslant 0, x + y \geqslant -1/2\}$ in $\mathbb{R}^2$.

EXAMPLE 6.3 (Toric varieties). Toric varieties are algebraic varieties which can be encoded in certain combinatorial objects, called fans. Special cases are $\mathbb{CP}^n$ and spaces obtained from them by taking products and complex blow-ups. They come equipped with completely integrable torus actions.

For a smooth compact toric variety, the quotient M/T is homeomorphic to a closed ball. By Theorem 5.2, for any closed invariant 2-form, the push-forward measure is given by the winding numbers of a map $S^{n-1} \to \mathfrak{t}^*$ around the points in $\mathfrak{t}^*$; also see [KT].

EXAMPLE 6.4 (Delzant spaces). Take a symplectic manifold with a completely integrable Hamiltonian torus action. By Delzant [De], this space is equivariantly symplectomorphic to a Kähler toric variety. The moment map descends to a diffeomorphism $\bar{\Phi}$ of the quotient M/T with a compact convex polytope in $\mathfrak{t}^*$, so the quotient is topologically a closed ball. The winding number of its boundary $\partial(M/T)$ around a point $\alpha \in \mathfrak{t}^*$ is equal to 1 for α in the interior of the polytope and 0 for α in the exterior. The push-forward measure is equal to Lebesgue measure on the polytope and zero outside.

EXAMPLE 6.5 (Bott towers). The spaces which initially motivated our work were Bott–Samelson manifolds. These are diffeomorphic to certain toric varieties which we call *Bott Towers* [GK] and which are higher dimensional analogues of *Hirzebruch surfaces*. For a Bott tower, the density function for the push-forward measure only takes the values ± 1 on its support.

EXAMPLE 6.6. The torus action on $\mathbb{CP}^2$ in Example 6.2 has three fixed points: $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$. Let M be the space obtained by performing a complex blow-up of $\mathbb{CP}^2$ at these three points. The action of T lifts to M. There is a closed invariant two-form ω on M for which the push-forward measure is given by Fig. 3, with winding numbers -1 and -2; see [KT]. (We get this picture if the integral of ω over $\mathbb{CP}^1$ is 1 and over every exceptional divisor it is < -1.)

EXAMPLE 6.7. Let a circle act on a two-torus by translations. The quotient M/T is a circle. There is a symplectic structure (the area form) but it has no moment map; an obstruction lies in $H^1(M, \mathbb{R})$. Any closed, invariant two-form which admits a moment map is exact. The degree of the map $\bar{\Phi}: M/T \to \mathbb{R}$ is zero at every point of $\mathbb{R}$. (The degree is constant because M/T is a closed manifold but $\bar{\Phi}$ is not onto because M/T is compact.) The push-forward of Liouville measure is the zero measure on $\mathfrak{t}^*$. This is guaranteed by the absence of fixed points; see Section 7.

EXAMPLE 6.8. *Locally toric spaces* were defined in 5.10. There is a useful criterion for being locally toric: We define a decomposition of our manifold by partitioning it into the connected components of points which have the same isotropy subgroup. We shall call these sets the *orbit-type strata* in M. If the closure of every stratum contains a fixed point then M

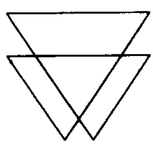

Fig. 3. $\mathbb{CP}^2$ blown up at three points.

is locally toric. This follows by noting that the T-action on the normal to the stratum does not change along the stratum and that a neighborhood of a fixed point is always toric.

EXAMPLE 6.9. One can construct new examples from old ones by taking *equivariant connected sums*. We now describe this construction.

Let M_1 and M_2 be two completely integrable T-spaces. Pick two points $p_i \in M_i$ which have the same stabilizer $H \subseteq T$ and for which the isotropy actions (i.e., the linear H-representations $T_{p_i}M_i$) are isomorphic. Each orbit $\mathcal{O}_i = T \cdot p_i$ has a tubular neighborhood U_i which is equivariantly diffeomorphic to $T \times_H D^{n+k}$, where D^{n+k} is a disk in the linear H-representation $T_{p_i}M_i$ and $k = \dim H$. The deleted neighborhood $U_i \backslash \mathcal{O}_i$ is equivariantly diffeomorphic to $T \times_H S^{n+k-1} \times I$, where I is the interval $(-\varepsilon, \varepsilon)$ and where H acts on the sphere S^{n+k-1}. We glue these neighborhoods by the equivariant diffeomorphism ψ which is the identity on $T \times_H S^{n+k-1}$ and which sends $t \mapsto -t$ on the I component. This produces the *equivariant connected sum of* M_1 *and* M_2 *along the orbits* $\mathcal{O}_1$ and $\mathcal{O}_2$. To produce an *oriented* manifold we reverse the orientation on the M_2 factor. We write

$$M_1 \sharp_T \bar{M}_2 = (M_1, \mathcal{O}_1) \sharp_T (\bar{M}_2, \mathcal{O}_2) = (M_1 \backslash \mathcal{O}_1) \cup_\psi (\bar{M}_2 \backslash \mathcal{O}_2).$$

The connected sum depends on the choice of the orbits $\mathcal{O}_i$. For example, connecting two copies of $\mathbb{CP}^2$ at a fixed point gives a manifold M such that M/T has one boundary component (see Fig. 4), whereas connecting along a free orbit gives an M/T with two boundary components (see Fig. 5).

We will use the short notation $M_1 \sharp_T M_2$, but we will always specify along which orbits we are gluing. The definition above depends not just on the orbits but also on the gluing map. We conjecture that if M_1 and M_2 are locally toric then different gluing maps should give rise to isomorphic spaces, and in fact that one only needs to specify the orbit-type strata in which the orbits lie.

The two extremal cases of equivariant connected sums are: (1) connecting at fixed points and (2) connecting along free orbits. Connecting at a

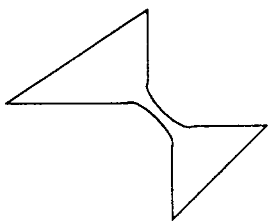

Fig. 4. Connecting two $\mathbb{CP}^2$'s at a fixed point.

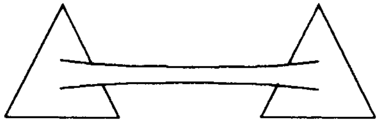

Fig. 5. Connecting two $\mathbb{CP}^2$'s at free orbits, or two kissing triangles.

fixed point produces an ordinary connected sum: $M_1 \sharp_T M_2 = M_1 \sharp M_2$. Connecting along a free orbit produces an ordinary connected sum *on the level of quotient spaces*: $(M_1 \sharp_T M_2)/T = (M_1/T) \sharp (M_2/T)$. This can be seen as follows. If D_i is a small disk in M_i which is transversal to the orbit $\mathcal{O}_i = T \cdot p_i$ then a tubular neighborhood of the orbit is isomorphic to the product $T \times D_i$. We glue the U_i's by taking the connected sums of the disks; locally we get $(T \times D_1) \sharp_T (T \times D_2) = T \times (D_1 \sharp D_2)$.

Any two closed invariant two-forms on M_1 and M_2 extend to the new manifold $M_1 \sharp_T \bar{M}_2$ in the following way. The restriction of ω_1 to U_1 is exact because U_1 retracts to the orbit $T \cdot p_1$ and ω_1 is zero on the tangents to the orbit. Thus we can write $\omega_1 = d\beta_1$ on U_1 and, similarly, $\omega_2 = d\beta_2$ on U_2. The manifolds M_1 and M_2 are connected by the "tube" $T \times_H S^{n+k-1} \times I$. Let β be a one-form on the tube which coincides with β_1 near one end of the tube and which coincides with β_2 on the other end. (Explicitly, $\beta = \rho_1 \beta - 1 + \rho_2 \beta_2$, where ρ_i are smooth cutoff functions, each supported on a small neighborhood of an endpoint of I.) The two-form $\omega = d\beta$ coincides with ω_1 near one end of the tube and with ω_2 near the other end and, therefore, extends to a two-form on the rest of the manifold.

We would like to extend not just the two-forms but also the moment maps. This can be done under the assumption:

$$\Phi_1^\xi(p_1) = \Phi_2^\xi(p_2) \qquad \text{for all} \quad \xi \in \mathfrak{h}, \tag{6.10}$$

where $\mathfrak{h}$ is the Lie algebra of the stabilizer. Note that this condition is empty if we glue along free orbits.

To extend the moment maps we need to modify the above construction. As before, we choose primitive one-forms $\tilde{\beta}_i$ such that $d\tilde{\beta}_i = \omega_i$ on U_i. Since $d\langle \tilde{\beta}_i, \xi_M \rangle = -\iota(\xi_M)\, \omega_i = d\Phi_i^\xi$, we have

$$\Phi_i^\xi = \langle \tilde{\beta}_i, \xi_M \rangle + f_i(\xi) \qquad \text{for all} \quad \xi \in \mathfrak{t} \tag{6.11}$$

for some $f_i \in \mathfrak{t}^*$. The linear functional $\xi \mapsto \langle \tilde{\beta}_i(p_i), \xi_M \rangle$ annihilates $\mathfrak{h}$. This fact, together with our assumption (6.10) and with (6.11), implies that $f_1(\xi) = f_2(\xi)$ for all $\xi \in \mathfrak{h}$. The linear functional $\Delta = f_1 - f_2$ annihilates $\mathfrak{h}$ so it defines a linear functional on the vector space $\mathfrak{t}/\mathfrak{h}$ and, hence, a one-form with constant coefficients on the orbit $T \cdot p_1 \cong T/H$. The pull-back to U_1 is a closed invariant one-form $\tilde{\Delta}$ with the property $\langle \tilde{\Delta}, \xi_M \rangle = \Delta(\xi)$. Define

$\beta_1 = \tilde{\beta}_1 + \tilde{\Delta}$ on U_1 and $\beta_2 = \tilde{\beta}_2$ on U_2; then $d\beta_i = \omega_i$ on U_i. One can easily check that

$$\Phi_1^\xi = f_2(\xi) + \langle \beta_1, \xi_M \rangle, \qquad \Phi_2^\xi = f_2(\xi) + \langle \beta_2, \xi_M \rangle.$$

On the tube we define $\beta = \rho_1 \beta_1 + \rho_2 \beta_2$, $\omega = d\beta$, and $\Phi^\xi = f_2(\xi) + \langle \beta, \xi_M \rangle$.

We note that as far as the push-forward measure is concerned we get nothing new.

PROPOSITION 6.12. *The push-forward measure for $M_1 \sharp_T \bar{M}_2$ is the same as that for the disjoint union $M_1 \dot{\cup} \bar{M}_2$.*

Proof. If we glue along free orbits then the proposition is an immediate consequence of the GLS formula (Theorem 7.1), because the fixed-point data for $M_1 \sharp_T \bar{M}_2$ and for the disjoint union $M_1 \dot{\cup} \bar{M}_2$ are the same.

Otherwise, Theorem 7.1 still implies that the push-forward measure for $M_1 \sharp_T \bar{M}_2$ is independent of how we glue the two-forms and the moment maps. In particular, by choosing the "gluing tube" arbitrarily small, we can arrange that the two-form on $M_1 \sharp_T \bar{M}_2$ is arbitrarily close to the two-forms ω_1 and ω_2 on the disjoint union $(M_1 \backslash \mathcal{O}_1) \dot{\cup} (M_2 \backslash \mathcal{O}_2)$ (embedded in $M_1 \sharp_T \bar{M}_2$). ▮

EXAMPLE 6.13. We can produce locally toric spaces which are not toric varieties by taking the equivariant connected sums of toric varieties. Recall, for a smooth compact toric variety, the quotient space M/T is homeomorphic to a closed ball. In particular, if M_1 and M_2 are toric varieties and we connect them along a free orbit then the quotient $(M_1 \sharp_T \bar{M}_2)/T$ is homeomorphic to the connected sum of two closed balls and, hence, $M_1 \sharp_T \bar{M}_2$ is not a toric variety.

EXAMPLE 6.14. If $M_1 = M_2 = \mathbb{CP}^2$ then the quotients M_i/T are (diffeomorphic to) triangles; see Example 6.2. If we glue along free orbits then the quotient $(M_1 \sharp_T \bar{M}_2)/T$ can be pictured as "two kissing triangles" (see Fig. 5) and is topologically an annulus $I \times S^1$. If we glue at fixed points then $(M_1 \sharp_T M_2)/T$ is topologically a disc. In fact, $M_1 \sharp_T M_2^-$ is then a toric variety; it is a Hirzebruch surface. On the other hand, if $M_i = S^2 \times S^2$ and we glue at fixed points then $M_1 \sharp_T M_2$ is not a toric variety, although $M_1 \sharp_T \bar{M}_2/T$ is a disc; see Example 6.16.

EXAMPLE 6.15. Take the six-dimensional manifold $M = \mathbb{CP}^1 \times (\mathbb{CP}^2 \sharp_T \mathbb{CP}^2)$. Then the boundary of M/T is a torus, $\partial(I \times (I \times S^1))$. In a similar way we can obtain examples with $\partial(M/T)$ being a surface of any genus. We start by connecting a sequence of $g+1$ copies of $\mathbb{CP}^2$ along free orbits. The quotient space is homeomorphic to a disc with g holes. If M is the product of this space with $\mathbb{CP}^1$ then $\partial(M/T)$ is a surface of genus g.

The rest of this section is devoted to an example which we learned from Chris Woodward and from Victor Guillemin. It occurs as a natural submanifold of a coadjoint orbit. It is not a toric variety but it is locally toric and it is isomorphic to the equivariant connected sum of two copies of $S^2 \times S^2$ at a fixed point.

In what follows the results which we state without proof are either explained in [Wo, Section 2]; [GS3, Section 45], or they can be checked by an easy computation.

EXAMPLE 6.16. Let $\mathcal{O}$ be a generic coadjoint orbit of the group $SU(3)$. The group $U(2)$ embeds as a subgroup of $SU(3)$ and thus acts on $\mathcal{O}$. Let $\Phi: \mathcal{O} \to \mathfrak{u}(2)^*$ be the moment map. Let T be the maximal torus of $U(2)$. The dual of its Lie algebra, $\mathfrak{t}^*$, embeds in $\mathfrak{u}(2)^*$ as the space of vectors fixed by T. The moment map Φ is transverse to $\mathfrak{t}^*$ and so the pre-image $M = \Phi^{-1}(\mathfrak{t}^*)$ is a submanifold of $\mathcal{O}$ on which T acts. This will be our space. We have $\dim M = 4$ and $\dim T = 2$.

Figure 6 shows the image of the moment map, $\Phi(M)$. It is the intersection $\mathfrak{t}^* \cap \Phi(\mathcal{O})$. The solid line separates the two Weyl chambers in $\mathfrak{t}^*$. We choose the positive Weyl chamber $\mathfrak{t}^*_+$ to be the bottom one. The rectangle in it is *Kirwan polytope* for the action of $U(2)$ on $\mathcal{O}$, i.e., the intersection $\Phi(\mathcal{O}) \cap \mathfrak{t}^*_+$. The rest of the picture is obtained by taking the images of this polytope under the action of the Weyl group $\mathbb{Z}/2\mathbb{Z}$.

If $\alpha \in \mathfrak{t}^*$ lies in the interior of one of these rectangles then the level set $\Phi^{-1}(\alpha)$ is a single T-orbit. For the α on the solid line, where the two rectangles intersect, $\Phi^{-1}(\alpha)$ is a 3-sphere in M.

The structure of M as a T-space is as follows. There are six fixed points. Their images in $\mathfrak{t}^*$ are the vertices which lie in the interiors of the two chambers; we call these *interior vertices*. Two circles in T occur as stabilizers (and T is the product of these circles). For each of these, the fixed point set has three components. The images of these components in $\mathfrak{t}^*$ are

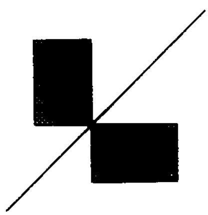

Fig. 6. The intersection of $\mathfrak{t}^*$ with the moment image in $\mathfrak{u}(2)^*$, for $U(2)$ acting on a coadjoint orbit of $SU(3)$.

the horizontal and vertical intervals which connect the interior vertices. At all other points of M the T-action is free. Since the closure of every orbit-type stratum of M contains a fixed point, M is locally toric; see Example 6.8. The space M is not a toric variety; otherwise the fixed point set of a circle in T could have at most two components.

M can be obtained from two copies of $S^2 \times S^2$ via an equivariant connected sum at fixed points. This follows from the Guillemin local model [Wo, Section 4]. The two rectangles in Fig. 6 are the images of the moment maps for the two copies of $S^2 \times S^2$.

Topologically, M/T is a closed two dimensional disk. Its two halves map to the two pieces in Fig. 6 but in opposite orientations. The disk twists as it maps to $\mathfrak{t}^*$ and the line which separates its two halves gets squashed to one point.

We choose an orientation on M arbitrarily. (The choice of a positive Weyl chamber determines an orientation on M.) Then the density function for the push-forward measure will be 1 on one piece of Fig. 6 and -1 on the other piece.

7. THE GUILLEMIN–LERMAN–STERNBERG FORMULA

Duistermaat and Heckman [DH] wrote a formula which expresses the Fourier transform of the push-forward measure in terms of the fixed point data of the torus action. Guillemin, Lerman, and Sternberg [GLS1, GLS2] have transformed this into a formula for the push-forward measure itself. We describe this formula in the special case of a completely integrable action.

Let M be a compact-oriented manifold of dimension $2n$ with an effective action of a torus T of dimension n and let ω be a closed and invariant two-form on M. If $p \in M$ is a fixed point then the action of T on the manifold M induces a linear action of T on the tangent space T_pM called the *isotropy representation*. As was explained in Section 4, we can identify the vector space T_pM with $\mathbb{C}^n$ and express the T-action as $\lambda \cdot (z_1, ..., z_n) = (\lambda^{\alpha_1} z_1, ..., \lambda^{\alpha_n} z_n)$. The elements $\alpha_j = \alpha_{j,p} \in \ell^*$ are called the *isotropy weights* at p.

Choose a vector $\xi \in \mathfrak{t}$ such that $\langle \alpha_{j,p}, \xi \rangle \neq 0$ for every $1 \leqslant j \leqslant n$ and for every fixed point p. Define $\varepsilon_{j,p} = \operatorname{sign}\langle \alpha_{j,p}, \xi \rangle$ and $\varepsilon_p = \prod_j \varepsilon_{j,p}$. Define the *polarized weights* by $\alpha'_{j,p} = \varepsilon_{j,p} \alpha_{j,p}$, so that $\langle \alpha'_{j,p}, \xi \rangle > 0$. Consider the following function on $\mathfrak{t}^*$:

$$\rho_p^\xi(\alpha) = \begin{cases} \varepsilon_p & \text{if} \quad \alpha = \Phi(p) - \sum_{j=1}^n r_j \alpha'_{j,p} \text{ for some } r_j \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Its support is a polyhedral cone emanating from $\Phi(p)$ and pointing in the direction of $-\xi$.

THEOREM 7.1 (Guillemin–Lerman–Sternberg). *The Duistermaat–Heckman measure is equal to Lebesgue measure on* $\mathfrak{t}^*$ *times the density function* $\sum_p \rho_p^\xi(\alpha)$.

Proof. This is a special case of the Theorem in [GLS1, Section 3]. ∎

8. EQUIVARIANT STRUCTURES

Background material for Sections 8–13 can be found in [AB1; AB2; LM]. We consider three types of structures on M, of increasing generality:

(i) a complex structure on M and a holomorphic line bundle $\mathbf{L} \to M$, or

(ii) an almost complex structure on M and a smooth line bundle $\mathbf{L} \to M$, or

(iii) a Spin^c structure on M.

Spin^c structure. The Spin group is the double covering:

$$\mathrm{Spin}(2n) \xrightarrow{p} \mathrm{SO}(2n) \tag{8.1}$$

with kernel $= Z_2$. We take the central extension,

$$\mathrm{Spin}^c(2n) = \mathrm{Spin}(2n) \times_{Z_2} U(1), \tag{8.2}$$

where $U(1)$ are the complex numbers of norm 1 and Z_2 is the subgroup $\{1, -1\}$. The map $[A, \lambda] \mapsto [p(A), \lambda^2]$ defines a double covering,

$$\mathrm{Spin}^c(2n) \to \mathrm{SO}(2n) \times \mathrm{SO}(2). \tag{8.3}$$

We think of $U(1)$ as a double covering of SO(2); this notation will be convenient later.

Let M be a smooth manifold. Choose a Riemann metric on M, an orientation, and a complex Hermitian line bundle $\tilde{\mathbf{L}} \to M$. This data determines a principal bundle over M with a structure group $\mathrm{SO}(2n) \times \mathrm{SO}(2)$. (Its fiber over $p \in M$ consists of pairs of an oriented orthogonal frame of $T_p M$ and an element of the unit circle in $\tilde{\mathbf{L}}_p$). A Spin^c *structure* on M is a principal Spin^c bundle and a double covering $P_{\mathrm{Spin}^c(2n)} \to P_{\mathrm{SO}(2n) \times \mathrm{SO}(2)}$ which is equivariant with respect to the group homomorphism (8.3).

EXAMPLE 8.4. A Spin structure determines a Spin^c structure in which the $U(1)$ bundle is trivial. We can twist this Spin^c structure by a line bundle $\mathbf{L} \to M$ to produce a new Spin^c structure in which $\tilde{\mathbf{L}}$ is replaced by

$\tilde{\mathbf{L}} \otimes \mathbf{L}^{\otimes 2}$. The converse is not true—given a Spin^c structure we cannot in general separate it into a $U(1)$ bundle and a Spin bundle. And even if this is possible, it might not be possible equivariantly. In fact, under a completely integrable torus action this is *never* possible.

EXAMPLE 8.5. Every almost complex manifold is also a Spin^c manifold. The associated line bundle $\tilde{\mathbf{L}}$ is the determinant bundle $\bigwedge_{\mathbb{C}}^n TM$. This is because there is a group homomorphisms $\beta\colon U(n) \to \mathrm{Spin}^c(2n)$ which allows us to define $P_{\mathrm{Spin}^c(2n)} = P_{U(n)} \times_\beta \mathrm{Spin}^c(2n)$.

Liftings of torus actions. Suppose that we have a torus action on a smooth manifold M. If M has an extra structure (for example, a Spin^c structure) then we define a *lifting of the torus action* in the following way.

A lifting of the torus action to a principal bundle $P \to M$ is defined to be a smooth left action of T on the total space of P, which commutes with the principal right action of the structure group and such that the bundle map $P \to M$ is T-equivariant.

A lifting of the torus action to a vector bundle $E \to M$ is defined to be a smooth action of T on E which sends each fiber linearly onto another fiber and such that the bundle map $E \to M$ is T-equivariant.

Suppose that M is a complex manifold of dimension n and $\mathbf{L} \to M$ is a holomorphic line bundle. A torus action on this structure is a holomorphic torus action on M, together with a lifting to a holomorphic action on $\mathbf{L}$.

Suppose that M has an almost complex structure and that $\mathbf{L} \to M$ is a smooth line bundle. A torus action on this structure is a smooth torus action on M which preserves the almost complex structure and a smooth lifting of the action to $\mathbf{L}$.

Finally, suppose that M has a Spin^c structure. A torus action on the Spin^c structure consists of a lifting of the torus action to the bundle $P_{\mathrm{Spin}^c(2n)}$. Such a lifting induces actions of T on the principal bundles $P_{\mathrm{SO}(2n)}$ and $P_{SO(2n)}$. We require the action on $P_{\mathrm{SO}(2n)}$ to be induced from the action on M (in particular, the action on M must preserve the Riemann metric). Note that the action on the principle circle bundle induces an action on the line bundle $\tilde{\mathbf{L}}$ which preserves the Hermitian metric.

9. THE DOLBEAULT AND DIRAC OPERATORS

An almost complex structure on a smooth manifold M determines a bigrading $\bigwedge^r (T^*M \otimes \mathbb{C}) = \bigoplus_{p+q=r} \bigwedge^{p,q} T^*M$ and, hence, a decomposition of the complex-valued differential forms:

$$\Omega^r(M, \mathbb{C}) = \bigoplus_{p+q=r} \Omega^{p,q}(M). \tag{9.1}$$

If the almost complex structure is integrable and $z_1, ..., z_n$ are local holomorphic coordinates then $\Omega^{p,q}(M)$ consists of the forms $\sum a_{I,J}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, where the coefficients $a_{I,J}$ are smooth complex-valued functions on M.

The Dolbeault operator $\bar{\partial}: \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ is defined to be the exterior differentiation d followed by the projection $\Omega^{p+q+1}(M, \mathbb{C}) \to \Omega^{p,q+1}(M)$.

In the presence of a smooth line bundle $\mathbf{L} \to M$ we consider the differential forms with coefficients in $\mathbf{L}$, denoted $\Omega^{p,q}(M, \mathbf{L})$; these are the smooth sections of $\bigwedge^{p,q} T^*M \otimes \mathbf{L}$. A connection on $\mathbf{L}$ determines a map $\nabla_{\mathbf{L}}: \Gamma(\mathbf{L}) \to \Gamma(T^*M \otimes \mathbf{L})$. It decomposes as $\nabla_{\mathbf{L}} = \nabla'_{\mathbf{L}} + \nabla''_{\mathbf{L}}$, where $\nabla'_{\mathbf{L}}$ and $\nabla''_{\mathbf{L}}$ take values in $\Omega^{1,0}(M, \mathbf{L})$ and $\Omega^{0,1}(M, \mathbf{L})$, respectively. We then define $\bar{\partial}_{\mathbf{L}}: \Omega^{p,q}(M, \mathbf{L}) \to \Omega^{p,q+1}(M, \mathbf{L})$ by

$$\bar{\partial}_{\mathbf{L}}(\varphi \otimes s) = (\bar{\partial}\varphi) \otimes s + (-1)^q \varphi \wedge \nabla''_{\mathbf{L}} s,$$

where φ is a smooth (p, q)-form on M and where s is a smooth section of $\mathbf{L}$. In general $\bar{\partial}^2_{\mathbf{L}} \neq 0$, so we cannot use it to define cohomology as we do for (integrable) complex structures. Instead, we consider the "rolled-up operator"

$$\mathscr{D}_{\mathbf{L}}: \Omega^{0,\text{even}}(M, \mathbf{L}) \to \Omega^{0,\text{odd}}(M, \mathbf{L}),$$

where $\mathscr{D}_{\mathbf{L}} = (\bar{\partial} + \bar{\partial}^*) \otimes 1 + 1 \otimes (\nabla''_{\mathbf{L}} + \nabla''^*_{\mathbf{L}})$ and $\bar{\partial}^*$ and $\nabla''^*_{\mathbf{L}}$ are adjoints of $\bar{\partial}$ and $\nabla''_{\mathbf{L}}$ with respect to some Hermitian metrics on M and $\mathbf{L}$.

Now assume that we have a Spin^c structure on M. The group Spin$(2n)$ has two real representations, called the *Spin representations* and denoted Δ^+ and Δ^-. These fit together with the scalar action of $U(1)$ to give representations of the group Spin$^c(2n)$ on the complex vector spaces $\Delta^{\pm}_{\mathbb{C}} = \Delta^{\pm} \otimes \mathbb{C}$.

The group Spin$(2n)$ acts on $\mathbb{R}^{2n}$ via the action of SO$(2n)$. There exists a natural map

$$\sigma: \mathbb{R}^{2n} \otimes \Delta^+ \to \Delta^- \tag{9.2}$$

which is equivariant with respect to the actions of Spin$(2n)$. (This map comes from multiplication in the Clifford algebra in which these objects actually live; see [LM].)

If M is a Spin^c manifold then the *Spinor bundles* over M are the associated bundles $S^{\pm}_{\mathbb{C}} = P \times_G \Delta^{\pm}_{\mathbb{C}}$, where $G = \text{Spin}^c(2n)$ and $P = P_{\text{Spin}^c(2n)}$. The Dirac operator $\not{D}: \Gamma(S^+_{\mathbb{C}}) \to \Gamma(S^-_{\mathbb{C}})$ is constructed in the following way. The map (9.2) induces a map on the spaces of sections, $\tilde{\sigma}: \Gamma(T^*M \otimes S^+_{\mathbb{C}}) \to \Gamma(S^-_{\mathbb{C}})$.

A connection on the principal Spin^c-bundle induces a connection on the associated spinor bundle $S^+_{\mathbb{C}}$ and thus a covariant differentiation map $\Gamma(S^+_{\mathbb{C}}) \xrightarrow{\nabla} \Gamma(T^*M \otimes S^+_{\mathbb{C}})$. The Dirac operator is the composition, $\Gamma(S^+_{\mathbb{C}}) \xrightarrow{\nabla} \Gamma(T^*M \otimes S^+_{\mathbb{C}}) \xrightarrow{\tilde{\sigma}} \Gamma(S^-_{\mathbb{C}})$. The map $\tilde{\sigma}$ is the symbol of this operator.

Recall that every almost complex manifold is also a Spin^c manifold. In this case, the spinor bundles are the anti-holomorphic forms, $S^+_{\mathbb{C}} \cong \bigwedge^{0,\,\mathrm{even}}(M)$ and $S^-_{\mathbb{C}} \cong \bigwedge^{0,\,\mathrm{odd}}(M)$.

10. THE EQUIVARIANT INDEX

We described two elliptic differential operators $\mathscr{D}\colon \mathscr{H}^+ \to \mathscr{H}^-$. In one case $\mathscr{D}$ was the twisted Dolbeault operator and $\mathscr{H}^+$, $\mathscr{H}^-$ were the spaces of smooth sections of $\bigwedge^{0,\,\mathrm{even}} M \otimes \mathbf{L}$ and $\bigwedge^{0,\,\mathrm{odd}} M \otimes \mathbf{L}$. In the other case $\mathscr{D}$ was the Dirac operator and $\mathscr{H}^+$, $\mathscr{H}^-$ were the spaces of smooth sections of the spinor bundles over M.

Ellipticity implies that $\mathscr{D}$ has a finite-dimensional kernel and cokernel. (We omit the analytic details; see [BB].) The *index* of $\mathscr{D}$ is the virtual vector space, $[\ker \mathscr{D}] - [\operatorname{coker} \mathscr{D}]$. This is an element of the K-theory of a point, and it is independent of the choices of metrics and connections used in defining the Dolbeault and Dirac operators. It should be thought of as a formal difference of vector spaces.

In the case of a complex manifold with a holomorphic line bundle, the index coincides with the alternating sum of the cohomology groups for the sheaf of holomorphic sections, $\sum(-1)^i H^i(M, \mathcal{O}_{\mathbf{L}})$. In the almost complex or Spin^c case the individual H^i's are not defined; however, the index provides us with the desired "cohomology." The first author, in his thesis [G], applied this technique of "rolling up the operator" to compute a certain Dolbeault index by deforming the complex structure into an almost complex structure for which the index of the rolled-up operator could be computed more easily.

In the presence of a torus action which preserves every structure in sight, the operator $\mathscr{D}$ is equivariant, so its kernel and cokernel are finite-dimensional representations of T. Again by forming their formal difference (modulo the appropriate equivalence relation) the index becomes a *virtual representation* of T. It is determined by its multiplicity function, $\mathrm{mult}\colon \ell^* \to \mathbb{Z}$, defined by: $\mathrm{mult}(\alpha) = \mathrm{mult}^+(\alpha) - \mathrm{mult}^-(\alpha)$, where $\mathrm{mult}^+(\alpha)$, $\mathrm{mult}^-(\alpha)$ are the multiplicities of α in the representations $\ker \mathscr{D}$ and $\operatorname{coker} \mathscr{D}$, respectively. Alternatively, the equivariant index is determined by its *virtual character*, $\chi\colon T \to \mathbb{C}$, defined by

$$\chi(\lambda) = \chi^+(\lambda) - \chi^-(\lambda),$$

where $\chi^+(\lambda)$ and $\chi^-(\lambda)$ are the traces of the respective operators $\{\lambda: \ker \mathcal{D} \to \ker \mathcal{D}\}$ and $\{\lambda: \operatorname{coker} \mathcal{D} \to \operatorname{coker} \mathcal{D}\}$. We have $\chi(\lambda) = \sum_{\alpha \in \ell^*} \operatorname{mult}(\alpha) \lambda^\alpha$, so the multiplicity function is simply the Fourier transform of the virtual character.

We now describe explicitly how the torus acts on the vector spaces $\mathcal{H}^\pm$. First assume that we have an almost complex structure on M and a smooth line bundle $\mathbf{L}$. The torus action on M induces an action on the space of smooth functions by

$$(\lambda \cdot f)(p) = f(\lambda^{-1} \cdot p). \tag{10.1}$$

Similarly, $\lambda \in T$ acts on differential forms by pulling-back via the action of λ^{-1}. This preserves the spaces $\Omega^{p,q}(M)$ because the T-action preserves the almost complex structure. Next, the T-action on the line bundle $\mathbf{L}$ induces an action on the space of sections by

$$(\lambda \cdot s)(p) = \lambda(s(\lambda^{-1} p)). \tag{10.2}$$

These two actions fit together to an action on the space of twisted forms $\Omega^{p,q}(M, \mathbf{L})$ and thus to actions of T on the spaces $\mathcal{H}^+ = \Omega^{0,\text{even}}(M, \mathbf{L})$ and $\mathcal{H}^- = \Omega^{0,\text{odd}}(M, \mathbf{L})$.

In the presence of a T-equivariant Spin^c structure, the action of T on the principal bundle $P_{\text{Spin}^c(2n)}$ induces an action on the associated Spinor bundles $S^\pm_{\mathbb{C}}$ and on their spaces of sections, $\mathcal{H}^\pm$.

11. THE ATIYAH–BOTT FORMULA

Let $\mathbf{E} \to M$ be a vector bundle and let $f: M \to M$ be a smooth map. Atiyah and Bott define a *lifting* of f to be a bundle map $\varphi: f^*\mathbf{E} \to \mathbf{E}$, so at each point $p \in M$ we get a linear isomorphism $\varphi_p: \mathbf{E}_{f(p)} \to \mathbf{E}_p$. This gives rise to a map on the space of sections, $T: \Gamma(\mathbf{E}) \to \Gamma(\mathbf{E})$, by $(Ts)(p) = \varphi(s(f(p)))$.

A torus action on M gives, for each element λ of the torus, a diffeomorphism $f_\lambda: M \to M$. A lifting of the torus action to a bundle $\mathbf{E} \to M$ gives linear isomorphisms $\mathbf{E}_p \to \mathbf{E}_{f(p)}$. In the notation of Atiyah and Bott this is a lifting of the inverse map $(f_\lambda)^{-1}$, not of f_λ.

In our setting we have two vector bundles, $\mathbf{E}^+$ and $\mathbf{E}^-$, and an elliptic differential operator $\mathcal{D}$ between their spaces of smooth sections. The torus T acts on everything. We get two representations of the torus, ρ^+ and ρ^-, on the kernel and cokernel of $\mathcal{D}$, respectively. The *virtual character* of the representation of T on the index of $\mathcal{D}$ was defined as

$$\chi(\lambda) = \operatorname{trace} \rho^+(\lambda) - \operatorname{trace} \rho^-(\lambda).$$

The torus T is of half the dimension of the manifold M and the action is effective, therefore the fixed points are isolated. For an open dense set of λ's in T, the graph of the action map $f_\lambda: M \to M$ is transversal to the diagonal in $M \times M$ and the set of fixed points of f_λ coincides with the set of fixed points for the torus action. For such λ's we can apply the formula of Atiyah and Bott, which computes the Lefschetz number as a sum over fixed points:

$$\chi(\lambda) = \sum_p \frac{\operatorname{trace}_{\mathbb{C}}(\varphi_\lambda^+)_p - \operatorname{trace}_{\mathbb{C}}(\varphi_\lambda^-)_p}{|\det_{\mathbb{R}}(1 - (df_\lambda^{-1})_p)|}, \tag{11.1}$$

where $(\varphi_\lambda^+)_p$ is the automorphism of the fiber E_p^+ which is given by the action of the element $\lambda \in T$, and $(\varphi_\lambda^-)_p$ is similar.

In the rest of this section we compute (11.1) for the Dirac and Dolbeault operators.

PROPOSITION 11.2 (The virtual character for the Dolbeault operator). *Take an almost complex manifold M and a smooth line bundle $\mathbf{L} \to M$. Suppose that we have a torus action on M which preserves the almost complex structure and a lifting of this action to $\mathbf{L}$. We form the twisted Dolbeault operator $\mathscr{D}_{\mathbf{L}}$. Its virtual character is then given by a sum over the fixed points, $\chi(\lambda) = \sum_p v(p)(\lambda)$, and the contribution of the fixed point p is*

$$v(p)(\lambda) = \frac{\lambda^\mu}{\prod_{i=1}^n (1 - \lambda^{-\alpha_i})},$$

where $\alpha_1, ..., \alpha_n$ are the weights for the torus action on the complex vector space $T_p M$ and where μ is the weight for the torus action on the complex line $\mathbf{L}_p$.

Proof. This computation was already done by Atiyah and Bott in [AB2, p. 457]. We will now translate their argument into our notation.

We have $E^+ = (\bigwedge^{0,\,\text{even}} T^*M) \otimes \mathbf{L}$ and $E^- = (\bigwedge^{0,\,\text{odd}} T^*M) \otimes \mathbf{L}$. An element of $(\bigwedge^{0,q} T_p^*M) \otimes \mathbf{L}_p$ can be written as $u\, d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}$, where $\partial/\partial z_i$ are a basis for the complex linear space $T_p M$ and where u is a vector in the fiber $\mathbf{L}_p$. If p is a fixed point we may make a linear change of basis so the dz_i's are eigenvectors for the torus action on T_p^*M then $\lambda \cdot dz_i = \lambda^{-\alpha_i}\, dz_i$. The minus sign is because the action of $\lambda \in T$ on the differential forms was defined by pulling back via the action of λ^{-1} on M. We get $\lambda \cdot (u\, d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}) = \lambda^{\mu + \alpha_{i_1} + \cdots + \alpha_{i_q}}(u\, d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q})$; i.e., the weights for the torus action on $E_p^{\pm}$ are $\{\mu+\alpha_{i_1}+\cdots+\alpha_{i_q}\}_{i_1<\cdots<i_q}$. The numerator in the term $v(p)(\lambda)$ is

$$\operatorname{trace}_{\mathbb{C}} \varphi_p^+ - \operatorname{trace}_{\mathbb{C}} \varphi_p^- = \sum_q (-1)^q \sum_{i_1<\cdots<i_q} \lambda^{\mu+\alpha_{i_1}+\cdots+\alpha_{i_q}} = \lambda^\mu(1-\lambda^{\alpha_1})\cdots(1-\lambda^{\alpha_n}).$$

Downstairs, if $f=f_\lambda^{-1}$ then $df_p=\operatorname{diag}(\lambda^{-\alpha_1}, ..., \lambda^{-\alpha_n})$ and so

$$|\det_{\mathbb{R}}(1-df_p)| = \prod_{i=1}^n (1-\lambda^{-\alpha_i})(1-\lambda^{\alpha_i})$$

(because $\det_{\mathbb{R}} w = w\bar{w}$ for every complex number w). ∎

Proposition 11.3 (The virtual character for the Dirac operator). *Let M be a Spin^c manifold with a torus action and with a lifting of this action to the Spin^c structure. Then the virtual character of the Dirac operator is given by a sum over the fixed points, $\chi(\lambda)=\sum_p v(p)(\lambda)$, and the contribution of the fixed point p is*

$$v(p)(\lambda) = \frac{\lambda^{\mu/2}\prod_{i=1}^n (\lambda^{-\alpha_i/2}-\lambda^{\alpha_i/2})}{\prod_{i=1}^n (1-\lambda^{\alpha_i})(1-\lambda^{-\alpha_i})}, \tag{11.4}$$

where $\alpha_1, ..., \alpha_n$ are weights for the linear action of T on T_pM and where μ is the weight for the torus action on the fiber $\tilde{\mathbf{L}}_p$ of the line bundle associated to the Spin^c structure.

Note that (11.4) only depends on pointwise data: the actions of our torus on the tangent space T_pM and on the fiber $\tilde{L}_p$. Also, recall that the α_i's are only determined up to a simultaneous sign change of an even number of them. Since $v(p)$ flips its sign when we flip the sign of an α_i, it is well defined, although the individual α_i's are not well defined.

Proof. The relevant Lie groups are Spin$^c(2n)$, Spin$(2n)$, and SO$(2n)$. The relations (8.1), (8.2), and (8.3), between these Lie groups give similar relations between their maximal tori. We denote by T_G the standard maximal torus in the group G which is one of the above. Then we have a double covering $T_{\mathrm{Spin}(2n)} \to T_{\mathrm{SO}(2n)}$, a central extension $T_{\mathrm{Spin}^c(2n)} = T_{\mathrm{Spin}(2n)} \times_{Z_2} U(1)$, and a double covering $T_{\mathrm{Spin}^c(2n)} \to T_{\mathrm{SO}(2n)} \times \mathrm{SO}(2)$.

We write $T_{\mathrm{SO}(2n)} = \mathbb{R}^n/\mathbb{Z}^n$. Let $x_1, ..., x_n$ be the standard basis for the weight lattice in $\mathfrak{t}^*_{\mathrm{SO}(2n)} = \mathbb{R}^n$. The double covering $T_{\mathrm{Spin}(2n)} \to T_{\mathrm{SO}(2n)}$ induces an isomorphism $\mathfrak{t}^*_{\mathrm{SO}(2n)} \xrightarrow{\cong} \mathfrak{t}^*_{\mathrm{Spin}(2n)}$. With this identification, the weight lattice of Spin$(2n)$ is generated by $x_1, ..., x_n$ and $\frac{1}{2}(x_1+\cdots+x_n)$. Let y be the standard generator for the weight lattice of SO(2) and remember that the map $U(1) \to \mathrm{SO}(2)$ is a double covering, then the weight lattice of $U(1)$ is generated by $y/2$. Finally, the weight lattice of $T_{\mathrm{Spin}^c(2n)} = T_{\mathrm{Spin}(2n)} \times_{Z_2} U(1)$ in $\mathbb{R}^n \times \mathbb{R}$ is generated by $x_1, ..., x_n, y$, and $\frac{1}{2}(x_1 + \cdots + x_n + y)$.

Let $\Delta^{\pm}_{\mathbb{C}} = W_1 \oplus \cdots \oplus W_{2^{n-1}}$ be the weight decomposition of the Spin representation with respect to the actions of the maximal torus. The weights are $\alpha_j = \frac{1}{2}(\pm x_1 \pm x_2 \pm \cdots \pm x_n)$. The weights occurring in $\Delta^+_{\mathbb{C}}$ are those with an even numbers of plus signs in α_j, and those occurring in $\Delta^-_{\mathbb{C}}$ are with an odd numbers of plus signs. With this convention the Dirac index coincides with the Dolbeault index if the manifold is almost complex and the orientation is determined by the almost complex structure.

The standard action of $U(1)$ on $\mathbb{C}$ is given by the weight $y/2$; hence, $T_{\mathrm{Spin}^c(2n)}$ acts on W_j with a weight of $\alpha_j + y/2$.

Remember that the Spin^c structure is a principal bundle P over M with structure group Spin^c and the spinor bundles are the associated bundles $E^{\pm} = P \times_{\mathrm{Spin}^c} \Delta^{\pm}_{\mathbb{C}}$. Let p be a fixed point in M. The torus T acts on the fiber $E^{\pm}_p$; we need to determine the weights for this representation.

Fix a basepoint in the fiber of P over p. This determines: (i) an identification of that fiber with the group $\mathrm{Spin}^c(2n)$, (ii) an identification of the fibers $E^{\pm}_p$ with the representation space $\Delta^{\pm}_{\mathbb{C}}$, and (iii) an identification of the tangent space T_pM with $\mathbb{R}^{2n}$. The isotropy representation is given by an inclusion $T \hookrightarrow \mathrm{SO}(2n)$ followed by the left action of $\mathrm{SO}(2n)$ on $\mathbb{R}^{2n} \cong T_pM$. Similarly, the action of T on $E^{\pm}_p$ is given by an inclusion $i_p: T \hookrightarrow \mathrm{Spin}^c(2n)$ followed by the left action of $\mathrm{Spin}^c(2n)$ on $\Delta^{\pm}_{\mathbb{C}}$. The images of T lie in the standard maximal tori $T_{\mathrm{Spin}^c(2n)}$ and $T_{\mathrm{SO}(2n)}$ if we chose the basepoint appropriately. Recall that the Spin^c structure has an associated line bundle $\tilde{\mathbf{L}}$. The action of T on the fiber $\tilde{\mathbf{L}}_p$ defines a map $T \to \mathrm{SO}(2)$. All these maps fit into a commuting diagram,

$$\begin{array}{ccccc} T & \xhookrightarrow{i_p} & T_{\mathrm{Spin}^c(2n)} & \hookrightarrow & \mathrm{Spin}^c(2n) \\ \big\downarrow \mathrm{id} & & \big\downarrow \text{double cover} & & \big\downarrow \text{double cover} \\ T & \hookrightarrow & T_{\mathrm{SO}(2n)} \times \mathrm{SO}(2) & \hookrightarrow & SO(2n) \times SO(2) \end{array}$$

Pick an element of the torus, $\lambda \in T$, and let $\lambda_p = i_p(\lambda)$ be its image in $T_{\mathrm{Spin}^c(2n)}$. The numerator of $v(p)(\lambda)$ in (11.1) is then

$$\sum_{\substack{\text{even number of} \\ \text{plus signs}}} \lambda_p^{(1/2)(\pm x_1 \pm x_2 \pm \cdots \pm x_n + y)} - \sum_{\substack{\text{odd number of} \\ \text{plus signs}}} \lambda_p^{(1/2)(\pm x_1 \pm x_2 \pm \cdots \pm x_n + y)}$$

$$= \lambda_p^{y/2} \prod_{i=1}^{n} (\lambda_p^{-x_i/2} - \lambda_p^{x_i/2}). \tag{11.5}$$

The denominator in (11.1) is

$$\prod_{i=1}^{n} (1-\lambda_p^{x_i})(1-\lambda_p^{-x_i}). \tag{11.6}$$

To see this, note that the complexified tangent space splits, under the torus action, into $2n$ weight spaces with weights $\pm x_i$, i.e., $T_p M \otimes \mathbb{C} \cong \mathbb{C}^n$, consider the linear operator $A = 1 - df_\lambda^{-1}$, and remember that $\det_{\mathbb{R}}(A)|_{T_pM} = \det_{\mathbb{C}}(A \otimes 1)|_{T_pM \otimes \mathbb{C}}$.

Now suppose that T acts on $T_p M$ with weights $\pm\alpha_1, ..., \pm\alpha_n$ and that T acts on the fiber $\tilde{\mathbf{L}}$ with a weight μ. This means that $\alpha_i = i_p^* x_i$ and $\mu = i_p^* y$. By taking the pullbacks of (11.5) and (11.6) we get the desired expression (11.4). ▮

12. EXPANDING THE TERMS

We have obtained localization formulas for the virtual character of the Dolbeault or Dirac operator as a sum over the fixed points p in M:

$$\chi(\lambda) = \sum_p \nu(p)(\lambda). \tag{12.1}$$

Recall that the virtual character is a Laurent polynomial in $\lambda \in T$:

$$\chi(\lambda) = \sum_{\alpha \in \ell^*} \mathrm{mult}(\alpha)\, \lambda^\alpha.$$

We are interested in its coefficients. In particular, we would like to express them as sums:

$$\mathrm{mult}(\alpha) = \sum_p \mathrm{mult}_p(\alpha).$$

Such expressions will come from Laurent series expansions for the individual terms,

$$\nu(p)(\lambda) = \sum_{\alpha \in \ell^*} \mathrm{mult}_p(\alpha)\, \lambda^\alpha.$$

The tricky point is that we can choose from several possible Laurent expansions for $\nu(p)$ and that if we are not careful about our choices then the sum of the $\mathrm{mult}_p(\alpha)$'s will not give the right answer. We will expand the terms using the recipe of Guillemin–Lerman–Sternberg [GLS1].

Expanding the terms for the Dolbeault operator. By Proposition 11.2 we have $v(p)(\lambda) = \lambda^{\mu} \prod_{i=1}^{n} 1/(1-\lambda^{-\alpha_i})$. Each factor can be expanded in two different ways:

$$\frac{1}{1-\lambda^{-\alpha_i}} = 1 + \lambda^{-\alpha_i} + \lambda^{-2\alpha_i} + \cdots \tag{12.2}$$

$$\frac{1}{1-\lambda^{-\alpha_i}} = \frac{-\lambda^{\alpha_i}}{1-\lambda^{\alpha_i}} = -\lambda^{\alpha_i} - \lambda^{2\alpha_i} - \lambda^{3\alpha_i} - \cdots. \tag{12.3}$$

Denote by $\alpha_{1p}, ..., \alpha_{np}$ the isotropy weights at p. (Remember, these are elements of the integral weight lattice ℓ^*.) Choose an element $\xi \in \mathfrak{t}$ such that $\langle \xi, \alpha_{ip} \rangle \neq 0$ for all i and p. We choose the expansion of $1/(1-\lambda^{-\alpha_{ip}})$, according to the sign of $\langle \xi, \alpha_{ip} \rangle$; if it is positive then we expand by (12.2); if it is negative then we expand by (12.3). By multiplying the power series for the $1/(1-\lambda^{-\alpha_i})$, we get a power series:

$$v(p)(\lambda) = \lambda^{\mu} \prod_{i=1}^{n} \frac{1}{1-\lambda^{-\alpha_i}} = \sum_{\alpha \in \ell^*} \text{mult}_p^{\xi}(\alpha)\, \lambda^{\alpha}. \tag{12.4}$$

PROPOSITION 12.5. *If* $\text{mult}_p^{\xi}(\alpha)$ *are defined as above then for every* $\alpha \in \ell^*$,

$$\text{mult}(\alpha) = \sum_p \text{mult}_p^{\xi}(\alpha).$$

Proof. We expand both sides of (12.1) as Laurent series in λ:

$$\sum_{\alpha} \text{mult}(\alpha)\, \lambda^{\alpha} = \sum_p \sum_{\alpha} \text{mult}_p^{\xi}(\alpha)\, \lambda^{\alpha}. \tag{12.6}$$

By our choice of the expansions, the summands on the right converge when we substitute $\lambda = \exp(i\eta)$ if η is close to ξ. (The character $\chi(\lambda)$ is defined for all λ in the *complexified* torus, $T_{\mathbb{C}}$, by analytic continuation.) Since (12.6) holds on an open set of η's, it also holds formally, as an equality between formal power series. ∎

Let C_p be the convex polyhedral cone with vertex at μ and with edges spanned by the vectors α'_{ip}, $i = 1, ..., n$, where $\alpha'_{ip} = \pm\alpha_{ip}$ according to the sign of $\langle \alpha_{ip}, \xi \rangle$. Denote $\varepsilon_p = (-1)^{w_p}$, where w_p is the number of indices i for which $\alpha'_{ip} = -\alpha_{ip}$, i.e., for which $\langle \alpha_{ip}, \xi \rangle < 0$. One can easily see that if α is in the interior of the cone C_p then its multiplicity, given in (12.4), is $\text{mult}_p^{\xi}(\alpha) = \varepsilon_p$ and that if α is in the exterior then $\text{mult}_p^{\xi}(\alpha) = 0$.

PROPOSITION 12.7. *Consider an effective action of a torus T on a manifold M, where* $\dim T = \frac{1}{2}\dim M$. *Let* $\mathbf{L} \xrightarrow{\pi} M$ *be a complex Hermitian line bundle and fix a lifting to* $\mathbf{L}$ *of the T-action. Suppose that M has an invariant almost complex structure. Consider the twisted Dolbeault operator (for any choice of metric and connection) and its virtual character,* $\chi(\lambda) = \sum \operatorname{mult}(\alpha)\, \lambda^{\alpha}$. *Now take any T-invariant connection on* $\mathbf{L}$ *and let ω be $-i$ times its curvature. The action of the torus on* $\mathbf{L}$ *determines a moment map $\Phi: M \to \mathfrak{t}^*$. Consider the Liouville measure on M and its push-forward via Φ, as defined in Section 2. Let $\rho(\alpha)$ be the density function for the push-forward measure. Then*

$$\operatorname{mult}(\alpha) = \rho(\alpha)$$

for every $\alpha \in \ell^*$ outside a finite union of hyperplanes in $\mathfrak{t}^*$.

Remark 12.8. A priori, the density function lies in the space L^1 so it is meaningless to evaluate it at a point. In our case, the density function can be represented by a function which is locally constant on $\mathfrak{t}^*$ minus a finite union of hyperplanes. This is the function ρ which appears in the above statement.

Proof of Proposition 12.7. Pick a generic element $\xi \in \mathfrak{t}$. We have shown that $\operatorname{mult}(\alpha) = \sum_p \operatorname{mult}_p^{\xi}(\alpha)$. The GLS formula (Theorem 7.1) gave $\rho(\alpha) = \sum_p \rho_p^{\xi}(\alpha)$. We need to show that $\operatorname{mult}_p^{\xi}(\alpha) = \rho_p^{\xi}(\alpha)$ for all $\alpha \in \ell^*$ outside a finite union of hyperplanes. We will recall the explicit descriptions of the functions $\operatorname{mult}_p^{\xi}$ and ρ_p^{ξ}.

In both cases, we considered the isotropy weights α_{ip}, the polarized weights $\alpha'_{ip} = \pm\alpha_{ip}$, and $\varepsilon_p = \pm 1 =$ the parity of the number of i's such that $\alpha'_{ip} = -\alpha_{ip}$. In both cases we considered a convex polyhedral cone $C_p \subseteq \mathfrak{t}^*$ whose rays were generated by the polarized weights.

For the multiplicities, the vertex of the cone was at the weight μ by which the torus acts on the fiber $\mathbf{L}_p$ over the fixed point. For the D-H measure, the vertex of the cone was at the value of the moment map at p. We will show that $\Phi(p) = \mu$. Assuming this, we showed that if α is in the interior of C_p then $\operatorname{mult}_p(\alpha) = \rho_p(\alpha) = \varepsilon_p$, and if α is in the exterior then $\operatorname{mult}_p(\alpha) = \rho_p(\alpha) = 0$. On the boundary of C_p, $\rho_p(\alpha)$ was not defined. This boundary is contained in the union of n hyperplanes.

Now we show that $\Phi(p) = \mu$. The moment map which we are using is determined by $\pi^*\Phi^{\xi} = \langle -i\beta, \xi_{\mathbf{L}} \rangle$, where $\xi \in \mathfrak{t}$ acts on $\mathbf{L}$ by the vector field $\xi_{\mathbf{L}}$ and where β is the connection form. The vector field $\xi_{\mathbf{L}}$ is tangent to the fiber $\mathbf{L}_p$ and it is equal to $\langle \mu, \xi \rangle\, \partial/\partial\theta$ in polar coordinates on the fiber. The pullback of the connection form to the fiber is the one-form dz/z. Hence, $\langle \beta, \xi_{\mathbf{L}} \rangle = i\langle \mu, \xi \rangle$ and we get $\Phi(p) = \mu$ as desired. ∎

Expanding the terms for the Dirac index. Each factor in (11.4) can be expanded either as

$$\frac{\lambda^{\mu/2}(\lambda^{-\alpha/2}-\lambda^{\alpha/2})}{(1-\lambda^{\alpha})(1-\lambda^{-\alpha})}=\frac{\lambda^{(\mu-\alpha)/2}(1-\lambda^{\alpha})}{(1-\lambda^{\alpha})(1-\lambda^{-\alpha})}$$
$$=\lambda^{(\mu-\alpha)/2}(1+\lambda^{-\alpha}+\lambda^{-2\alpha}+\cdots) \qquad (12.9)$$

or as

$$\frac{\lambda^{\mu/2}(\lambda^{-\alpha/2}-\lambda^{\alpha/2})}{(1-\lambda^{\alpha})(1-\lambda^{-\alpha})}=\frac{\lambda^{(\mu+\alpha)/2}(\lambda^{-\alpha}-1)}{(1-\lambda^{\alpha})(1-\lambda^{-\alpha})}$$
$$=\lambda^{(\mu+\alpha)/2}(-1-\lambda^{\alpha}-\lambda^{2\alpha}-\cdots). \qquad (12.10)$$

Again as in [GLS1] we choose an element ξ of $\mathfrak{t}\backslash\bigcup_{i,p}\ker\alpha_{ip}$ and we choose each expansion (12.9) or (12.10) according to the sign of $\langle\alpha_{ip},\xi\rangle$. When we multiply these we get the Laurent expansion

$$v(p)(\lambda)=\varepsilon_p\lambda^{(1/2)(\mu_p-\alpha'_{ip}-\cdots-\alpha'_{np})}\prod_{i=1}^{n}(1+\lambda^{-\alpha'_{ip}}+\lambda^{-2\alpha'_{ip}}+\cdots). \qquad (12.11)$$

Let C_p be the polyhedral cone with a vertex at $\frac{1}{2}\mu_p$ and with rays generated by the vectors $\alpha'_{ip}=\pm\alpha_{ip}$. We claim that if α is in the weight lattice ℓ^* then the coefficient of α in the Laurent expansion (12.11) is $\varepsilon_p(=\pm 1)$ for α in the interior of C_p and is 0 for α in the exterior and that the boundary of the cone contains no lattice points. All this follows easily from (12.11) together with

LEMMA 12.12. *$\alpha=\mu/2+\sum r_i\alpha_i$ is in the weight lattice ℓ^* if and only if every r_i is equal to $n_i+\frac{1}{2}$ for some integer n_i.*

Proof. Recall that the weight lattice of $T_{\mathrm{Spin}^c(2n)}$ is generated over $\mathbb{Z}$ by $x_1, ..., x_n$, y, and $\frac{1}{2}(x_1+\cdots+x_n+y)$. Since $\mu/2+\sum r_i\alpha_i$ is the pull-back of $\frac{1}{2}(x_1+\cdots+x_n+y)+\sum(r_i-1/2)\,x_i$ under an inclusion $i_p\colon T\to T_{\mathrm{Spin}^c(2n)}$, it is in the weight lattice if $r_i-1/2$ is an integer. Conversely, since the torus action on M is effective, the α_i's generate the weight lattice of T, and hence, if $\mu/2+\sum r_i\alpha_i$ is integral then the r_i's are as required. ∎

PROPOSITION 12.13. *Consider an effective action of a torus T on a manifold M where* $\dim T=\frac{1}{2}\dim M$. *Suppose that we have a Spin^c structure on M and a lifting of the torus action to this structure. Consider the equivariant Dirac index and its virtual character, $\chi(\lambda)=\sum\mathrm{mult}(\alpha)\,\lambda^{\alpha}$. Let $\tilde{\mathbf{L}}$ be the line bundle associated to the Spin^c structure on M. Pick any connection on it and let ω be $-i/2$ times its curvature. Let $\Phi\colon M\to\mathfrak{t}^*$ be the moment map which is determined by the lifting of the torus action to $\tilde{\mathbf{L}}$. Consider the*

Liouville measure on M and its push-forward via Φ, as defined in Section 2. Let $\rho(\alpha)$ be the density function of the push-forward measure. Then

$$\text{mult}(\alpha) = \rho(\alpha)$$

for every $\alpha \in \ell^*$.

Remark 12.14. As in Remark 12.8, the density function is an element of the space L^1 and is represented by the function ρ. As before, this function is locally constant on $\mathfrak{t}^*$ minus a finite union of hyperplanes. We show that the hyperplanes do not intersect the weight lattice ℓ^* and hence $\rho(\alpha)$ is defined for all α in ℓ^*.

Proof of Proposition 12.13. By the same arguments as in Proposition 12.5, $\text{mult}(\alpha) = \sum_p \text{mult}_p^{\xi}(\alpha)$, and we only need to show that $\text{mult}_p^{\xi}(\alpha) = \rho_p^{\xi}(\alpha)$ for all $\alpha \in \ell^*$. As in the proof of Proposition 12.7, both of these functions are supported on a convex polyhedral cone C_p whose rays are generated by the polarized weights and whose vertex is $\Phi(p) = \mu/2$, where μ is the weight by which the torus acts on the fiber $\tilde{\mathbf{L}}_p$ of the line bundle associated to the Spin^c structure. We have $\rho_p^{\xi}(\alpha) = \varepsilon_p$ for all α in the interior of C_p. By Lemma 12.12 and the remarks immediately above it, $\text{mult}_p^{\xi}(\alpha)$ is also equal to ε_p for all α in the intersection of ℓ^* with the interior of C_p, and the boundary of C_p is disjoint from ℓ^*. Hence, $\rho_p^{\xi}(\alpha) = \text{mult}_p^{\xi}(\alpha)$ for all $\alpha \in \ell^*$. ∎

Remark 12.15. A holomorphic line bundle $\mathbf{L}$ over a complex manifold M determines a Spin^c structure on M for which $\tilde{\mathbf{L}} = \mathbf{L}^{\otimes 2} \otimes \det$, where $\det = \bigwedge_{\mathbb{C}}^n TM$ is the determinant line bundle. The Dolbeault index then coincides with the Dirac index, but the two-form $\tilde{\omega}$ associated with the Spin^c structure is different from the two-form ω associated with the almost complex structure: we have $\tilde{\omega} = \omega + 1/2\omega_{\det}$, where $\omega_{\det}$ is a curvature for the determinant line bundle. Let $\tilde{\rho}$ and ρ be the density functions for the Duistermaat–Heckman measures which correspond to $\tilde{\omega}$ and to ω. Denote by $\tilde{H}_i$ and by H_i the hyperplanes along which $\tilde{\rho}$ and ρ are discontinuous, respectively. Then the $\tilde{H}_i$'s are parallel shifts of the H_i's; see Fig. 7.

Fig. 7. The moment image for CP^2 and its holomorphic tangent bundle, as a complex manifold and as a Spin^c manifold.

EXAMPLE 12.16. Consider the action of the circle group S^1 on the complex manifold $M = \mathbb{CP}^1$ with the line bundle $\mathbf{L} = TM$. If we think of M as a Spin^c manifold then the associated line bundle is $\tilde{\mathbf{L}} = \det(M) \otimes \mathbf{L}^{\otimes 2}$, where $\det(M) = \bigwedge^n TM$. In the case $\mathbf{L} = T\mathbb{CP}^1$ we have $\tilde{\mathbf{L}} = (T\mathbb{CP}^1)^{\otimes 3}$. There are two fixed points: the north and south poles. The circle acts on the fiber over the north pole with a weight $\mu_n = 3$ and over the south pole with a weight $\mu_s = -3$. The isotropy weights at the poles are $\alpha_n = 1$ and $\alpha_s = -1$. From Proposition 11.3 we get

$$\chi(\lambda) = \frac{\lambda^{3/2}(\lambda^{-1/2} - \lambda^{1/2})}{(1-\lambda)(1-\lambda^{-1})} + \frac{\lambda^{-3/2}(\lambda^{1/2} - \lambda^{-1/2})}{(1-\lambda^{-1})(1-\lambda)} = \lambda + 1 + \lambda^{-1}. \quad (12.17)$$

The D-H measure is Lebesgue measure on the interval $[-3/2, 3/2]$.

We can now present our main theorem.

THEOREM 12.1. *Let M be a smooth manifold with a Spin^c structure. Assume that we have an effective action of a torus T on M, with* $\dim T = \frac{1}{2} \dim M$ *and that the torus action lifts to the Spin^c structure. Consider the Dirac operator (with respect to an arbitrary choice of connection) and its virtual character* $\chi(\lambda) = \sum_\alpha \operatorname{mult}(\alpha)\, \lambda^\alpha$. *Let $\tilde{\mathbf{L}} \xrightarrow{\pi} M$ be the Hermitian line bundle associated to the Spin^c structure. Let β be a T-invariant connection on $\tilde{\mathbf{L}}$. Define a moment map $\Phi: M \to \mathfrak{t}^*$ by $\pi^* \Phi^\xi = \langle -(i/2)\beta, \xi_{\mathbf{L}} \rangle$. Consider the descended moment map $\bar{\Phi}: M/T \to \mathfrak{t}^*$. Take an integral weight $\alpha \in \ell^* \subset \mathfrak{t}^*$:*

If α is a regular value for Φ then $\operatorname{mult}(\alpha)$ *is equal to the number of points in $\bar{\Phi}^{-1}(\alpha)$ counted with appropriate signs.*

If (M, T) is locally toric then $\operatorname{mult}(\alpha)$ *is equal to the winding number around α of the descended moment map, restricted to the boundary.*

Proof. These statements are immediate consequences of Proposition 12.13, Theorem 5.1, and Theorem 5.2. ∎

13. EXAMPLES

In Section 6 we defined the equivariant connected sum of two spaces with completely integrable torus actions. We now describe how, in some cases, the extra structure (almost complex structure, line bundles, Spin^c structure) on two spaces carries over to their equivariant connected sum. In this way we produce new examples of spaces to which our theorems apply.

Extending almost complex structures equivariantly. Take two manifolds M_1 and M_2 with completely integrable torus actions and with invariant almost complex structures J_1 and J_2. It is impossible to extend these to an almost complex structure on $M_1 \sharp_T \bar{M}_2$ because J_2 is not compatible with the reversed orientation on $\bar{M}_2$. Nevertheless, if n is odd then we can remove this obstruction by replacing J_2 by $-J_2$.

If n is even and we wish to glue at a free orbit then we can use an alternative definition of an equivariant connected sum, which does not require us to reverse the orientation of M_2. Here it is. A neighborhood U_i of a free orbit in M_i looks like a product of the torus with a disc, $T \times D_i$. A neighborhood of the boundary ∂U_i looks like $T \times S^{n-1} \times (-\varepsilon, \varepsilon)$. Define the gluing map ϕ to be the identity on T, the map that sends $t \to -t$ on $(-\varepsilon, \varepsilon)$, and a map of degree -1, $S^{n-1} \to S^{n-1}$, on the boundary spheres (instead of the identity map which we used in the previous construction). We call the resulting manifold $M_1 \sharp'_T M_2$.

We now discuss the case where $\dim M_i = 2n = 4$. Assume that we are given almost complex structures on M_1 and M_2. We will show that these always extend to $M = M_1 \sharp'_T M_2$. The almost complex structures determine a principal $\mathrm{GL}(2, \mathbb{C})$ bundle on each M_i on which T acts from the left. Since this group retracts onto $U(2)$, we can find a metric on each M_i which is compatible with the almost complex structure. We can extend the metrics on M_1 and M_2 to a metric on $M = M_1 \sharp'_T M_2$; this determines a principal $SO(4)$ bundle P whose fiber over $p \in M$ is the set of orthonormal oriented bases of $T_p M$. The subgroup $U(2) \subseteq SO(4)$ acts on these fibers from the right by the restriction of the principal action. The choice of an almost complex structure compatible with the metric amounts to choosing a global section of $Q = P/U(2)$. Moreover, we can make all our choices T-equivariant, and then we get a left action of the torus T on the bundle Q. Over the set M_{free} in M where the torus acts freely, choosing an invariant almost complex structure compatible with the metric amounts to choosing a section of the (double) quotient $Q/T = T \backslash P/U(2)$ over M_{free}/T. The almost complex structures on M_1 and M_2 determine invariant sections of the Q_i's over the M_i's. A neighborhood of the gluing locus in $M = M_1 \sharp'_T M_2$ is a tube of the form $T \times S^1 \times I$ where I is an interval. The bundle Q/T is trivial over the quotient tube $S^1 \times I$. (A trivialization comes, for example, from viewing this tube as punctured disc in M_1/T, whereas the bundle Q_i/T is trivial over the whole disc.) The almost complex structures on M_1 and M_2 give sections of the bundles Q_i/T which are defined near the boundary of the tube $S^1 \times I$ in M/T. We need to extend these sections across the rest of the tube. Over the tube, Q/T is a trivial bundle with fiber $SO(4)/U(2)$, which is topologically S^2. The boundaries of the tube are circles. Since every loop in S^2 is homotopically trivial, the sections can be extended.

Extending line bundles equivariantly. Suppose that $L_i \to M_i$ are equivariant line bundles. We connect the M_i's along some T-orbits, as described in Section 6, to form the equivariant connected sum $M_1 \sharp_T \bar{M}_2$. Let H be the isotropy group of the orbits at which we glue. Then H acts on the fiber of L_1 over that orbit by a character, say, $\alpha_1 \in \mathfrak{h}^*$. We define α_2 similarly. If $\alpha_1 = \alpha_2$ then we can fit the bundles together into an equivariant bundle over $M_1 \sharp_T \bar{M}_2$, in the following way. Locally, we write $L_i|_{U_i} = T \times_H (D^{n+k} \times \mathbb{C})$, where U_i are neighborhoods of the orbits at which we glue. Over the deleted neighborhoods we have $L_i|_{U_i \setminus T \cdot p_i} = T \times_H (S^{n+k-1} \times \mathbb{C}) \times I$, where I is the interval $(-\varepsilon, \varepsilon)$. We glue these by the identity on $T \times_H (S^{n+k-1} \times \mathbb{C})$ and by the map $t \mapsto -t$ on the I component. If $\alpha_1 \neq \alpha_2$ then we can patch the line bundles after changing the action on L_2 as follows: we compose the T-action on L_2 by the action on the fibers via any character of T whose restriction to H is $\exp(\alpha_2 - \alpha_1)$.

Extending equivariant Spin^c structures. Suppose that M_1 and M_2 have Spin^c structures. These can never extend to $M_1 \sharp_T \bar{M}_2$ because, just as for almost complex structures, the orientations do not fit. Nevertheless, there is a canonical way to define a new Spin^c structure on M_2 which induces the opposite orientation. We now describe this. We later specify the condition on the Spin^c structures which guarantees that we can glue.

There is a commuting diagram of homomorphisms of Lie groups:

$$\begin{array}{ccc} \mathrm{Spin}(2n) & \longrightarrow & \mathrm{Pin}(2n) \\ \downarrow & & \downarrow \\ \mathrm{SO}(2n) & \longrightarrow & O(2n) \end{array}$$

We define $\mathrm{Pin}^c(2n) = \mathrm{Pin}(2n) \times_{Z_2} U(1)$. Suppose that M has a Spin^c structure, given by a principal bundle P with a structure group $G = \mathrm{Spin}^c$. We form the associated bundle $P \times_G \mathrm{Pin}^c(2n)$. This bundle has two components which lie over the two components of the orthogonal frame bundle over M. Each of these components is a principal bundle with respect to the right action of the group Spin^c. One of the components is the Spin^c structure that we started with. The other component provides us with a new Spin^c structure which is compatible with the opposite orientation on M. The associated line bundle $\tilde{\mathbf{L}}$ is the same as for the original structure.

Now take two Spin^c manifolds with completely integrable torus actions and consider their equivariant connected sum along some T-orbits, $M_1 \sharp_T \bar{M}_2$. Let $H \subset T$ be the stabilizer of the orbits at which we glue. Then H acts on the fiber $\tilde{\mathbf{L}}_p$ of the bundle associated to the Spin^c structure by some character. If this character is the same for M_1 and for M_2 then the Spin^c structures fit together. We now explain why.

This condition implies that the associated bundles $\tilde{L}_i$ fit together equivariantly; this we showed earlier. The bundles $SO(2n)$ also fit together equivariantly; one can use a partition of unity to define a metric on the equivariant connected sum and then take its average with respect to the torus action to get an invariant metric. Hence we get a principal bundle over the "tube", $T \times_H S^{n+k-1} \times I$, with fiber $SO(2n) \times SO(2)$. This bundle is topologically trivial because it is the restriction to the tube of an equivariant bundle over $T \times_H D^{n+k}$. The Spin^c structures on M_1 and M_2 provide double coverings of this bundle over the ends of the "tube." These are topologically trivial for the same reason. Hence, we can extend these to a double covering over the whole tube, i.e., to a Spin^c structure. We get a torus action by lifting the vector fields which generate the torus action via the double covering. Since this integrates to a torus action near the ends of the tube, it must integrate throughout the tube.

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