On the growth of meromorphic solutions of the Schwarzian differential equations

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Abstract
We study the properties of meromorphic solutions of the Schwarzian differential equations in the complex plane by using some techniques from the study of the class $W_p$. We find some upper bounds of the order of meromorphic solutions for some types of the Schwarzian differential equations. We also show that there are no wandering domains nor Baker domains for meromorphic solutions of certain Schwarzian differential equations.

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1. Introduction

Let $f(z)$ be a meromorphic function. The Schwarzian derivative of $f(z)$ is defined by

$$S(f, z) = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.$$
It is known that \( S(f, z) \equiv 0 \) if and only if \( f \) is a Möbius transformation, and \( S(f, z) \equiv S(g, z) \) if and only if \( f \) is a Möbius transformation of \( g \).

Let \( p \) be a positive integer, and

\[
R(f, z) = \frac{P(f, z)}{Q(f, z)} = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_m(z)f(z)^m}{b_0(z) + b_1(z)f(z) + \cdots + b_n(z)f(z)^n},
\]

where \( a_i(z) \) and \( b_j(z) \) \((i = 0, \ldots, m; j = 0, \ldots, n)\) are polynomials, and \( P(f, z) \) and \( Q(f, z) \) are irreducible. In the sequel, we denote a rational function of \( f \) by \( R(f) \). A general form of the Schwarzian differential equation can be written as

\[
S(f, z)^p = R(f, z).
\]

Standard references for the Schwarzian derivatives and equations in the complex plane are [7, 11].

The Schwarzian derivative of meromorphic functions and the Schwarzian differential equations are closely connected to the study of the second order linear differential equation

\[
f'' + A(z)f = 0,
\]

where \( A(z) \) is a meromorphic function. In fact, the Schwarzian derivative of the quotient of its two linearly independent meromorphic solutions in (3) is equal to \( 2A(z) \). Steinmetz [18] studied factorization of solutions of Eq. (2). Ishizaki [8] proved some theorems about the value distributions of meromorphic solutions of Eq. (2) and classification of \( Q(f, z) \) in Eq. (2). There is always an interest to know the order of meromorphic solutions of any differential equations in the complex plane. Recently, Bergweiler [2], Frank–Wang [4], Hayman [6], and Liao–Yang [12, 13] studied the order of meromorphic solutions of algebraic differential equations. Liao–Su–Yang [14] studied the Malmquist–Yosida type of theorems for second-order algebraic differential equations. However, to our knowledge, there is no any published work to discuss the order of meromorphic solutions of Eq. (2).

In this paper, we use some techniques from the study of the class of meromorphic functions \( W_p \), similar to the argument used by the first author and his coworkers in [14], to investigate Eq. (2). We generalize some Makhmutov’s results in the class \( W_p \) and not only prove the order of various Schwarzian differential equations is finite but also find their upper bounds. In particular, we prove that the order of meromorphic solutions of the equation \( S(f, z)^p = R(f) \) is not greater than two. As by-products of our theorems, we also get two results about complex dynamics of meromorphic solutions of some Schwarzian differential equations.

### 2. The class \( W_p \) and lemmas

Let \( f \) be a meromorphic function. We denote the spherical derivative of \( f \) by

\[
f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.
\]
Definition. Let $p \geq 1$ be an integer. A meromorphic function $f$ defined in the complex plane is in $W_p$ if and only if

$$\lim_{|z| \to +\infty} |z|^{2-p} f^#(z) < \infty.$$  \hfill (4)

It is clear that if $p < 1$ and $f$ satisfies the inequality (4), then $f$ is a rational function.

The classes $W_1$ and $W_2$ are classical. The distribution of values of functions in $W_1$ and $W_2$ has been thoroughly investigated by Julia [10], Yosida [19] and Ostrowski [17]. Gavrilov [5] and Makhmutov [15,16] have made a substantial contribution to the study of $W_p$.

Lemma 1 [16]. A meromorphic function $f(z)$ on $\mathbb{C}$ is not in $W_p$, $p \geq 1$, if and only if there exist a sequence $\{z_\nu\}$ with \( \lim_{\nu \to \infty} z_\nu = \infty \) and a sequence of positive numbers $\{\varepsilon_\nu\}$ with \( \lim_{\nu \to \infty} \varepsilon_\nu = 0 \) such that $g_\nu(z) = f(z_\nu + \varepsilon_\nu |z_\nu|^{2-p}z)$ converges uniformly to a non-constant meromorphic function $g(z)$ on compact subsets of $\mathbb{C}$.

Lemma 2 [16]. If, for any five different values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C} \cup \{\infty\}$,

$$\sup\{|z|^{2-p} f^#(z); \ f(z) = a_k, \ k = 1, 2, 3, 4, 5\} < \infty,$$

then $f \in W_p$, $p \geq 1$.

Remark 1. The original statement of the theorem is for $p > 1$. But we can see from the proof of the theorem that the conclusion is also true when $p = 1$.

Makhmutov in [16] also proved that for any integer $k \neq 0$ the meromorphic functions $f$ and $f^k$ on $\mathbb{C}$ belong to $W_p$ only simultaneously. We extend the result and obtain

Proposition 1. Let $p$ be a positive integer and let $f$ be a meromorphic function, $R$ a non-constant rational function and $F(z) = R(f(z))$. Then $f \in W_p$ if and only if $F \in W_p$.

Proof. Assume $f \in W_p$. We choose a number $A \in \mathbb{C}$ such that $R(w) = A$ has only simple roots, i.e., if $R(a) = A$, then $R'(a) \neq 0$. Now assume that

$$\lim_{|z| \to \infty, \ z \in F^{-1}(A)} |z|^{2-p} F^#(z) = \infty.$$ 

This is

$$\lim_{|z| \to \infty, \ z \in F^{-1}(A)} \frac{|z|^{2-p} |F'(z)|}{1 + |A|^2} = \lim_{|z| \to \infty, \ z \in F^{-1}(A)} \frac{|z|^{2-p} |R'(f(z))||f'(z)|}{1 + |A|^2} = \infty.$$ 

Let $a_1, a_2, \ldots, a_n$ be all roots of $R(w) = A$. Thus

$$z \in F^{-1}(A) \quad \text{if and only if} \quad z \in f^{-1}(a_1, a_2, \ldots, a_n).$$

Noting $|R'(a_i)| > 0$, $i = 1, 2, \ldots, n$, we have

$$\lim_{|z| \to \infty, \ z \in f^{-1}(a_1, a_2, \ldots, a_n)} |z|^{2-p} |f'(z)| = \infty.$$
Thus, there exists at least one \( a_i \), say \( a_1 \), such that
\[
\lim_{|z| \to \infty, z \in f^{-1}(a_1)} |z|^{2-p} f^\#(z) = \lim_{|z| \to \infty, z \in f^{-1}(a_1)} |z|^{2-p} \frac{|f'(z)|}{1 + |a_1|^2} = \infty.
\]
This contradicts to the fact \( f \in W_p \). Therefore, for such an \( A \), we have
\[
\lim_{|z| \to \infty, z \in F^{-1}(A)} |z|^{2-p} F^\#(z) < \infty.
\]
Since there are infinitely many such \( A \)'s, it follows from Lemma 2 that \( F \in W_p \).

Now we assume \( f \notin W_p \). It is straightforward from Lemma 1 to see \( F \notin W_p \). Thus the proof of the proposition is complete.

**Proposition 2.** Let \( f \) be a non-constant meromorphic function and \( P \) a non-constant polynomial with degree \( n \geq 1 \) and \( F(z) = f(P(z)) \). Then \( f \in W_p \) if and only if \( F \in W_n(p-1)+1 \).

**Proof.** Using the facts
\[
F^\#(z) = |P'(z)| f^\#(w), \quad \text{where } w = P(z)
\]
and
\[
\lim_{z \to \infty} \frac{|P'(z)|}{|P(z)|^{2-p}} |z|^{2-(np-n+1)} = A,
\]
where \( A \) is a finitely positive number, we can easily derive the conclusions of the proposition by verifying the definition of \( W_p \).

The Nevanlinna characteristic function of \( f \) is defined by
\[
T(f, r) = \frac{1}{\pi} \int_0^r \int_0^t \int_{|z| \leq r} \left( f^\#(z) \right)^2 \, dx \, dy
\]
and we denote the order of a meromorphic function \( f \) by
\[
\rho(f) = \lim_{r \to \infty} \frac{\log T(f, r)}{\log r}.
\]
When \( f \) is an entire function, the order of \( f \) can also be
\[
\rho(f) = \lim_{r \to \infty} \frac{\log \log M(f, r)}{\log r} = \lim_{r \to \infty} \frac{\log \nu(f, r)}{\log r},
\]
where \( M(f, r) = \max_{|z| \leq r} |f(z)| \) and \( \nu(f, r) \) is the central index of the power series of \( f \) at \( |z| = r \).

**Lemma 3** [14]. If \( f \in W_p \), then \( \rho(f) \leq 2p - 2 \).

**Remark 2.** The bound \( 2p - 2 \) in Lemma 3 is sharp. It is known that the Weierstrass \( \wp \)-function belongs to \( W_2 \) and the order is two. Thus, for any integer \( p \geq 2 \), \( \wp(z^{p-1}) \) has the order \( 2p - 2 \). By Proposition 2, \( \wp(z^{p-1}) \) is in \( W_p \).

**Remark 3.** The converse of Lemma 3 is not true. For instance, we take \( f(z) = e^{2z^{p-2}} \) for any \( p > 1 \). Then \( \rho(f) = 2p - 2 \). However, by a little computation, we get \( f \notin W_p \).
Example 1. Let \( k \) be a positive integer and \( f(z) = \int e^{zk} \, dz \). Then \( f \) satisfies the Schwarzian differential equation
\[
S(f, z) = (k - 1)zk - 2 - \frac{1}{2}z^{2k-2}.
\]

Example 2. Let \( f(z) = \sin z \). Then \( f(z) \) satisfies the Schwarzian differential equation
\[
S(f, z) = -2 - \frac{f^2(z)}{2 - f^2(z)}.
\]

Lemma 4 [11, p. 51]. Let \( f \) be a transcendental entire function, and \( 0 < \delta < \frac{1}{4} \). Suppose that at the point \( z \) with \( |z| = r \) the inequality
\[
|f(z)| > M(f, r)\nu(f, r) - \frac{1}{4} + \delta
\]
holds. Then there exists a set \( F \) in \( R^+ \) of finite logarithmic measure, i.e., \( \int_F 1/t \, dt < +\infty \) such that
\[
f^{(m)}(z) = \left( \frac{\nu(f, r)}{z} \right)^m (1 + o(1)) f(z)
\]
holds whenever \( m \) is a fixed non-negative integer and \( r \notin F \).

3. Main results

Recall that \( a_i(z) \) and \( b_j(z) \) are polynomials defined as in (1), and let
\[
k = \max\{ \deg a_i(z), \ i = 1, 2, \ldots, m \}; \quad l = \max\{ \deg b_j(z), \ j = 1, 2, \ldots, n \};
\]
and
\[
\Omega(t) = \lim_{z \to \infty} \frac{b_0(z) + \cdots + b_n(z) t^n}{z^t}.
\]

Theorem 1. Let \( f \) be a transcendental meromorphic solution of Eq. (2). If one of the following conditions holds:

(a) the polynomial \( \Omega \) has two distinct zeros;
(b) there exists \( j \) with \( 0 < j < n \) such that \( \deg b_j(z) > \max\{ \deg b_n(z), \deg b_0(z) \} \);
(c) if \( m > n \) and there exists an integer \( i \) (\( 0 < i \leq n \)) such that \( \deg b_i(z) \geq \deg b_0(z) \);
(d) the meromorphic solution \( f \) has two ramified values;

then \( \rho(f) \leq 2 \) when \( k \leq l \); and \( \rho(f) \leq 2 + (k - l)/p \) when \( k > l \).

Proof. Set \( \alpha = \max\{2, 2 + (k - l)/p\} \) and \( \beta = \frac{\alpha}{2} + 1 \). Now we prove that if \( f \) is a meromorphic solution of Eq. (2) and the one of the hypotheses in the theorem holds, then \( f \in W_\beta \). In fact, if \( f \) is not in \( W_\beta \), then by Lemma 1 there exist a sequence \( \{z_v\} \) with \( \lim_{v \to \infty} z_v = \infty \) and a sequence of positive numbers \( \{\varepsilon_v\} \) with \( \lim_{v \to \infty} \varepsilon_v = 0 \) such that
\[
g_v(\xi) = f(\xi_v + \varepsilon_v |z_v|^2 - \beta \xi)
\]
converges uniformly to a non-constant meromorphic function \( g(\zeta) \) on compact subsets of \( \mathbb{C} \). Denote \( \rho_v = \varepsilon_0 |z_\nu|^2 - \beta \). Then
\[
\begin{align*}
  g'_v(\zeta) &= \rho_v f'(z_\nu + \rho_v \zeta), \\
  g''_v(\zeta) &= \rho_v^2 f''(z_\nu + \rho_v \zeta), \\
  g'''_v(\zeta) &= \rho_v^3 f'''(z_\nu + \rho_v \zeta),
\end{align*}
\]
locally uniformly converge to \( g'(\zeta) \), \( g''(\zeta) \) and \( g'''(\zeta) \), respectively. Substituting these into Eq. (2), we have
\[
\left[ g'''_v(\zeta) - \frac{3}{2} \left( \frac{g''_v(\zeta)}{g'_v(\zeta)} \right)^2 \right]^p = \varepsilon_v^2 \rho_v O \left( \frac{1}{|z_\nu - (\beta - 2)p - (k - l)|} \right).
\]
Noting that \( (\beta - 2)2p - (k - l) \geq 0 \) and letting \( v \to \infty \), we have \( S(g, z) = 0 \). Therefore,
\[
g(\zeta) = \frac{A \zeta + B}{C \zeta + D},
\]
where \( A, B, C, \) and \( D \) are constants and \( AD - BC \neq 0 \). Set
\[
h_v(\zeta) = \frac{Q(z_\nu + \rho_v \zeta, f(z_\nu + \rho_v \zeta))}{(z_\nu + \rho_v \zeta)^l} = \frac{b_0(z_\nu + \rho_v \zeta) + \cdots + b_m(z_\nu + \rho_v \zeta) f(z_\nu + \rho_v \zeta)^m}{(z_\nu + \rho_v \zeta)^l}.
\]
Then \( h_v(\zeta) \) locally uniformly converges to \( \Omega(g(\zeta)) \) in the complex plane.

If the condition (a) is satisfied, then \( \Omega(g(\zeta)) \) has at least one finite zero, say, at \( \zeta_* \). Hence there exists a sequence of \( \zeta_\nu \) such that \( \zeta_\nu \to \zeta_* \) as \( n \to \infty \) and \( h_v(\zeta_\nu) = 0 \). It follows that \( z_\nu + \rho_v \zeta_\nu \) are the zeros of \( Q(f(z), z) \). By Jank and Volkmann [9], there are at most finitely many common zeros of \( P(f(z), z) \) and \( Q(f(z), z) \). Thus, we obtain that all zeros of \( Q(f(z), z) \), except finitely many, are the zeros of \( f'(z) \). Therefore, without lost of generality, we have \( f'(z_\nu + \rho_v \zeta_\nu) = 0 \). It follows that

\[
g'(\zeta_*) = \lim_{v \to \infty} g'_v(\zeta_\nu) = \lim_{v \to \infty} \rho_v f'(z_\nu + \rho_v \zeta_\nu) = 0.
\]
This contradicts to the fact that \( g'(\zeta) \) has no finite zero. Hence, \( f \in W_\beta \). By Lemma 3, we have \( \rho(f) \leq 2\beta - 2 = \alpha \). We finish our proof in this case.

If the condition (b) is satisfied, set \( \deg(\Omega(t)) = q \), then, \( 0 < q < m \). If \( \Omega(g(\zeta)) \) has a finite zero, then the conclusion of the theorem follows from the proof of the case (a). In the case when \( \Omega(g(\zeta)) \) does not have any finite zero, then we can write
\[
\Omega(t) = a(t - b)^q \quad \text{and} \quad g(\zeta) - b = \frac{E}{C \zeta + D},
\]
where \( a, b, C, D, E \) are constants with \( CE \neq 0 \). Set
\[
F(z) = \frac{f(z) - 2}{f(z) - 1}, \quad \text{so} \quad f(z) = \frac{F(z) - 2}{F(z) - 1}.
\]
Since \( S(F, z) = S(f, z) \), \( F(z) \) is a solution of the following Schwarzian differential equation
\[
S(F, z)^p = \frac{\hat{P}((F - 2)/(F - 1), z)}{\hat{Q}((F - 2)/(F - 1), z)} = \frac{\hat{P}(F, z)}{\hat{Q}(F, z)}.
\]
where \( \tilde{P}(F, z) \) and \( \tilde{Q}(F, z) \) are polynomials in \( z \) and \( F \). Indeed, when \( m > n \),

\[
\tilde{Q}(F, z) = Q\left(z, \frac{F(z) - 2}{F(z) - 1}\right)(F(z) - 1)^m = \sum_{j=0}^{n} b_j(F(z) - 2)^j(F(z) - 1)^{m-j};
\]

and when \( m \leq n \),

\[
\tilde{Q}(F, z) = Q\left(z, \frac{F(z) - 2}{F(z) - 1}\right)(F(z) - 1)^n = \sum_{j=0}^{n} b_j(F(z) - 2)^j(F(z) - 1)^{n-j}.
\]

Now we take the same sequences of \( \{z_n\} \) and \( \{\rho_n\} \) as before. Then, \( G(\zeta) = F(z_\nu + \rho_\nu \zeta) \) locally uniformly converges to the function

\[
G(\zeta) = g(\zeta) - 2 g(\zeta) - 1.
\]

Let

\[
\tilde{h}_\nu(\zeta) = \tilde{Q}(F(z_\nu + \rho_\nu \zeta), z_\nu + \rho_\nu \zeta) (z_\nu + \rho_\nu \zeta)^l.
\]

Then, \( \tilde{h}_\nu \) locally uniformly converges to a function \( T \). Moreover, set \( \tau = \max\{m, n\} \), we have

\[
T(\zeta) = \Omega(\zeta) = a(g(\zeta) - b)\left(\frac{-1}{g(\zeta) - 1}\right)^\tau = (C\zeta + D)^{-q} \frac{(C\zeta + D)^{-q}}{(C - A)(\frac{C}{A} + (D - B))^r}.
\]

Since \( \tau > q \), \( T(\zeta) \) has a finite zero \( \zeta_\ast = -D/C \). By a similar argument used in the proof of the case (a), one can derive that \( G'(\zeta_\ast) = 0 \), which leads a contradiction to the fact \( G(\zeta) \) is a Möbius transformation. Thus we finish our proof in the case (b).

Suppose the condition (c) is satisfied. We let \( z_0 \) be a pole \( f(z) \) and not a zero of \( a_0(z), a_1(z), \ldots, a_m(z); b_0(z), b_1(z), \ldots, b_n(z) \). Since \( m > n \), \( z_0 \) is a pole of \( R(f, z) = P(f, z)/Q(f, z) \) and also a pole of \( S(f, z) \). By a little computation, we have that \( z_0 \) must be a pole of \( f \) with multiplicity \( \frac{2p}{m-n} \). Thus, \( f \) has only multiple poles in \( \{\zeta \mid |\zeta| \leq R \} \) for sufficiently large \( R \). Therefore, all poles of \( f \), except at most finitely many, are multiple poles. Thus, for a fixed \( R > 0 \), \( g_\nu(\zeta) \) has only multiple poles in \( \{\zeta \mid |\zeta| \leq R \} \). Hence, the Möbius transformation \( g(\zeta) = A\zeta + B \) and \( A \neq 0 \). By the hypothesis, we have that \( \Omega(\zeta) \) is a non-constant polynomial, hence, \( \Omega(g(\zeta)) \) has a finite zero. Again, this leads to a contradiction as we see in the proof of the case (a). It follows that we prove the case (c).

Assume the condition (d) is satisfied. Let \( \tau_1 \) and \( \tau_2 \) be the ramified values of \( f \). Since \( g \) is a Möbius transformation, there exists at least one \( \tau_i \) \( (i = 1, 2) \), say, \( \tau_1 \), such that \( g(\zeta) - \tau_1 \) has a finite zero at \( \zeta_\ast \). Thus, there exists a sequence \( \zeta_\nu \) such that \( \zeta_\nu \) goes to \( \zeta_\ast \) and \( g_\nu(\zeta_\nu) - \tau_1 = 0 \), i.e., \( f(z_\nu + \varepsilon_\nu|z_\nu|^{2-\beta}\zeta_\nu) - \tau_1 = 0 \). Hence, \( f'(z_\nu + \varepsilon_\nu|z_\nu|^{2-\beta}\zeta_\nu) = 0 \). It follows that \( g'(\zeta_\ast) = 0 \). We get a contradiction again. □

**Theorem 2.** Let \( f \) be a meromorphic solution of (2). If \( f \) has a Picard exceptional value, then the order of \( f \) is finite. Furthermore, if \( \infty \) is the Picard exceptional value of \( f \), then
\[ n = m, \ \deg a_m > 2p + \deg b_n \text{ and } \rho(f) = (\deg a_m - \deg b_n)/(2p) - 1; \text{ if } f \text{ has a finite Picard exceptional value } a, \text{ then } P(a, z) \not\equiv 0, \ Q(a, z) \not\equiv 0, \ \deg_z P(a, z) > 2p + \deg_z Q(a, z) \]

and \[ \rho(f) = (\deg_z P(a, z) - \deg_z Q(a, z))/(2p) - 1. \]

**Proof.** First, we consider the case that \( \infty \) is the Picard exceptional value of \( f \). Otherwise, we may consider a composition of a Möbius transformation and the function \( f \). Thus we can express \( f \) as

\[ f(z) = \frac{g(z)}{P(z)}, \]

where \( g(z) \) is a transcendental entire function and \( P(z) \) is a polynomial. For any \( r > 0 \), let \( |g(z_0)| = M(g, r), \ (|z_0| = r) \). Then, by Lemma 4, there is a set \( F \) of a finite logarithmic measure such that

\[ f'(z_0) f(z_0) = \frac{g'(z_0)}{g(z_0)} - \frac{P'(z_0)}{P(z_0)} - \nu(g, r) \frac{z_0}{1 + o(1)}, \]

\[ f''(z_0) f(z_0) = \frac{g''(z_0)}{g(z_0)} - \frac{P''(z_0)}{P(z_0)} - 2 \frac{P'(z_0)}{P(z_0)} f'(z_0) - \nu(g, r) \frac{z_0}{1 + o(1)}; \]

and

\[ f'''(z_0) f(z_0) = \frac{g'''(z_0)}{g(z_0)} - \frac{P'''(z_0)}{P(z_0)} - 3 \frac{P''(z_0)}{P(z_0)} f'(z_0) - 3 \frac{P'(z_0)}{P(z_0)} f''(z_0) - \nu(g, r) \frac{z_0}{1 + o(1)}; \]

for all sufficient large \( r \notin F \). Thus Eq. (2) can be changed to

\[ \left( \frac{\nu(g, r)}{z_0} \right)^2 (1 + o(1)) = R(f(z_0), z_0). \tag{5} \]

If \( m > n \), then it follows from Eq. (5) that

\[ \left( \frac{\nu(g, r)}{r} \right)^{2p} \sim A M(r, g)^{m-n} \nu^{(m-n)} \deg P + \deg a_m - \deg b_n, \]

where \( A \) is some constant. Hence,

\[ \lim_{r \to \infty} \frac{\log v(g, r)}{\log M(g, r)} > 0. \]

This is a contradiction.

If \( m < n \), then

\[ \left( \frac{\nu(g, r)}{r} \right)^{2p} \sim A M(r, g)^{n-m} \nu^{(m-n)} \deg P + \deg a_m - \deg b_n, \]

where \( A \) is some constant. It follows from this that \( \nu(g, r) \to 0 \) as \( r \to \infty \). This is a contradiction.
Hence \( n = m \). Thus

\[
\left( \frac{v(g, r)}{r} \right)^{2p} \sim A_r \deg a_n - \deg b_m \quad \text{or} \quad v(g, r) \sim A_r (\deg a_n - \deg b_m)/(2p) - 1.
\]

It follows that \( (\deg a_n - \deg b_m)/(2p) - 1 > 0 \), i.e., \( \deg a_n > 2p + \deg b_m \); and

\[
\rho(f) = \rho(g) = \frac{\deg a_n - \deg b_m}{2p} - 1.
\]

Secondly, we consider the case that \( f \) has a finite Picard exceptional value \( a \). Let \( g(z) = \frac{1}{f'(z) - a} \). Thus \( \infty \) is a Picard exceptional value of \( g \) and Eq. (2) is changed into

\[
S(g, z)^p = A_m(z) g^m + \cdots + A_0(z) g^n + B_0(z) g^n - m,
\]

where \( A_m(z) = P(a, z) \) and \( B_n(z) = Q(a, z) \). If \( A_m(z) \equiv B_n(z) \equiv 0 \), then \( P(f, z) \) and \( Q(f, z) \) has a common factor \( f(z) - a \), a contradiction. Using the arguments as in the first case, we have the conclusion. \( \square \)

**Theorem 3.** Let \( P \) and \( Q \) be polynomials with \( \deg P = m \) and \( \deg Q = n \) and let \( R(z) = P(z)/Q(z) \) and \( p \) a positive integer. If \( f \) is a transcendental meromorphic solution of the equation

\[
S(f, z)^p = R(z), \tag{6}
\]

then \( m - n + 2p > 0 \) and the order \( \rho(f) = (m - n + 2p)/2p. \)

**Proof.** By comparing the poles of the both sides of Eq. (6), we get that \( f' \) has only finitely many zeros. Set \( g = 1/f' \). Thus \( 1/f' \) has only finitely many poles and Eq. (6) can be changed into

\[
\left( -\frac{g''}{g} + \frac{1}{2} \left( \frac{g'}{g} \right)^2 \right)^p = R(z). \tag{7}
\]

Furthermore, there is a polynomial \( p \) such that \( g(z)p(z) = h(z) \) is a transcendental entire function. Let \( |h(z_0)| = M(h, r) \). Then, by Lemma 4, we have

\[
\frac{g'(z_0)}{g(z_0)} = \frac{h'(z_0)}{h(z_0)} - \frac{p'(z_0)}{p(z_0)} = \frac{v(h, r)}{z_0} (1 + o(1)), \quad r \notin F,
\]

and

\[
\frac{g''(z_0)}{g(z_0)} = \frac{h''(z_0)}{h(z_0)} - \frac{p''(z_0)}{p(z_0)} = \frac{v(h, r)}{z_0} \left( \frac{v(h, r)}{z_0} \right)^2 (1 + o(1)), \quad r \notin F,
\]

where \( F \) is a set of a finite logarithmic measure. Thus Eq. (7) can be changed to

\[
\left( -\frac{v(h, r)}{z_0} \right)^2 (1 + o(1)) + \frac{1}{2} \left( \frac{v(h, r)}{z_0} \right)^2 (1 + o(1)) \right)^p = R(z_0).
\]

It turns out that there is a positive constant \( A \) such that

\[
v(h, r) = A r^{1+(m-n)/(2p)}
\]
for all sufficient large \( r \) outside a set of finite logarithmic measure. Therefore, \( m - n + 2p > 0 \) and the order \( \rho(g) = \rho(h) = 1 + (m - n)/(2p) \). So \( \rho(f) = 1 + (m - n)/(2p) \). □

Remark 4. It is easy to see that if \( f \) is a meromorphic solution of Eq. (6), then \( f \) is a quotient of two linear independent solutions of the equation

\[
w'' + A(z)w = 0,
\]

where \( A(z) \) is a rational function. Thus, the finiteness of the growth order of \( f \) follows from the results of Nevanlinna et al.

Now we want to consider the order of solutions of the following Schwarzian differential equation

\[
S(f, z) = R(f).
\]

In 1991, Ishizaki [8] studied the structure of \( R(f, z) \) in Eq. (8) and proved

Theorem A. Suppose that the Schwarzian differential equation (8) admits a transcendental meromorphic solution. Then for some Möbius transformation \( u = (af + b)/(cf + d) \), \( ab - cd \neq 0 \), (8) reduces into one of the following types:

\[
\begin{align*}
S(u, z) &= c\frac{(u - \sigma_1)(u - \sigma_2)(u - \sigma_3)(u - \sigma_4)}{(u - \tau_1)(u - \tau_2)(u - \tau_3)(u - \tau_4)}, \\
S(u, z)^3 &= c\frac{(u - \sigma_1)^3(u - \sigma_2)^3}{(u - \tau_1)^3(u - \tau_2)^3(u - \tau_3)^3}, \\
S(u, z)^3 &= c\frac{(u - \sigma_1)^3(u - \sigma_2)^3}{(u - \tau_1)^3(u - \tau_2)^2(u - \tau_3)^2}, \\
S(u, z)^2 &= c\frac{(u - \sigma_1)^2(u - \sigma_2)^2}{(u - \tau_1)^2(u - \tau_2)(u - \tau_3)}, \\
S(u, z) &= c\frac{(u - \sigma_1)(u - \sigma_2)}{(u - \tau_1)(u - \tau_2)}, \\
S(u, z) &= c,
\end{align*}
\]

where \( \tau_j \ (j = 1, \ldots, 4) \) are distinct constants, and \( \sigma_j \ (j = 1, \ldots, 4) \) are constants, not necessarily distinct.

In fact, Ishizaki also obtained a more general result about the structure of \( R(f, z) \) in Eq. (2) when \( R(f, z) \) has meromorphic coefficients.

Theorem 4. Let \( f \) be a meromorphic solution of the Schwarzian differential equation (8), then \( \rho(f) \leq 2 \).

Proof. By Theorem A, it suffices to show that if \( f \) is a solution of Eqs. (9)–(14), then the order \( \rho(f) \leq 2 \). In fact, if \( f \) satisfies Eqs. (9)–(13), then \( f \) has at least two ramified values \( \tau_1 \) and \( \tau_2 \). Thus by Theorem 1, we have \( \rho(f) \leq 2 \). If \( f \) satisfies Eq. (14), then we have \( \rho(f) = 1 \) by Theorem 3. □
4. Complex dynamics of solutions of the Schwarzian differential equations

We say a meromorphic function $f$ in the class $S$ if $f$ has only finitely many critical and asymptotic values. It is well known (e.g., see Bergweiler [1]) functions in $S$ do not have wandering domains or Baker domains. In 1995, Bergweiler and Eremenko [3] proved

**Theorem B.** If $f$ is a meromorphic function of finite order $\rho$ and $E$ is the set of its critical values, then the number of asymptotic values of $f$ is at most $2\rho + \text{card } E'$, where $E'$ stands for the derived set of $E$.

**Corollary 1.** If $f$ is a meromorphic solution of Eq. (6), then $f \in S$ and has no wandering domain and no Baker domain.

**Proof.** Since $f$ is a solution of Eq. (6), $f'$ has only finitely many zeros. By Theorems 3 and B, $f$ has only finitely many asymptotic values. Therefore, $f$ is in the class $S$. □

**Corollary 2.** If $f$ is a meromorphic solution of Eq. (8), then $f \in S$ and has no wandering domain and no Baker domain.

**Proof.** Since $f$ is a meromorphic solution of (8), $z$ is a zero of $f'$ if and only if $z$ is a zero of $Q(f(z))$. It turns out from Theorem A that $f$ has only finitely many critical values. Therefore, the corollary follows from Theorems 4 and B. □

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**References**