Bilateral generating functions
and operational methods

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Abstract
Bilateral generating functions are those involving products of different types of polynomials. We show that operational methods offer a powerful tool to derive these families of generating functions. We study cases relevant to products of Hermite polynomials with Laguerre, Legendre and other polynomials. We also propose further extensions of the method which we develop here. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

One of the by-products of the method of quasi-monomials, proposed in Refs. [1,2], is the possibility of dealing, in a fairly straightforward way, with different classes of generating functions. An example is provided by the Mehler type formula [3]

\[ S(x, y; z, w|t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y) H_n(z, w), \]  

(1)

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with $H_n(x, y)$ being Hermite–Kampé de Fériet polynomials defined by

$$H_n(x, y) = n! \sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r}}{r!(n-2r)!},$$

(2)

by the operational rule [4]

$$H_n(x, y) = \exp \left( y \frac{\partial^2}{\partial x^2} \right)(x^n),$$

(3)

and by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y) = \exp(x t + y t^2).$$

(4)

In fact, in terms of the classical Hermite polynomials $H_n(x)$, we have the following relationship:

$$H_n(x, y) = i^{-n} y^{n/2} H_n \left( \frac{i x}{2\sqrt{y}} \right).$$

(5)

The use of the identity (3) allows to cast Eq. (1) in the form

$$S(x, y; z, w | t) = \exp \left( y \frac{\partial^2}{\partial x^2} + w \frac{\partial^2}{\partial z^2} \right) \sum_{n=0}^{\infty} \frac{(x z t)^n}{n!}$$

$$= \exp \left( y \frac{\partial^2}{\partial x^2} + w \frac{\partial^2}{\partial z^2} \right) e^{x z t}.$$  

(6)

Furthermore, since

$$\exp \left( y \frac{\partial^2}{\partial x^2} \right) f(x) = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4y} \right) f(\xi) \, d\xi,$$  

(7)

we can transform the derivation of the generating function in Eq. (1) in the evaluation of a Gaussian integral, thus getting [3]

$$S(x, y; z, w | t) = \frac{1}{\sqrt{1 - 4y t^2 w}} \exp \left( \frac{x z t + t^2 (w x^2 + y z^2)}{1 - 4t^2 y w} \right).$$ 

(8)

The use of analogous techniques may be useful to state other results concerning, e.g., the generating functions associated with Laguerre polynomials, which are quasi-monomials under the action of the operators [5]

$$\hat{M} = y - \hat{D}_x^{-1}, \quad \hat{P} = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x},$$

(9)
where $\hat{D}_x^{-1}$ is the inverse of the derivative operator and accordingly the Laguerre polynomials can be defined as

$$\mathcal{L}_n(x, y) = \hat{M}^n = \sum_{s=0}^{n} \binom{n}{s} (-1)^s y^{n-s} x^s \hat{D}_x^{-s} = n! \sum_{s=0}^{n} \frac{(-1)^s y^{n-s} x^s}{(n-s)! (s!)^2}. \quad (10)$$

Here the operators $\hat{M}$ and $\hat{P}$ act on a quasi-monomial $p_n(x)$ in a multiplicative-like and derivative-like form; namely

$$\hat{M} p_n(x) = p_{n+1}(x), \quad \hat{P} p_n(x) = np_n(x). \quad (11)$$

For further comments see Refs. [2,5].

Equations (9) and (10) can also be applied in order to obtain the following generating function [5]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}_n(x, y) = \exp(yt) C_0(xt), \quad (12)$$

with

$$C_n(x) = x^{-n/2} J_n(2\sqrt{x}) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}. \quad (13)$$

being the $n$th-order Tricomi (or Bessel) function [6].

In the following sections we will prove that the previously outlined procedure offers a useful tool for the derivation of bilateral generating functions involving products of polynomials of different nature given by (3). We will, indeed, derive generating functions of Mehler type, involving products of Hermite–Laguerre, Hermite–Legendre polynomials, using methods of operational nature.

### 2. Bilateral generating functions

The first example we will consider is provided by

$$F(x, y; z, w|t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y) \mathcal{L}_n(z, w) \quad (14)$$

which, according to Eqs. (3) and (12), can be rewritten as

$$F(x, y; z, w|t) = \exp\left( y \frac{\partial^2}{\partial x^2} \right) \exp(w xt) C_0(z xt). \quad (15)$$

It is evident that the evaluation of the infinite sum (14) amounts to the derivation of the integral

$$F(x, y; z, w|t) = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{\infty} \exp\left[ -\frac{(x - \xi)^2}{4y} \right] \exp(w \xi t) C_0(z \xi t) d\xi. \quad (16)$$
By using integration techniques based on the generating function method (see Appendix A) we find the following result:

\[
F(x, y; z, w|t) = \exp\left(\frac{y(wt)^2 + x wt}{2}\right)H_C_0\left(2zt\left(\frac{x}{2} + y wt\right), y(zt)^2\right),
\]

(17)

where

\[
H_C_n(x, y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r(x, y)}{r!(n + r)!}
\]

(18)

is the \(n\)th-order Hermite–Tricomi function whose properties have been discussed in Ref. [7].

The previously outlined method can be exploited to extend the above result, thus getting, e.g.,

\[
F_{\lambda}(x, y; z, w|t) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_{n+\lambda}(x, y) \mathcal{L}_n(z, w)
\]

\[
= A_{\lambda}(x, y; z, w|t) \exp\left(y(wt)^2 + x wt\right),
\]

\[
A_{\lambda}(x, y; z, w|t) = \sum_{p=0}^{\lambda} \binom{\lambda}{p} (-2yzt)^p H_{\lambda-p}(x + 2y wt, y)
\]

\[
\times \left(2zt\left(\frac{x}{2} + y wt\right), y(zt)^2\right)
\]

(19)

whose derivation is detailed in Appendix A. It is also worth noting that the same result can be achieved by exploiting Eqs. (4), (9), and (10) which yield

\[
F(x, y; z, w|t) = \exp\left(x t \left(w - \hat{D}_z^{-1}\right) + y t^2 \left(w - \hat{D}_z^{-2}\right)\right)
\]

\[
= \exp\left(y(wt)^2 + wxt\right) \exp\left(-(xt + 2y t^2 w)\hat{D}_z^{-1} + y t^2 \hat{D}_z^{-2}\right)
\]

(20)

which, according to the operational identity

\[
\exp(-x\hat{D}_z^{-1} + y\hat{D}_z^{-2}) = \left(H_C_0(x z, y z^2)\right),
\]

(21)

provides us with a further tool to derive Eq. (17).

In Ref. [5] the following polynomials have been introduced:

\[
\mathcal{L}_n(x, y) = n! \sum_{s=0}^{[n/2]} \frac{y^{n-2s}(-x)^s}{(n - 2s)! s!^2}
\]

(22)

whose nature, in between Laguerre and Hermite type polynomials, ensures their monomiality under the action of the operators.
\[ \hat{M} = y - 2\hat{D}_x^{-1} \frac{\partial}{\partial y}, \]
\[ \hat{P}_y = \frac{\partial}{\partial y}, \quad \hat{P}_x = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x}. \]  \tag{23}

The relevant generating function can be directly deduced from the first of Eqs. (23) thus finding
\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} (2\mathcal{L}_n(x, y)) = \exp(yt)C_0(x^2 t^2). \]  \tag{24}

We can now derive a further bilateral generating function involving Hermite and \( 2\mathcal{L}_n(x, y) \) polynomials. We find indeed
\[ 2F(x, y; z, w|t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y) 2\mathcal{L}_n(z, w) \]
\[ = \exp\left( y \frac{\partial^2}{\partial x^2} \right) \exp(wxt)C_0(z^2 t^2). \]  \tag{25}

By following the integration method discussed in Appendix A we end up with
\[ 2F(x, y; z, w|t) = \exp(xwt + y(wt)^2) \times HJ_n\left(4y\sqrt{zt}\left(\frac{x}{2y} + wt\right), 4zyt^2\right), \]  \tag{26}
where
\[ HJ_n(x, y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}(x, y)}{2^{n+2r} r! (n + r)!} \]  \tag{27}
is a Bessel–Hermite function [7].

An important by product of the identity (26) follows from the fact that \( 2\mathcal{L}_n(x, y) \) implicitly contain the Legendre polynomials [8]; namely
\[ 2\mathcal{L}_n\left(-\frac{1}{4}(1 - w^2), -w\right) = P_n(w). \]  \tag{28}

It is therefore clear that, by setting \( z = -1/4(1 - w^2) \) and by replacing \( w \) with \( -w \), we are able to derive a bilateral Mehler-type generating function involving products of Hermite and Legendre polynomials.
3. Concluding remarks

It is evident that the method we have proposed offers a fairly efficient means to evaluate families of bilateral generating functions. A further example involves the polynomials

\[ R_n(x, y) = (n!)^2 \sum_{k=0}^{n} \frac{(-1)^{n-k} k^n x^{n-k}}{(k!)^2((n-k)!)^2} \]  

defined in Ref. [9] and quasi-monomials under the action of

\[ \hat{M} = \hat{D}_y^{-1} - \hat{D}_x^{-1}, \]
\[ \hat{P}_x = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x}, \quad \hat{P}_y = \frac{\partial}{\partial y} y \frac{\partial}{\partial y}. \]  

(30)

It is relatively easy to prove that

\[ \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} R_n(x, y) = C_0(-yt)C_0(xt). \]  

(31)

According to the previous relations, we can easily state that

\[ G(x, y; z, w|t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} H_n(x, y) R_n(z, w) \]
\[ = \exp\left( y \frac{\partial^2}{\partial x^2} \right) C_0(-wxt)C_0(zxt). \]  

(32)

The problem is again reduced to the evaluation of Gaussian integrals; the specific details of the integration method are given in Appendix A, and we get

\[ G(x, y; z, w|t) = H C_{0,0}(-xwt, y(wt)^2; zxt, y(zt)^2; -2wyzt^2), \]  

where \[ H C_{0} (\alpha, \beta; \gamma, \delta|\varepsilon) \] is the (0th, 0th) order of the two-index Tricomi–Bessel functions defined as

\[ H C_{m,n}(x, y; z, w|\tau) = \sum_{(r,s)=0}^{\infty} \frac{(-1)^{r+s} H_{r,s}(x, y; z, w|\tau)}{r!s!(m+r)!(n+r)!}, \]  

(34)

with \[ H_{m,n}(x, y; z, w|\tau) \] being two-index Hermite polynomials defined in Appendix A and studied in [4] and references therein. This last result is relatively important, since it offers an interesting conclusion, based on the fact that the Legendre polynomials are a particular case of the polynomials (27); according to Ref. [7] we have

\[ P_n(x) = R_n \left( \frac{1-x}{2}, \frac{1+x}{2} \right). \]  

(35)
Therefore by setting
\[ z = \frac{1 - \zeta}{2}, \quad w = \frac{1 + \zeta}{2} \] (36)
in Eqs. (30) and (31), we can derive a bilateral generating function of the type
\[ G(x, y; \zeta | t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y) P_n(\zeta). \] (37)

In this paper we have outlined a general method to derive families of generating functions, hardly achievable with ordinary means. In a forthcoming investigation, we will discuss the extension of the method to multiindex polynomials.

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**Appendix A**

Most of the results of the present paper are based on integrals of the type [10]
\[ I_n(a, b, c) = \int_{-\infty}^{\infty} \exp(-ax^2 + bx) J_n(cx) \, dx, \] (A.1)
where \( J_n(x) \) is a cylindrical Bessel function of the first kind. By recalling that the generating function of ordinary Bessel functions is
\[ \sum_{n=-\infty}^{\infty} t^n J_n(x) = \exp\left(\frac{x}{2} \left( t - \frac{1}{t} \right) \right), \quad |t| < \infty, \ t \neq 0, \] (A.2)
we can derive the integral (A.1) by following the method proposed in Ref. [10] and summarized below:

1. We use Eq. (A.2) to write
\[ \sum_{n=-\infty}^{\infty} t^n I_n(a, b, c) = \int_{-\infty}^{\infty} \exp[-ax^2] \exp\left[ b + c \left( t - \frac{1}{t} \right) \right] x \, dx. \] (A.3)

2. From the integral on the r.h.s. we get
\[ \sum_{n=\infty}^{\infty} t^n I_n(a, b, c) = \sqrt{\frac{\pi}{a}} \exp \left( \frac{b^2}{4a} \right) \exp \left( \frac{bc}{4a} \left( t - \frac{1}{t} \right) \right) \times \exp \left( \frac{c^2}{16a} \left( t - \frac{1}{t} \right)^2 \right). \] \quad (A.4)

3. By recalling that the Hermite–Bessel function is just provided by
\[ \sum_{n=\infty}^{\infty} t^n (H J_n(x, y)) = \exp \left( \frac{x}{2} \left( t - \frac{1}{t} \right) + \frac{y}{4} \left( t - \frac{1}{t} \right)^2 \right), \]
\[ H J_n(x, y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}(x, y)}{2^{n+2r} r!(n+r)!}, \] \quad (A.5)

it is therefore evident that
\[ I_n(a, b, c) = \sqrt{\frac{\pi}{a}} \exp \left( \frac{b^2}{4a} \right) (H J_n \left( \frac{bc}{2a}, \frac{c^2}{4a} \right)). \]

The same method applies to the derivation of integrals involving the Tricomi functions, characterized by the generating functions
\[ \sum_{n=\infty}^{\infty} t^n C_n(x) = \exp \left( t - \frac{x}{t} \right), \quad C_n(x) = x^{-n/2} J_n \left( 2\sqrt{x} \right), \] \quad (A.6)

thus finding
\[ I_n(a, b, c) = \int_{-\infty}^{\infty} \exp(-ax^2) \exp(bx) C_n(cx) \, dx \]
\[ = \sqrt{\frac{\pi}{a}} \exp \left( \frac{b^2}{4a} \right) (H C_n \left( \frac{bc}{2a}, \frac{c^2}{4a} \right)), \] \quad (A.7a)

where
\[ H C_n(x, y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r(x, y)}{r!(n+r)!}. \] \quad (A.7b)

As to Eq. (17) we note that its derivation depends on integrals of the type
\[ I_{n,\lambda}(a, b, c) = \int_{-\infty}^{\infty} \exp(-ax^2) \exp(bx) x^\lambda C_n(cx) \, dx \] \quad (A.8)

whose evaluation requires a slight extension of the generating function method; we can indeed write
\[
\sum_{n=-\infty}^{\infty} \sum_{\lambda=0}^{\infty} t^n \frac{u^\lambda}{\lambda!} I_{n,\lambda}(a, b, c) = e^t \int_{-\infty}^{\infty} \exp(-ax^2) \exp\left(\left(b + \frac{c}{t}\right)x\right) dx \\
= \exp\left(\frac{b^2}{4a}\right) \sqrt{\frac{\pi}{a}} \exp\left(t - \frac{bc}{2at} + \frac{c^2}{4at^2}\right) \exp\left(\frac{u^2}{4a} + \frac{bu}{2a}\right) \exp\left(-\frac{uc}{2at}\right).
\]
(A.9)

Thus, finally, we get
\[
I_{n,\lambda}(a, b, c) = \sqrt{\frac{\pi}{a}} \frac{\lambda!}{\lambda} \sum_{p=0}^{\lambda} \left(\frac{-\frac{c}{2a}\right)^p C_{n+p} \left(\frac{bc}{2a} + \frac{c^2}{4a}\right) H_{\lambda-p} \left(\frac{b}{2a} + \frac{1}{4a}\right) \exp\left(\frac{b^2}{4a}\right). \right.
\]
(A.10)

The case relevant to integrals containing products of Bessel functions, namely
\[
I_{m,n}(a, b, c, d) = \int_{-\infty}^{\infty} \exp(-ax^2 + bx) J_m(cx) J_n(dx) dx,
\]
(A.11)
requires an extension of the method leading to Eq. (A.11) which yields
\[
\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u^m v^n I_{m,n}(a, b, c, d) \\
= e^t \int_{-\infty}^{\infty} \exp(-ax^2) \exp\left\{\left[b + \frac{c}{2}\left(u - \frac{1}{u}\right) + \frac{d}{2}\left(v - \frac{1}{v}\right]\right]x\right\} dx,
\]
so that one finally gets
\[
I_{m,n}(a, b, c, d) = \sqrt{\frac{\pi}{a}} \exp(b^2/4a) \left(\mathcal{H}_{m,n}\left(\frac{bc}{2a}, \frac{c^2}{4a}; \frac{bd}{2a}, \frac{d^2}{4a} \right)\right),
\]
(A.12)

where
\[
\mathcal{H}_{m,n}\left(\frac{bc}{2a}, \frac{c^2}{4a}; \frac{bd}{2a}, \frac{d^2}{4a}\right)
\]
denotes a two-index Hermite–Bessel function defined by
\[
\mathcal{H}_{m,n}(x, y; z, w | \tau) = \sum_{(r,s)=0}^{\infty} \frac{(-1)^{r+s} H_{m+2r,n+2s}(x, y; z, w | \tau)}{2^{m+n+2(r+s)} r! s! (m+r)! (n+s)!},
\]
where
\[
H_{m,n}(x, y; z, w | \tau) = m! n! \sum_{r=0}^{\min(m,n)} \frac{\tau^r H_{m-r}(x, y) H_{n-r}(z, w)}{r!(m-r)! (n-r)!}.
\]
(A.13)
This family of functions is characterized by the generating function

\[
\sum_{m,n=\infty}^{\infty} u^m v^n \left( J_{m,n}(x, y; z, w|\tau) \right) = \exp \left( \frac{x}{2} \left( u - \frac{1}{u} \right) + \frac{y}{4} \left( u - \frac{1}{u} \right)^2 + \frac{z}{2} \left( v - \frac{1}{v} \right) + \frac{w}{4} \left( v - \frac{1}{v} \right)^2 \right.
\]
\[
+ \frac{\tau}{4} \left( u - \frac{1}{u} \right) \left( v - \frac{1}{v} \right) \right)
\]  
(A.14)

The Tricomi version of the functions (A.5) can be cast in the form

\[
H_{m,n}(x, y; z, w|\tau) = \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s} H_{r,s}(x, y; z, w|\tau)}{r!s!(m+r)!(n+s)!}
\]  
(A.15)

and the relevant generating function reads

\[
\sum_{m,n=\infty}^{\infty} u^m v^n \left( H_{m,n}(x, y; z, w|\tau) \right) = \exp \left( \left( u - \frac{x}{u} \right) + \frac{y}{u^2} + \left( v - \frac{z}{v} \right) + \frac{w}{v^2} + \tau uv \right).
\]  
(A.16)

These last relations justify the results contained in the concluding section of the present paper.

References