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Inheritance principle and non-renormalization theorems at finite temperature

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Abstract

We present a general proof of an "inheritance principle" satisfied by a weakly coupled $SU(N)$ gauge theory with adjoint matter on a class of compact manifolds (like S^3). In the large N limit, finite temperature correlation functions of gauge invariant single-trace operators in the low temperature phase are related to those at zero temperature by summing over images of each operator in the Euclidean time direction. As a consequence, various non-renormalization theorems of $\mathcal{N}=4$ super-Yang–Mills theory on S^3 survive at finite temperature despite the fact that the conformal and supersymmetries are both broken.

1. Introduction

It has been shown that weakly coupled $SU(N)$ gauge theories with adjoint matter on a class of compact manifolds (including S^3) have a large N "deconfinement" transition at a temperature T_c [1–3]. In the low temperature ("confined") phase $T < T_c$, the free energy is of order $O(1)$, while in the high temperature ("deconfined") phase the free energy becomes of order $O(N^2)$.

The main purpose of the Letter is to give a general proof of an "inheritance principle" satisfied by these gauge theories in the low temperature phase and point out some consequences of it. More explicitly, suppose at zero temperature the *Euclidean* n-point function of some gauge invariant single trace operators $\mathcal{O}_1, \dots, \mathcal{O}_n$ is given by

$$G_0(\tau_1, e_1; \tau_2, e_2; \dots; \tau_n, e_n) = \langle \mathcal{O}_1(\tau_1, e_1)\mathcal{O}_2(\tau_2, e_2)\cdots\mathcal{O}_n(\tau_n, e_n)\rangle,$$

where τ_i denote the Euclidean time and e_i denote a point on the compact manifold. Then one finds that the corresponding correlation function at finite temperature $T = \frac{1}{\beta}$ is given by[1]

$$G_\beta(\tau_1, \tau_2, \dots, \tau_n) = \sum_{m_1,\dots,m_n=-\infty}^{\infty} G_0(\tau_1 - m_1\beta, \tau_2 - m_2\beta, \dots, \tau_n - m_n\beta), \tag{1.1}$$

where for notational simplicity we have suppressed the spatial coordinates. In other words, one adds images for each operator $\mathcal{O}_i$ in the Euclidean time direction. Note that the statement is not trivial, since in thermal gauge theory computations one is supposed to add images for each fundamental field in the operator, not the operator as a whole.

Eq. (1.1) was first derived in [4], where the case of a scalar field on $\mathbb{R}^3$ was explicitly considered. The crucial ingredients for establishing (1.1) were the dominance of planar graphs in the large N limit and the saddle point configuration of the time component

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[1] The following expression assumes that all $\mathcal{O}_i$ are bosonic. If an $\mathcal{O}_i$ is fermionic, then one multiplies an additional factor $(-1)^{m_i}$.

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A_0 of the gauge field characteristic of the confined phase. While not shown explicitly, the discussion there could in principle be generalized to compact spaces and including dynamical gauge fields and fermions.

In this Letter we will present an alternative proof of (1.1) for gauge theories on compact spaces including dynamical gauge fields and fermions. Our framework has the advantages that the discussions can be easily generalized to the study of non-planar diagrams and other saddle points of finite temperature gauge theories. We also discuss some subtleties involving self-contractions at finite temperature and present an explicit argument that (1.1) holds to *all orders* in perturbative expansions in the 't Hooft coupling.

Eq. (1.1) implies that properties of correlation functions of the theory at zero temperature can be inherited at finite temperature in the large N limit. For example, for those correlation functions which are independent of the 't Hooft coupling in the large N limit at zero temperature, the statement remains true at finite temperature. For $\mathcal{N}=4$ super-Yang–Mills theory (SYM) on S^3, which was the main motivation of our study, it was conjectured in [5] that two- and three-point functions of chiral operators are non-renormalized from weak to strong coupling.[2] The conjecture, if true, will also hold for $\mathcal{N}=4$ SYM theory at finite temperature despite the fact that the conformal and supersymmetries are broken. (1.1) also suggests that at leading order in $1/N$ expansion, the one-point functions of all gauge invariant operators (including the stress tensor) at finite temperature are zero.

While (1.1) is somewhat non-intuitive from the gauge theory point of view, it has a simple interpretation[3] from the string theory dual [4]. Suppose that the gauge theory under consideration is described by a string theory on some target space M. Then (1.1) translates into the statement that the theory at finite temperature is described by propagating strings on M with Euclidean time direction compactified with a period β. The leading order expression for G_β in large N limit is mapped to the sphere amplitude of the dual string theory. The "inheritance" principle in the gauge theory then follows simply from that of the tree level orbifold string theory.

We note that given a perturbative string theory, it is not a priori obvious that the theory at finite temperature is described by the same target space with time direction periodically identified.[4] For perturbative string theory in flat space at a temperature below the Hagedorn temperature, this can be checked by explicit computation of the free energy at one-loop [11]. Eq. (1.1) provides evidences that this should be the case for string theories dual to the class of gauge theories we are considering at a temperature $T < T_c$.

For $\mathcal{N}=4$ SYM theory on S^3, the result matches well with that from the AdS/CFT correspondence [12–14]. When the curvature radius of the anti-de Sitter (AdS) spacetime is much larger than the string and Planck scales (which is dual to the YM theory at large 't Hooft coupling) the correspondence implies that IIB string in $AdS_5 \times S_5$ at $T < T_c$ is described by compactifying the time direction (so-called thermal AdS) [14,15]. The result from the weakly coupled side suggests that this description can be extrapolated to weak coupling.[5]

The plan of the Letter is as follows. In section two we make some general statements regarding finite temperature correlation functions in free theory. In section three we prove the "inheritance principle" in free theory limit. In section four we generalize the discussion to include interactions. In section five we conclude with a discussion of string theory interpretation and some other remarks.

For the rest of the Letter unless stated explicitly, by finite temperature we always refer to finite temperature in the low temperature phase of the theory.

2. Correlation functions of free Yang–Mills theory on S^3

In this section we discuss some general aspects of free gauge theories with adjoint matter on S^3 at finite temperature. We will assume that the theory under consideration has a vector field A_μ and a number of scalar and fermionic fields[6] all in the adjoint representation of $SU(N)$. The discussion should also be valid for other simply-connected compact manifolds. We use the Euclidean time formalism with time direction τ compactified with a period $\beta = \frac{1}{T}$. Spacetime indices are denoted by $\mu = (\tau, i)$ with i along directions on S^3.

The theory on S^3 can be written as a $(0+1)$-dimensional (Euclidean) quantum mechanical system by expanding all fields in terms of spherical harmonics on S^3. Matter scalar and fermionic fields can be expanded in terms of scalar and spinor harmonics respectively. For gauge field, it is convenient to use the Coulomb gauge $\nabla_i A^i = 0$, where ∇ denotes the covariant derivative on S^3. In this gauge, A_i can be expanded in terms of transverse vector harmonics, A_τ and the Fadeev–Popov ghost c can be expanded in terms of scalar harmonics. At quadratic level, the resulting action has the form

$$S_0 = N\,\mathrm{Tr}\int d\tau\left[\left(\frac{1}{2}(D_\tau M_a)^2 - \frac{1}{2}\omega_a^2 M_a^2\right) + \xi_a^\dagger(D_\tau + \tilde{\omega}_a)\xi_a + \frac{1}{2}m_a^2 v_a^2 + m_a^2 \bar{c}_a c_a\right], \tag{2.1}$$

[2] See also [6–9] for further evidence.

[3] It also has a natural interpretation from the point of view of large N reduction. This was pointed out to us by S. Shenker and K. Furuuchi. See also [10].

[4] A counter example is IIB string in $AdS_5 \times S_5$ at a temperature above the Hawking–Page temperature. Also in curved spacetime this implies one has to choose a particular time slicing of the spacetime.

[5] Also note that it is likely that $AdS_5 \times S_5$ is an exact string background [16–18].

[6] We also assume that the scalar fields are conformally coupled.

where we have grouped all harmonic modes into three groups:

(1) Bosonic modes M_a with non-trivial kinetic terms. Note that in the Coulomb gauge, the harmonic modes of the dynamical gauge fields have the same $(0+1)$d action as those from matter scalar fields. We thus use M_a to denote collectively harmonic modes coming from both the gauge field A_i and matter scalar fields.
(2) Fermionic modes ξ_a with non-trivial kinetic terms.
(3) v_a and c_a are from non-zero modes of A_τ and the Fadeev–Popov ghost c, which have no kinetic terms.

The explicit expressions of various $(0+1)$-dimensional masses $\omega_a,\ \tilde{\omega}_a,\ m_a$ can in principle be obtained from properties of various spherical harmonics and will not be used below. In (2.1), following [2] we separated the zero mode $\alpha(\tau)$ of A_τ on S^3 from the higher harmonics and combine it with ∂_τ to form the covariant derivative D_τ of the $(0+1)$-dimensional theory, with

$$D_\tau M_a = \partial_\tau M_a - i[\alpha, M_a], \qquad D_\tau \xi_a = \partial_\tau \xi_a - i[\alpha, \xi_a].$$

$\alpha(\tau)$ plays the role of the Lagrange multiplier which imposes the Gauss law on physical states. In the free theory limit the ghost modes c_a do not play a role and v_a only give rise to contact terms (i.e. terms proportional to delta functions in the time direction) in correlation functions.[7] Also note that M_a, ξ_a satisfy periodic and anti-periodic boundary conditions respectively

$$M_a(\tau+\beta) = M_a(\tau), \qquad \xi_a(\tau+\beta) = -\xi_a(\tau). \tag{2.2}$$

Upon harmonic expansion, correlation functions of gauge invariant operators in the four-dimensional theory reduce to sums of those of the one-dimensional theory (2.1). More explicitly, a four-dimensional operator $\mathcal{O}(\tau, e)$ can be expanded as

$$\mathcal{O}(\tau, e) = \sum_i f_i^{(\mathcal{O})}(e) Q_i(\tau), \tag{2.3}$$

where e denotes a point on S^3 and Q_i are operators formed from M_a, ξ_a, v_a and their time derivatives. The functions $f_i^{(\mathcal{O})}(e)$ are given by products of various spherical harmonics. A generic n-point function in the four-dimensional theory can be written as

$$\langle \mathcal{O}_1(\tau_1, e_1)\mathcal{O}_2(\tau_2, e_2)\cdots \mathcal{O}_n(\tau_n, e_n)\rangle = \sum_{i_1,\ldots,i_n} f_{i_1}^{\mathcal{O}_1}(e_1)\cdots f_{i_n}^{\mathcal{O}_n}(e_n)\langle Q_{i_1}(\tau_1) Q_{i_2}(\tau_2)\cdots Q_{i_n}(\tau_n)\rangle \tag{2.4}$$

where $\langle\cdots\rangle$ on the right-hand side denotes correlation functions in the one-dimensional theory (2.1). Note that (2.4) applies to all temperatures.

The theory (2.1) has a residue gauge symmetry

$$M_a \to \Omega M_a \Omega^\dagger, \qquad \xi_a \to \Omega \xi_a \Omega^\dagger, \qquad \alpha \to \Omega\alpha\Omega^\dagger + i\Omega\partial_\tau\Omega^\dagger. \tag{2.5}$$

At zero temperature, the τ direction is uncompact. One can use the gauge symmetry (2.5) to set $\alpha = 0$. Correlation functions of the theory (2.1) can be obtained from the propagators of $M_a,\ \xi_a$ by Wick contractions. Note that[8]

$$\langle M_{ij}^a(\tau) M_{kl}^b(0)\rangle_0 = \frac{1}{N} G_s(\tau; \omega_a)\delta_{ab}\delta_{il}\delta_{kj}, \qquad \langle \xi_{ij}^a(\tau)\xi_{kl}^b(0)\rangle_0 = \frac{1}{N} G_f(\tau; \tilde{\omega}_a)\delta_{ab}\delta_{il}\delta_{kj}, \tag{2.6}$$

where

$$G_s(\tau; \omega) = \frac{1}{2\omega} e^{-\omega|\tau|}, \qquad G_f(\tau; \omega) = (-\partial_\tau + \omega) G_s(\tau; \omega) \tag{2.7}$$

and i, j, k, l denote $SU(N)$ indices.

At finite temperature, one can again use a gauge transformation to set $\alpha(\tau)$ to zero. The gauge transformation, however, modifies the boundary conditions from (2.2) to

$$M_a(\tau+\beta) = U M_a U^\dagger, \qquad \xi_a(\tau+\beta) = -U\xi_a U^\dagger. \tag{2.8}$$

The unitary matrix U can be understood as the Wilson line of α wound around the τ direction, which cannot be gauged away. It follows that the path integral for (2.1) at finite T can be written as

$$\langle\cdots\rangle_\beta = \frac{1}{Z(\beta)} \int dU \int DM(\tau)\, D\xi(\tau) \cdots e^{-S_0[M_a, \xi_a; \alpha=0]} \tag{2.9}$$

[7] Also note that since v_a, c_a do not have kinetic terms, at free theory level they only contribute to the partition function by an irrelevant temperature-independent overall factor.

[8] We use $\langle\cdots\rangle_0$ and $\langle\cdots\rangle_\beta$ to denote the correlation functions of (2.1) at zero and finite temperature, respectively.

with M_a, ξ_a satisfying boundary conditions (2.8) and Z the partition function.

Since the action (2.1) has only quadratic dependence on M_a and ξ_a, the functional integrals over M_a and ξ_a in (2.9) can be carried out straightforwardly, reducing (2.9) to a matrix integral over U. For example, the partition function can be written as

$$Z(\beta) = \int dU\, e^{S_{\mathrm{eff}}(U)}, \tag{2.10}$$

where $S_{\mathrm{eff}}(U)$ was computed in [1,2]

$$S_{\mathrm{eff}}(U) = \sum_{n=1}^{\infty} \frac{1}{n} V_n(\beta)\, \mathrm{Tr}\, U^n\, \mathrm{Tr}\, U^{-n} \tag{2.11}$$

with

$$V_n(\beta) = z_s(n\beta) + (-1)^{n+1} z_f(n\beta), \qquad z_s(\beta) = \sum_a e^{-\beta\omega_a}, \qquad z_f(\beta) = \sum_a e^{-\beta\tilde{\omega}_a}.$$

Similarly, correlation functions at finite temperature are obtained by first performing Wick contractions and then evaluating the matrix integral for U. With boundary conditions (2.8), the contractions of M_a and ξ_a become

$$\begin{aligned}
\underbrace{M^a_{ij}(\tau) M^b_{kl}(0)} &= \frac{\delta_{ab}}{N} \sum_{m=-\infty}^{\infty} G_s(\tau - m\beta; \omega_a) U^{-m}_{il} U^m_{kj}, \\
\underbrace{\xi^a_{ij}(\tau) \xi^b_{kl}(0)} &= \frac{\delta_{ab}}{N} \sum_{m=-\infty}^{\infty} (-1)^m G_f(\tau - m\beta; \tilde{\omega}_a) U^{-m}_{il} U^m_{kj}.
\end{aligned} \tag{2.12}$$

(2.12) are obtained from (2.6) by summing over images in τ-direction and can be checked to satisfy (2.8).

As an example, let us consider the planar expression of one- and two-point functions of a normal-ordered operator $Q = \mathrm{Tr}\, M^4$, with M being one of the M_a in (2.1). One finds that

$$\langle \mathrm{Tr}\, M^4 \rangle_\beta = \frac{2}{N^2} \sum_{m\neq 0, n\neq 0} G_s(-m\beta) G_s(-n\beta) \langle \mathrm{Tr}\, U^m\, \mathrm{Tr}\, U^n\, \mathrm{Tr}\, U^{-m-n} \rangle_U \tag{2.13}$$

and the connected part of the two-point function is

$$\begin{aligned}
&\langle \mathrm{Tr}\, M^4(\tau)\, \mathrm{Tr}\, M^4(0) \rangle_\beta \\
&\quad = \frac{4}{N^4} \sum_{m,n,p,q} G_s(\tau - m\beta) G_s(\tau - n\beta) G_s(\tau - p\beta) G_s(\tau - q\beta) \langle \mathrm{Tr}\, U^{q-m}\, \mathrm{Tr}\, U^{m-n}\, \mathrm{Tr}\, U^{n-p}\, \mathrm{Tr}\, U^{p-q} \rangle_U \\
&\qquad + \frac{16}{N^4} \sum_{m,n\neq 0, p,q} G_s(-m\beta) G_s(-n\beta) G_s(\tau - p\beta) G_s(\tau - q\beta) \\
&\qquad \times \langle \mathrm{Tr}\, U^m\, \mathrm{Tr}\, U^n \left(\mathrm{Tr}\, U^{-m-p+q}\, \mathrm{Tr}\, U^{-n+p-q} + \mathrm{Tr}\, U^{-m-p-n+q}\, \mathrm{Tr}\, U^{p-q} \right) \rangle_U.
\end{aligned} \tag{2.14}$$

In (2.13)–(2.14) all sums are from $-\infty$ to $+\infty$ and

$$\langle \cdots \rangle_U = \frac{1}{Z} \int dU \cdots e^{S_{\mathrm{eff}}(U)} \tag{2.15}$$

with Z given by (2.10). We conclude this section by noting some features of (2.13)–(2.14):

(1) Since the operators are normal-ordered, the zero temperature contributions to the self-contractions (corresponding to $m, n = 0$) are not considered. In general, the one-point function is not zero at finite T because of the sum over images; this is clear from (2.13).

(2) The first term of (2.14) arises from contractions in which all Ms of the first operator contract with those of the second operator. The second term of (2.14) contains partial self-contractions[9], i.e. two of Ms in $\mathrm{Tr}\, M^4$ contract within the operator. The non-vanishing of self-contractions is again due to the sum over non-zero images.

[9] Full self-contractions correspond to disconnected contributions.

3. Correlation functions in the low temperature phase

It was found in [1,2] that (2.1) has a first order phase transition at a temperature T_c in the $N = \infty$ limit. $\operatorname{Tr} U^n$ can be considered as order parameters of the phase transition. In the low temperature phase, one has

$$\langle \operatorname{Tr} U^n \rangle_U \approx N\delta_{n,0} + O(1/N), \tag{3.1}$$

while for $T > T_c$, $\operatorname{Tr} U^n, n \neq 0$, develop non-zero expectation values. It follows from (3.1) that in the low temperature phase, to leading order in $1/N$ expansion

$$\begin{aligned} &\langle \operatorname{Tr} U^{n_1} \operatorname{Tr} U^{n_2} \cdots \operatorname{Tr} U^{n_k} \rangle_U \\ &\quad \approx \langle \operatorname{Tr} U^{n_1} \rangle_U \langle \operatorname{Tr} U^{n_2} \rangle_U \cdots \langle \operatorname{Tr} U^{n_k} \rangle_U \\ &\quad \approx N^k \delta_{n_1,0} \cdots \delta_{n_k,0}, \end{aligned} \tag{3.2}$$

where in the second line we have used the standard factorization property at large N.

We now look at the implications of (3.2) on correlation functions. Applying (3.2) to (2.13) and (2.14), one finds

$$\begin{aligned} &\langle \operatorname{Tr} M^4 \rangle_\beta = 0 + O(1/N), \\ &\langle \operatorname{Tr} M^4(\tau) \operatorname{Tr} M^4(0) \rangle_\beta = 4 \sum_m G_s^4(\tau - m\beta) + O(1/N^2) = \sum_m \langle \operatorname{Tr} M^4(\tau - m\beta) \operatorname{Tr} M^4(0) \rangle_0. \end{aligned} \tag{3.3}$$

Note that the second term of (2.14) due to partial self-contractions vanishes and the finite temperature correlators are related to the zero temperature ones by adding the images for the whole operator.

The conclusion is not special to (3.3) and can be generalized to any correlation functions of single-trace (normal-ordered) operators in the large N limit. Now consider a generic n-point function for some single-trace operators. At zero temperature, the contribution of a typical contraction can be written in a form

$$\frac{1}{N^{n-2+2h}} \prod_{i<j=1}^{n} \prod_{p=1}^{I_{ij}} G_s^{(p)}(\tau_{ij}), \quad \tau_{ij} = \tau_i - \tau_j, \tag{3.4}$$

where i, j enumerate the vertices, I_{ij} is the number of propagators between vertices i, j, $G^{(p)}(\tau_{ij})$ is the pth propagator between vertices i and j, and h is the genus of the diagram. At finite temperature, one uses (2.12) to add images for each propagator and finds the contribution of the same diagram is given by

$$\frac{1}{N^I} \left(\prod_{i<j=1}^{n} \prod_{p=1}^{I_{ij}} \sum_{m_{ij}^{(p)}=-\infty}^{\infty} \right) \left(\prod_{i<j=1}^{n} \prod_{p=1}^{I_{ij}} G_s^{(p)}\big(\tau_{ij} - m_{ij}^{(p)}\beta\big) \right) \langle \operatorname{Tr} U^{s_1} \operatorname{Tr} U^{s_2} \cdots \rangle_U, \tag{3.5}$$

where $m_{ij}^{(p)}$ label the images of $G^{(p)}(\tau_{ij})$. When involving contractions of fermions, one replaces $G_s^{(p)}(\tau_{ij} - m_{ij}^{(p)}\beta)$ by $(-1)^{m_{ij}^{(p)}} G_f^{(p)}(\tau_{ij} - m_{ij}^{(p)}\beta)$ for the relevant p's. The powers $s_1, s_2, \ldots$ in the last factor of (3.5) can be found as follows. To each propagator in the diagram we assign a direction, which can be chosen arbitrarily and similarly an orientation can be chosen for each face. For each face A in the diagram, we have a factor $\operatorname{Tr} U^{s_A}$, with s_A given by

$$s_A = \sum_{\partial A} (\pm) m_{ij}^{(p)}, \quad A = 1, 2, \ldots, F, \tag{3.6}$$

where the sum ∂A is over the propagators bounding the face A and F denotes the number of faces of the diagram. In (3.6) the plus (minus) sign is taken if the direction of the corresponding propagator is the same as (opposite to) that of the face.

In the low temperature phase, due to Eq. (3.2) one has constraints on $m_{ij}^{(p)}$ associated with each face

$$s_A = \sum_{\partial A} (\pm) m_{ij}^{(p)} = 0, \quad A = 1, 2, \ldots, F. \tag{3.7}$$

Note that not all equations in (3.7) are independent. The sum of all the equations gives identically zero. One can also check that this is the only relation between the equations, thus giving rise to $F - 1$ constraints on $m_{ij}^{(p)}$'s. For a given diagram, the number I of propagators, the number F of faces and the number n of vertices[10] satisfy the relation $F + n - I = 2 - 2h$, where h is the genus of the diagram. It then follows that the number of independent sums over images is $K = I - (F - 1) = n - 1 + 2h$.

[10] Note that since we are considering the free theory, the number of vertices coincides with the number of operators in the correlation functions.

For *planar* diagrams, we have the number of independent sums over images given by

$$K = n - 1, \tag{3.8}$$

i.e. one less than the number of vertices. Also for any loop L in a planar diagram, one has[11]

$$\sum_{\partial L} \pm m_{ij}^{(p)} = 0, \tag{3.9}$$

where one sums over the image numbers associated with each propagator that the loop contains with the relative signs given by the relative directions of the propagators. Eq. (3.9) implies that all propagators connecting the same two vertices should have the same images, i.e. $m_{ij}^{(p)} = m_{ij}$ (up to a sign), which are independent of p. Furthermore, this also implies that one can write

$$m_{ij} = m_i - m_j. \tag{3.10}$$

In other words, the sums over images for each propagator reduce to the sums over images for each operator. We thus find that (3.5) becomes (for $h = 0$)

$$\frac{1}{N^{n-2}} \sum_{m_1,\ldots,m_n=-\infty}^{\infty} \prod_{i<j=1}^{n} \prod_{p=1}^{I_{ij}} G_s^{(p)}\big((\tau_i - m_i\beta) - (\tau_j - m_j\beta)\big). \tag{3.11}$$

In the above we considered contractions between different operators. As we commented at the end of Section 2, at finite temperature generically self-contractions do not vanish despite the normal ordering. One can readily convince himself using the arguments above that all *planar* self-contractions reduce to those at zero temperature and thus are canceled by normal ordering. For example, for one-point functions, $n = 1$, from (3.8) there is no sum of images. Thus the finite temperature results are the same as those of zero temperature, which are zero due to normal ordering.

When the operators contain fermions, we replace G_s by G_f in appropriate places and multiply (3.11) by a factor

$$\prod_{i<j=1}^{n} (-1)^{m_{ij} I_{ij}^{(f)}}, \tag{3.12}$$

where $I_{ij}^{(f)}$ is the number of fermionic propagators between vertices i, j. Using (3.10), we have

$$(-1)^{\sum_{i<j} m_{ij} I_{ij}^{(f)}} = (-1)^{\sum_{i,j} m_i I_{ij}^{(f)}} = (-1)^{\sum_i m_i \epsilon_i}, \tag{3.13}$$

where $\epsilon_i = 0$ (1) if the ith operator contains even (odd) number of fermions.

Since (3.11) and (3.13) do not depend on the specific structure of the diagram, we conclude that to leading order in $1/N$ expansion the full correlation function should satisfy

$$G_\beta(\tau_1, \ldots, \tau_n) = \sum_{m_1,m_2,\ldots,m_n=-\infty}^{\infty} (-1)^{m_i\epsilon_i} G_0(\tau_1 - m_1\beta, \ldots, \tau_n - m_n\beta). \tag{3.14}$$

Note that (3.14) applies also to the correlation functions in the four-dimensional theory since the harmonic expansion is independent of the temperature.

4. Including interactions

In the sections above we have focused on the free theory limit. We will now present arguments that (3.14) remains true order by order in the expansion over a small 't Hooft coupling λ. In addition to (2.1) the action also contains cubic and quartic terms which can be written as

$$S_{\text{int}} = N \int_0^\beta d\tau \left(\lambda^{\frac{1}{2}} \sum_\alpha b_\alpha \mathcal{L}_{3\alpha} + \lambda \sum_\alpha d_\alpha \mathcal{L}_{4\alpha} \right), \tag{4.1}$$

where $\mathcal{L}_{3\alpha}$ and $\mathcal{L}_{4\alpha}$ are single-trace operators made from ξ_a, M_a, v_a, c_a and their time derivatives. b_α and d_α are numerical constants arising from the harmonic expansion. Again the precise form of the action will not be important for our discussion below. The corrections to free theory correlation functions can be obtained by expanding the exponential of (4.1) in the path integral. For

[11] The following equation also applies to contractible loops in a non-planar diagram.

example, a typical term will have the form

$$\int_0^\beta d\tau_{n+1}\cdots\int_0^\beta d\tau_{n+k}\big\langle \mathcal{O}_1(\tau_1)\cdots\mathcal{O}_n(\tau_n)\mathcal{L}_{3\alpha_1}(\tau_{n+1})\cdots\mathcal{L}_{4\alpha_k}(\tau_{n+k})\big\rangle_{\beta,0}, \tag{4.2}$$

where to avoid causing confusion we used $\langle\cdots\rangle_{\beta,0}$ to denote the correlation function at zero coupling and finite temperature. Using (3.14), (4.2) can be written as[12]

$$\sum_{m_1,\dots,m_n}\int_{-\infty}^{\infty} d\tau_{n+1}\cdots\int_{-\infty}^{\infty} d\tau_{n+k}\big\langle \mathcal{O}_1(\tau_1-m_1\beta)\cdots\mathcal{O}_n(\tau_n-m_n\beta)\,\mathcal{L}_{3\alpha_1}(\tau_{n+1})\cdots\mathcal{L}_{4\alpha_k}(\tau_{n+k})\big\rangle_{0,0}, \tag{4.3}$$

where $\langle\cdots\rangle_{0,0}$ denotes correlation function at zero coupling and zero temperature and we have extended the integration ranges for $\tau_{n+1},\dots,\tau_{n+k}$ into $(-\infty,+\infty)$ using the sums over the images of these variables. Eq. (4.3) shows that (3.14) can be extended to include corrections in λ.

5. String theory argument and discussions

Eq. (3.14), while surprising from a gauge theory point of view, has a simple interpretation in terms of string theory dual. Suppose the gauge theory under consideration has a string theory dual described by some sigma-model M at zero temperature and some other sigma-model M' at finite temperature. The correlation functions in gauge theory to leading order in the $1/N$ expansion should be mapped to sphere amplitudes of some vertex operators in the M or M' theory. Eq. (3.14) follows immediately if we postulate that M' is identical to M except that the target space time coordinate is compactified to have a period β. To see this, it is more transparent to write (3.14) in momentum space. Fourier transforming τ_i to ω_i in (3.14) we find that

$$G_\beta(\omega_1,\dots,\omega_n)=G_0(\omega_1,\dots,\omega_n), \tag{5.1}$$

with all ω_i to be quantized in multiples of $\frac{2\pi}{\beta}$. Thus in momentum space to leading order in large N, finite temperature correlation functions are simply obtained by those at zero temperature by restricting to quantized momenta. From the string theory point of view, this is the familiar inheritance principle for tree-level amplitudes.

To use the above argument in the opposite direction, our result suggests that in the confined phase, M' should be given by M with time direction periodically identified. For $\mathcal{N}=4$ SYM, this gives further support that the thermal AdS description can be extrapolated to zero coupling.[13]

We conclude this Letter by some remarks:

(1) The inheritance principle (3.14) no longer holds beyond the planar level. For non-planar diagrams, it is possible to have images running along the non-contractible loops of the diagram. These may be interpreted in string theory side as winding modes for higher genus diagrams.
(2) One consequence of (1.1) is that for those correlation functions which are independent of the 't Hooft coupling in the large N limit at zero temperature the non-renormalization theorems remain true at finite temperature despite the fact that the conformal and supersymmetries are broken. For $\mathcal{N}=4$ SYM theory on S^3, in addition to the non-renormalizations of two and three-point functions of chiral operators [5] mentioned in the introduction, other examples include extremal correlation functions of chiral operators [19].
(3) In the high temperature (deconfined) phase, where $\mathrm{Tr}\,U^n$ generically are non-vanishing at leading order, (3.14) no longer holds, as can be seen from the example of (2.14). This suggests that in the deconfined phase M' should be more complicated. In the case of $\mathcal{N}=4$ SYM theory at strong coupling, the string dual is given by an AdS Schwarzschild black hole [14,15]. It could also be possible that the deconfined phase of the class of gauge theories we are considering describe some kind of stringy black holes [1,2].

We finally note that the argument of the Letter is but an example of how the inheritance property for the sphere amplitude in an orbifold string theory can have a non-trivial realization in the dual gauge theory. In particular it should also apply to cases where the cycle in question is spatial rather than temporal, like the cases discussed in [20].[14] It would also be interesting to understand what happens in the BTZ case.

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[12] Note $\mathcal{L}_{3\alpha}$ and $\mathcal{L}_{4\alpha}$ also contain ghosts c_a whose contractions are temperature independent and so will not affect our results in the last section.

[13] See also the discussion of [2] on the extrapolation of phase diagrams and [4] which discusses the relation between thermal AdS and free theory correlation functions.

[14] Pointed out to us by S. Minwalla.

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