On the spectra of hypertrees

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Abstract

In this paper, we study the spectral properties of a family of trees characterized by two main features: they are spanning subgraphs of the hypercube, and their vertices bear a high degree of (connectedness) hierarchy. Such structures are here called binary hypertrees and they can be recursively defined as the so-called hierarchical product of several complete graphs on two vertices.
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1. Introduction

Many networks associated to real-life complex systems have a hierarchical organization which is useful in their communication processes. This hierarchical structure leads very often to the existence of nodes with a relatively high degree (known as hubs) and to a low average distance in the graph. The characterization of graphs with these properties has therefore attracted much interest in the recent literature, see for example [8] and references therein.

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Some classical graphs also display a modular or hierarchical structure. Perhaps the best known example is the hypercube or $n$-cube which has useful communication properties: it is a minimum broadcast graph allowing optimal broadcasting and gossiping under standard communication models. However, it has a relatively large number of edges and many of them are not used in these communication schemes as the communication paths usually conform to a spanning tree [6]. It is of interest to have graph operations allowing the construction of these spanning trees. In [1], the authors introduce the hierarchical product of graphs which produces graphs with a strong (connectedness) hierarchy in their vertices. In fact, the obtained graphs turn out to be subgraphs of the Cartesian product of the corresponding factors. Some well-known properties of the cartesian product, such as a reduced mean distance and diameter, simple routing algorithms and some optimal communication protocols are inherited by the hierarchical product. When all the factors are the complete graph $K_2$, the resulting graph is a spanning tree of the hypercube. Another example of hierarchical product is the deterministic tree obtained by Jung et al. [7] which corresponds to the case when the factors are star graphs.

On the other hand, the study of the spectrum of a graph is relevant for estimating important structural properties, which provide information on the topological and communication properties of the corresponding network [3]. Among these properties, which usually are very hard to obtain by other methods, we have edge expansion and node-expansion, bisection width, diameter, maximum cut, connectivity, and partitions.

In this paper, we study the spectral properties of a family of trees, which we call binary hypertrees, characterized by two main features: they are spanning subgraphs of the hypercube, and their vertices bear a high degree of (connectedness) hierarchy. The binary hypertree of dimension $m$, $T_m$, is defined as the hierarchical product [1] of $m$ copies of $K_2$. Among other properties, the hypertrees are shown to be good examples of graphs with distinct eigenvalues. This fact has some structural consequences, such as the Abelianity of its automorphism group [9]. Indeed, we show that the automorphism group of $T_m$ is the symmetric group $S_2$. This, together with the high degree of hierarchy of our family of trees, results in a number of nice properties of their spectra.

More precisely, because of the recurrence relation satisfied by the characteristic polynomial of $T_m$, every eigenvalue of a hypertree of a given dimension yields two eigenvalues of the hypertree of the next dimension. Consequently, there is a strong relationship between the eigenvalues and the eigenvectors of the hypertrees of different dimensions.

Concerning the eigenvalues of $T_m$, we study their asymptotic behavior and how they are distributed with respect to some intervals defined by the eigenvalues of $T_{m'}$, for $m' < m$. Finally, by using the techniques in [4,5], we compute the eigenvectors of $T_m$. The result is based on obtaining a charge distribution on the vertices of $T_m$ from a charge distribution on the vertices of $T_{m-1}$.

**2. Definition and basic properties of the hypertree**

The hierarchical product of graphs is defined in [1] as follows.

**Definition 1.** Let $G_i = (V_i, E_i)$ be $N$ graphs with each vertex set $V_i, i = 1, 2, \ldots, N$, having a distinguished or root vertex, labeled 0. The hierarchical product $H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$ is the graph with vertices the $N$-tuples $x_N \cdots x_3 x_2 x_1, x_i \in V_i$, and edges defined by the adjacencies:
The binary hypertree of dimension \( m \), \( T_m \), is the hierarchical product of \( m \) copies of \( K_2 \). That is,

\[
T_m = K_2^m = K_2 \sqcap \cdots \sqcap K_2.
\]

An equivalent definition of the binary hypertree is as follows.

**Definition 2.** Given an integer \( m > 0 \), the binary hypertree of dimension \( m \), \( T_m \), is the rooted tree with vertex set \( \mathbb{Z}_2^m \) and the adjacencies defined by the following rule: two vertices are adjacent if and only if their labels differ in exactly one position and the maximum common suffix is either empty or it contains only zeroes.

The root of \( T_m \) is \( 0 = 00 \cdots 0 \). (We naturally take \( T_0 = K_1 \).)

As an example, Fig. 1 shows the hierarchical products of two, four and six copies of the complete graph \( K_2 \).

Since the graphs obtained from the hierarchical product are spanning subgraphs of the corresponding Cartesian products, we have that \( T_m \) is a spanning subgraph of the hypercube \( Q_m \). (Recall that \( Q_m \) has set of vertices \( \mathbb{Z}_2^m \), and two vertices are adjacent if and only if they differ in exactly one position.)

Every \( i = i_{m-1} \cdots i_1 i_0 \in \mathbb{Z}_2^m \) can be viewed as the expression in base two, with fixed length \( m \), of \( i = \sum_{k=0}^{m-1} i_k 2^k \) with \( i \in [0, 2^m - 1] \). In particular, we will consider the vertices of \( T_m \) labeled

\[
x_N \cdots x_3 x_2 x_1 \sim \begin{cases} 
  x_N \cdots x_3 x_2 y_1, & \text{if } y_1 \sim x_1 \text{ in } G_1, \\
  x_N \cdots x_3 y_2 x_1, & \text{if } y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 = 0, \\
  x_N \cdots y_3 x_2 x_1, & \text{if } y_3 \sim x_3 \text{ in } G_3 \text{ and } x_1 = x_2 = 0, \\
  \vdots & \\
  y_N \cdots x_3 x_2 x_1, & \text{if } y_N \sim x_N \text{ in } G_N \text{ and } x_1 = \cdots = x_{N-1} = 0.
\end{cases}
\]
[0, 1, \ldots, 2^m - 1]. In this sense, we use \( i = i \), provided that \( m \) is fixed. By convention, we take \( \mathbb{Z}_2^0 = \{0\} \).

Let us now recall some basic properties of the hypertree \( T_m \), which are drawn from a previous study of the authors [1] dealing with the hierarchical product:

- For every \( m \geq 0 \), the hypertree \( T_m \) has order \( n = 2^m \) and size \( 2^m - 1 \).
- \( T_m = T_{m-1} \cap K_2 \) (as the hierarchical product has the associative property).
- \( T_m^* = T_m - 0 = \bigcup_{k=0}^{m-1} T_k \).
- \( T_m - e \), where \( e \) is the edge \( \{0, 10 \cdots 0\} \), is isomorphic to the disjoint union of two copies of \( T_{m-1} \). In fact, such copies of \( T_{m-1} \) are the subgraphs induced by the sets of vertices \( V_0 = \{0w|w \in \mathbb{Z}_2^{m-1}\} \) and \( V_1 = \{1w|w \in \mathbb{Z}_2^{m-1}\} \).
- \( T_m \) has 2 vertices of degree \( m \) and \( 2^m - j \) vertices of degree \( j \), for \( 1 \leq j \leq m - 1 \). Namely,
  - \( \delta(0) = \delta(10^{m-1}0) = m \);
  - \( \delta(w10^{j-1}0) = j \), for every \( w \in \mathbb{Z}_2^{m-j} \), for \( 1 \leq j \leq m - 1 \).

With respect to the symmetries of the hypertrees, we have the following result.

**Proposition 3.** For every \( m \geq 1 \) the automorphism group of \( T_m \) is \( S_2 \).

**Proof.** Let \( \phi : T_m \rightarrow T_m \) be defined by \( \phi(0i) = 1i \) and \( \phi(1i) = 0i \) for every \( i \in \mathbb{Z}_2^{m-1} \). We claim that \( Aut(T_m) = \{Id, \phi\} \). Thus, we have to prove that \( \phi \) is the only non-trivial automorphism of \( T_m \).

Let us first show that \( \phi \) is a \( T_m \)-automorphism. From its definition, it is clear that \( \phi \) is an involutive bijection, that is, \( \phi(\phi(v)) = v \) for every vertex \( v \) of \( T_m \). Now, let \( u \) and \( v \) be two vertices of \( T_m \). We have to show that if \( u \sim v \), then \( \phi(u) \sim \phi(v) \). With this aim, assume without loss of generality that \( u \) starts by a zero. If \( u = 0 \) and \( v = 10^{m-1}0 \), then \( \phi(u) = v \) and \( \phi(v) = u \). Otherwise, \( v \) also starts by a zero. By symmetry, we can take \( u = 0w0 \cdots 0 \) and \( v = 0w10 \cdots 0 \).

In this case, \( \phi(u) = 1w0 \cdots 0 \) and \( \phi(v) = 1w10 \cdots 0 \), which are clearly adjacent in \( T_m \).

Finally, we prove that \( \phi \) is the only non-trivial \( T_m \)-automorphism by using induction on \( m \). For \( m = 1 \), \( T_1 = K_2 \), and \( Aut(K_2) = S_2 \). Let \( m > 1 \). As mentioned above, \( V_0 = \{0w|w \in \mathbb{Z}_2^{m-1}\} \) and \( V_1 = \{1w|w \in \mathbb{Z}_2^{m-1}\} \) induce two disjoint subgraphs of \( T_m \), which are isomorphic to \( T_{m-1} \). We denote these subgraphs by \( G_0 = G[V_0] \) and \( G_1 = G[V_1] \). Assume that \( Aut(T_{m-1}) = \{Id, \phi\} \), and let \( \gamma \) be a \( T_m \)-automorphism. Because of the degree sequence of \( T_m \), either

- \( \gamma(0) = 0 \) and \( \gamma(10^{m-1}0) = 10^{m-1}0 \), or
- \( \gamma(0) = 10^{m-1}0 \) and \( \gamma(10^{m-1}0) = 0 \).

In the first case, \( \gamma \) maps \( G_0 \) and \( G_1 \) onto themselves. Moreover, the induced automorphisms let the root fixed. Hence, by induction hypothesis, \( \gamma = Id \). In the second case, \( \gamma \) maps \( G_0 \) onto \( G_1 \) and \( G_1 \) onto \( G_0 \). For every \( i \in \mathbb{Z}_2^{m-1} \), we define \( \gamma_0 \) and \( \gamma_1 \) in the following way:

- if \( \gamma(0i) = 1v \), then \( \gamma_0(i) = v \);
- if \( \gamma(1i) = 0w \), then \( \gamma_1(i) = w \).
It can be easily checked that \( \gamma_0 \) and \( \gamma_1 \) are both \( T_{m-1} \)-automorphisms that let the root fixed. By induction hypothesis, \( \gamma_0 = \gamma_1 = \text{Id} \), which implies that \( \gamma = \phi \). This completes the proof. \( \square \)

3. Spectral properties

The adjacency matrix of \( T_m \) is

\[
A_m = \begin{pmatrix} A_{m-1} & I \\ I & 0 \end{pmatrix},
\]

where the dimensions of each block are \( 2^{m-1} \times 2^{m-1} \). Its characteristic polynomial is

\[
\phi_m(x) = \det(xI - A_m) = \det\left( \begin{pmatrix} xI & -I \\ -I & xI \end{pmatrix} \right).
\]

By using elementary column operations, we get

\[
\det\left( \begin{pmatrix} xI & -I \\ -I & xI \end{pmatrix} \right) = \det\left( \begin{pmatrix} x - \frac{1}{x} & 0 \\ 0 & xI - A_{m-1} \end{pmatrix} \right)
\]

and, thus, \( \phi_m(x) \) satisfies the recurrence

\[
\phi_m(x) = \det\left( \begin{pmatrix} x - \frac{1}{x} & 0 \\ 0 & xI - A_{m-1} \end{pmatrix} \right) \det xI = x^n \phi_{m-1}\left( x - \frac{1}{x} \right),
\]

(see, for instance, [10]). Recall that \( n = 2^m \).

3.1. Eigenvalues

Given a graph \( G \) on \( n \) vertices, we denote by \( \text{ev}_G \) the set of its eigenvalues in increasing order, say \( \text{ev}_G = \{ \lambda_0, \lambda_1, \ldots, \lambda_{2^n-1} \} \) (notice that this notation presumes that all eigenvalues are distinct). The recurrence equation satisfied by the characteristic polynomial of \( T_m \) gives rise to a number of spectral properties.

From (2) we have that, if \( \lambda_i \in \text{ev}_{T_{m-1}} \), then both solutions of \( x - \frac{1}{x} = \lambda_i \) are in \( \text{ev}_{T_m} \). This equation is equivalent to

\[
x^2 - \lambda_i x - 1 = 0.
\]

A useful notation for these solutions is \( \lambda_{0i} \) and \( \lambda_{1i} \) since, by using the functions

\[
f_0(\lambda) := \frac{1}{2} (\lambda - \sqrt{\lambda^2 + 4}), \quad f_1(\lambda) := \frac{1}{2} (\lambda + \sqrt{\lambda^2 + 4}),
\]

they can be computed as \( \lambda_{0i} = f_0(\lambda_i) \) and \( \lambda_{1i} = f_1(\lambda_i) \).

Moreover, by recursively applying these functions starting from \( \lambda_{0\emptyset} := 0 \) (here \( \emptyset \) represents the empty sequence), we obtain the whole set of eigenvalues \( \text{ev}_{T_m} = \{ \lambda_i | i \in \mathbb{Z}_2^m \} \) where, if \( i = i_{m-1}i_{m-2}\cdots i_0 \), then

\[
\lambda_i = (f_{i_{m-1}} \circ \cdots \circ f_{i_1} \circ f_{i_0})(0).
\]

This presentation provides a natural ordering, in such a way that the higher the number \( i \) (whose binary representation of length \( m \) is \( i \)), the larger the eigenvalue \( \lambda_i \) [1]. As a direct consequence, all the \( 2^m \) eigenvalues are different, a property which has some far-reaching consequences. In particular, any automorphism of \( T_m \) is involutive [9]. With this regard, we have shown in Proposition 3 that \( \text{Aut}(T_m) = S_2 \).
As $T_m$ is trivially bipartite, its eigenvalue mesh is symmetric [2] and hence, for every $i \in \mathbb{Z}_2^m$

$$\lambda_i = -\lambda_{\bar{i}},$$

where $\bar{i}(= n - i)$ denotes the ones’ complement of $i$ (that is, the bitwise NOT operation). For instance, Fig. 2 shows the spectra of the hypertrees $T_m$ for the cases $0 \leq m \leq 6$ and how every eigenvalue of $T_m$ gives rise to two eigenvalues of $T_m$.

It is shown in [1] that the asymptotic behaviors of the maximum eigenvalue (spectral radius), $\rho_m = \max_{0 \leq i \leq n-1} |\lambda_i| = \lambda_{111\ldots1}$, and the minimum positive eigenvalue, $\sigma_m = \min_{0 \leq i \leq n-1} |\lambda_i| = \lambda_{100\ldots0}$, of the hypertree $T_m$ are

$$\rho_m \sim \sqrt{2}^m, \quad \sigma_m \sim 1/\sqrt{2}^m. \quad (7)$$

Let us now consider the eigenvalues of $T_m$ in decreasing order $\lambda_m^{(0)} > \lambda_m^{(1)} > \lambda_m^{(2)} > \cdots$ That is, the first (0th) eigenvalue is $\lambda_m^{(0)} = \lambda_{111\ldots1} (= \rho_m)$, the second (1st) one is $\lambda_m^{(1)} = \lambda_{111\ldots10}$ and so on. In general, for any fixed integer $r \geq 0$, consider the $r$th largest eigenvalue $\lambda_m^{(r)}$ of $T_m$, with $m \geq k = \lceil \log_2(r + 1) \rceil$. Let $r$ be the binary representation of $r$. Then $k$ is the length of $r$ and $\lambda_m^{(r)} = \lambda_{111\ldots1\bar{r}}$. In this context, as in the case of the spectral radius, a natural question is to ask about the (asymptotic) behavior of the sequence $\{\lambda_m^{(r)} = \lambda_{111\ldots1\bar{r}}\}_{m \geq k}$.

**Proposition 4.** For every fixed $r \geq 1$, let $\gamma_m$ denote the $r$th largest eigenvalue of $T_m$, that is, $\lambda_m^{(r)} = \gamma_m$. Then, the asymptotic behavior of $\gamma_m$ is

$$\gamma_m \sim \sqrt{2}^m.$$ 

**Proof.** For $m \geq k$

$$\gamma_{m+1} = f_1(\gamma_m) = \frac{1}{2} \left( \gamma_m + \sqrt{\gamma_m^2 + 4} \right).$$

This function tends to a power law, $\gamma_m \sim \alpha m^\beta$ for $m \to \infty$, for some constants $\alpha$ and $\beta$. Indeed, if we put this expression of $\gamma_m$ in the equation, we get

$$\alpha(m + 1)^\beta \sim \frac{\alpha m^\beta + \sqrt{\alpha^2 m^{2\beta} + 4}}{2} \Rightarrow \alpha^2 (m + 1)^\beta [(m + 1)^\beta - m^\beta] \sim 1.$$
It is easy to check that \( \gamma_m = \sqrt{2m} \) is a solution when \( m \to \infty \) since
\[
2(m+1)^{\frac{1}{2}} [(m+1)^{\frac{1}{2}} - m^{\frac{1}{2}}] = \frac{2(m+1)^{\frac{1}{2}}}{(m+1)^{\frac{1}{2}} + m^{\frac{1}{2}}} \to 1.
\]
This solution corresponds to \( \alpha = \sqrt{2} \) and \( \beta = \frac{1}{2} \).

The following result is derived from the fact that \( \lambda_0i \) and \( \lambda_1i \) are the roots of the quadratic polynomial in (3), \( x^2 - \lambda_ix - 1 = 0 \).

**Lemma 5.** For every \( \alpha \in \mathbb{Z}_2 \) and \( i \in \mathbb{Z}_2^{m-1} \)
\[
\begin{align*}
\lambda_0i + \lambda_1i & = \lambda_i, \\
\lambda_0i\lambda_{1i} & = -1, \\
\lambda_{ai}\lambda_{ai} & = 1, \\
\lambda_{ai} & = \lambda_{ai} + \lambda_i.
\end{align*}
\]

**Proof.** The two equalities (8) and (9) come from (3). Equality (10) is a consequence of (9) and the “symmetry property” (6).

From (8)–(10) we get
\[
\begin{align*}
\lambda_i &= \lambda_0i - \lambda_1i, \\
\lambda_i &= \lambda_1i - \lambda_0i.
\end{align*}
\]

As an illustration of the equality (8), see Fig. 3 (to be compared with Fig. 2).

In particular, by (10) the maximum \( \rho_m = \lambda_{11\ldots1} \) and the minimum \( \sigma_m = \lambda_{10\ldots0} \) absolute values in ev \( T_m \) are inverse of each other, that is, \( \rho_m\sigma_m = 1 \), in agreement with (7).

Moreover, by applying recursively (11) we have that the sum of the first \( m \) minimum positive eigenvalues yields the spectral radius of \( T_m \):
\[
\rho_m = \lambda_{11\ldots1} = \lambda_{10\ldots0} + \lambda_{11\ldots1} = \cdots = \sigma_m + \sigma_{m-1} + \cdots + \sigma_1.
\]

![Fig. 3. The digraph of the eigenvalues of \( T_m \).](image)
A more particular example could be the following:

\[
\lambda_{10110} = \lambda_{11001} + \lambda_{0110} = \lambda_{11001} + \lambda_{0001} + \lambda_{110} = \lambda_{11001} + \lambda_{0001} + \lambda_{101} + \lambda_{11} + \lambda_0.
\]

Now we concentrate on the distribution of the eigenvalues of \( T_m \) with respect to some intervals defined by the eigenvalues of \( T_{m'} \), for \( m' < m \). Let us first prove that all the eigenvalues are distinct, even if they belong to hypertrees of different dimensions.

**Lemma 6.** For any pair of binary sequences \( i \in \mathbb{Z}_2^* \), \( j \in \mathbb{Z}_2^* \)

\[ i = j \iff \lambda_i = \lambda_j. \]  

**Proof.** The sufficiency is trivial by (5). With respect to the necessity, assume that we have \( \lambda_i = \lambda_j \) for \( i = i_r^{-1}i_r^{-2} \cdots i_1i_0 \) and \( j = j_s^{-1}j_{s-2} \cdots j_1j_0 \). Hence

\[
f_{i_{r-1}}(f_{i_{r-2}} \circ \cdots \circ f_{i_1} \circ f_{i_0}(0)) = f_{j_{s-1}}(f_{j_{s-2}} \circ \cdots \circ f_{j_1} \circ f_{j_0}(0)).
\]  

(13)

Then, as \( f_0(x) = \frac{1}{2}(x - \sqrt{x^2 + 4}) < 0 \) and \( f_1(x) = \frac{1}{2}(x + \sqrt{x^2 + 4}) > 0 \) for any \( x \), it must be \( i_{r-1} = j_{s-1} \) and hence, \( f_{i_{r-2}} \circ \cdots \circ f_{i_1} \circ f_{i_0}(0) = f_{j_{s-2}} \circ \cdots \circ f_{j_1} \circ f_{j_0}(0) \). Following the same reasoning we get \( i_{r-2} = j_{s-2} \) and \( f_{i_{r-3}} \circ \cdots \circ f_{i_1} \circ f_{i_0}(0) = f_{j_{s-3}} \circ \cdots \circ f_{j_1} \circ f_{j_0}(0) \), and so on.

Now, we only need to show that \( r = s \). Assume without loss of generality that \( r > s \). Then, repeating \( s \) times the above process we will have \( i_{r-1} = j_{s-1}, i_{r-2} = j_{s-2}, \ldots, i_{r-s} = j_0 \) and \( f_{i_{r-s-1}} \circ \cdots \circ f_{i_1} \circ f_{i_0}(0) = 0 \). This contradicts the fact that, for every \( x \) and \( i = 0, 1, f_i(x) \neq 0 \). This completes the proof. \( \Box \)

To consider the eigenvalues of the hypertrees of all possible dimensions, we need to consider \( \mathbb{Z}_2^* \), that is, the union of \( \mathbb{Z}_2^m \) for all \( m \) (or the set of all the sequences over \( \mathbb{Z}_2 \)).

**Definition 7.** Let \( i, j \in \mathbb{Z}_2^* \) and let \( w \) be their (possibly void) maximum common prefix. We say that \( i \prec_T j \) if and only if one of the following condition holds:

1. \( i = w0i'_1 \) and \( j = w1j'_1 \);
2. \( i = w \) and \( j = w1j'_1 \);
3. \( i = w0i'_1 \) and \( j = w \).

We say that \( i \preceq_T j \) if and only if \( i \prec_T j \) or \( i = j \).

Note that two different binary sequences could represent the same natural number. Hence, the relation \( \prec_T \), which turns out to be a total ordering of \( \mathbb{Z}_2^* \), is not equivalent to the natural order.

**Definition 8.** For any \( \ell \geq 0 \) and \( w \in \mathbb{Z}_2^\ell \), the \( w \)-translation, \( \tau_w \), is the function

\[
\tau_w = f_{w\ell-1} \circ \cdots \circ f_{w1} \circ f_{w0},
\]

where \( w = w\ell-1 \cdots w1w0 \).
The functions $f_0$ and $f_1$ are both monotone increasing. This implies that, for every $w$, $\tau_w$ is monotone increasing. On the other hand, it is worth mentioning that, since $\tau_w$ does not preserves distances, it is not a translation in a geometric sense.

In addition, we have the following lemma.

**Lemma 9.** For every $i = 0, 1$ and any arbitrary $x, y$

$$|f_i(x) - f_i(y)| < |x - y|.$$  \hfill (14)

**Proof.** For $i = 1$, we can assume without loss of generality that $x < y$. Then

$$f_1(x) < f_1(y) \quad \text{and} \quad f_1(y) - f_1(x) = \frac{1}{2}(y + \sqrt{y^2 + 4} - x - \sqrt{x^2 + 4}).$$

Now, we only need to notice that $\sqrt{y^2 + 4} - \sqrt{x^2 + 4} < y - x$, because

$$\left(\sqrt{y^2 + 4} - \sqrt{x^2 + 4}\right) \left(\sqrt{y^2 + 4} + \sqrt{x^2 + 4}\right) = (y - x)(y + x)$$

$$< (y - x) \left(\sqrt{y^2 + 4} + \sqrt{x^2 + 4}\right).$$

This yields $|f_1(x) - f_1(y)| < |x - y|$. By a similar reasoning we get $|f_0(x) - f_0(y)| < |x - y|$. \hfill \Box

We use the $w$-translations and Lemma 9 to prove the following result. (See Fig. 4.)

**Theorem 10.** The set of all the eigenvalues

$$\bigcup_{m=0}^{\infty} \text{ev } T_m = \{\lambda_i | i \in \mathbb{Z}_2^*\}$$

satisfies the following properties:

(a) For every $i, j \in \mathbb{Z}_2^*$, $i <_T j$ if and only if $\lambda_i < \lambda_j$;

(b) The interval determined by two consecutive eigenvalues of $T_m$ contains exactly $2^k$ consecutive eigenvalues of $T_{m+k}$, for $k \geq 1$;

(c) The two successions $\{\lambda_{w00...0}^k\}_{k>0}$ and $\{\lambda_{w011...1}^k\}_{k>0}$ have both limit $\lambda_w^*$.

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**Fig. 4.** The distribution of the eigenvalues of $T_m$, for $m = 0, \ldots, 9$. 

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Proof. The translation $\tau_w$ maps the set
\[ \{\lambda_i | i \in \mathbb{Z}_2^+ \} = \{\lambda_0, \lambda_0, \lambda_1, \lambda_{00}, \ldots\} \]
on to the set
\[ \{\lambda_{w_i} | i \in \mathbb{Z}_2^+ \} = \{\lambda_w, \lambda_{w0}, \lambda_{w1}, \lambda_{w00}, \ldots\} \]
preserving the order. This implies that in (a) we can assume that $i$ and $j$ have no common prefix. And this together with Lemma 9 implies that in (c) we can assume that $w = \emptyset$.

(a) If $i$ and $j$ have no common prefix, then $i <_T j$ if and only if one of the following conditions hold:
- $i = 0i1$ and $j = 1j1$;
- $i = \emptyset$ and $j = 1j1$;
- $i = 0i1$ and $j = \emptyset$.
The first condition is equivalent to $\lambda_i < 0$ and $\lambda_j > 0$. The second condition is equivalent to $\lambda_i = 0$ and $\lambda_j > 0$. Finally, the third condition is equivalent to $\lambda_i < 0$ and $\lambda_j = 0$. Hence, we have that if $i$ and $j$ have no common prefix, then $i <_T j$ if and only if $\lambda_i < \lambda_j$.

(b) Let $i \in \mathbb{Z}_2^n$ and $j = i + 1 \in \mathbb{Z}_2^n$. Using (a) we only have to prove that
\[ |\{w \in \mathbb{Z}_2^{m+k} | i <_T w <_T j\}| = 2^k. \]
By definition of $<_T$
\[ \{w \in \mathbb{Z}_2^{m+k} | i <_T w <_T j\} = \{ii_1 | i_1 \in \mathbb{Z}_2^{k-1}\} \cup \{jj_1 | j_1 \in \mathbb{Z}_2^{k-1}\}, \]
whose cardinality is $2^{k-1} + 2^{k-1} = 2^k$.

(c) The result is a direct consequence of (7) and the fact that $\lambda_{100\ldots0} = \sigma_k$, the minimum positive eigenvalue of $T_k$, and $\lambda_{011\ldots1} = -\sigma_k$. □

3.2. Eigenvectors

For any (di)graph, it is known that the components of its eigenvalues can be seen as charges on each vertex (see [4,5]). More precisely, suppose that $G = (V, A)$ is a digraph (a graph can be seen as a symmetric digraph where every edge $(i, j)$ is represented by two opposite arcs $(i, j), (j, i)$) with adjacency matrix $A$ and $\lambda$-eigenvector $v$. Then the charge of a vertex $i \in V$ is the corresponding entry $v_i$ of $v$, and the equation $Av = \lambda v$ means that
\[ \sum_{i \rightarrow j} v_j = \lambda v_i \quad \text{for every } i \in V. \] (15)
That is, each vertex “absorbs” the charges of its out-neighbors to get a final charge $\lambda$ times the one it had originally.

This approach allows us to compute the eigenvectors of $T_m$ from the eigenvectors of $T_{m-1}$, as the next result shows.

Proposition 11. Every $\lambda_i$-eigenvector $u_i$ of the hypertree $T_{m-1}$ gives rise to the following eigenvectors of $T_m$:
Fig. 5. The construction of the eigenvectors of \( T_m \) from the eigenvectors of \( T_{m-1} \).

\[
\mathbf{u}_{0i} = (\mathbf{u}_i, \alpha_{0i} \mathbf{u}_i)\top, \quad \mathbf{u}_{1i} = (\mathbf{u}_i, \alpha_{1i} \mathbf{u}_i)\top,
\]

where \( \alpha_{0i} = f_0(-\lambda_i) \) and \( \alpha_{1i} = f_1(-\lambda_i) \), with corresponding eigenvalues \( \lambda_{0i} = \alpha_{0i}^{-1} \) and \( \lambda_{1i} = \alpha_{1i}^{-1} \).

**Proof.** The basic idea of the proof is shown in Fig. 5. From the eigenvector \( \mathbf{u} \) of \( T_{m-1} \), we construct the eigenvector \( \mathbf{u}' = (\mathbf{u}, \alpha \mathbf{u})\top \) of \( T_m \), for some \( \alpha \) to be determined. Formally, let \( A \) be the adjacency matrix of the binary hypertree \( T_{m-1} \), such that \( A\mathbf{u} = \lambda_i \mathbf{u} \). Then, the eigenvalue \( \lambda'_i \) of \( T_m \) corresponding to the eigenvector \( \mathbf{u}' \) satisfy

\[
\begin{pmatrix} A & I \\ I & \alpha u_0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \alpha \mathbf{u} \end{pmatrix} = \begin{pmatrix} A\mathbf{u} + \alpha \mathbf{u} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} (\lambda_i + \alpha)\mathbf{u} \\ \mathbf{u} \end{pmatrix} = \lambda'_i \begin{pmatrix} \mathbf{u} \\ \alpha \mathbf{u} \end{pmatrix},
\]

whence

\[
\lambda_i + \alpha = \frac{1}{\alpha} \Rightarrow \alpha = \frac{1}{\alpha} = -\lambda_i.
\]

Notice that the last equation in (18) and the first one in (3) coincide except for the sign of \( \lambda_i \).

Consequently, the possible values of \( \alpha \), denoted \( \alpha_{0i} \) and \( \alpha_{1i} \), are obtained by applying, respectively, the functions \( f_0 \) and \( f_1 \) in (4) to \( -\lambda_i \). \( \Box \)

Observe that the above proof is based on obtaining a charge distribution in \( T_m \) from a charge distribution in \( T_{m-1} \). Thus, the first equation in (18) corresponds to the two ways (depending on the type of vertex considered) of computing the new eigenvalue \( \lambda'_i \) by using (15).

![Diagrams of hypertrees T_0, T_1, and T_2](image-url)
By way of example, Fig. 6 shows how to obtain the eigenvectors of the (binary) hypertree $T_m$ for $m = 0, 1, 2$.

References