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New Upper Bounds for the Football Pool Problem for 11 and 12 Matches

PATRIC R. J. ÖSTERGÅRD

Helsinki University of Technology, Digital Systems Laboratory, Otakaari 1, 02150 Espoo, Finland

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We consider the problem of minimizing the number of words in a code with the property that all words in the space F_3^n are within Hamming distance 1 from some codeword. This problem is called the football pool problem, since the words in such a code can be used in a football pool to guarantee that at least one forecast has at least $n-1$ correct results. In this note we show that for 11 and 12 matches, there are 9477 and 27702 words, respectively, having the aforementioned property. Simulated annealing has played an important role in the search for these words.

1. INTRODUCTION

Let F_q^n denote the set of all n-tuples $(x_1, x_2, ..., x_n)$ with $x_i \in F_q = \{0, 1, ..., q-1\}$. We define the Hamming distance $d(x, y)$ between two words, $x \in F_q^n$ and $y \in F_q^n$, as the number of coordinates in which the two n-tuples differ. A code $C \subseteq F_q^n$ is said to cover F_q^n with covering radius R if every element in the space is within Hamming distance R from at least one element in C and there is at least one word that is at distance at least R from every codeword. Such a code is called a $(q, n, |C|)R$ code. We further denote

$$K_q(n, R) = \min\{M \mid \text{there is a } (q, n, M)R \text{ code}\}.$$

The index is usually omitted in the binary case. Other notations that have been used include $\sigma(n, q) = K_q(n, 1)$ and $\sigma_n = K_3(n, 1)$. Few exact values for $K_q(n, R)$ are known, so efforts have been concentrated on improving lower and upper bounds. Upper bounds can be improved by explicitly constructing covering codes. In this note we restrict our attention to the *football pool problem*, i.e., the case $q = 3$, $R = 1$. During the years, this problem has been treated, e.g., in [2, 4, 5, 7, 11, 15, 18, 20].

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TABLE I

$K_3(n, 1)$, $n \leqslant 14$

n	$K_3(n, 1)$	μ
1	1	1
2	3	1.667
3	5	1.296
4	9	1
5	27	1.222
6	63–73	1.302
7	147–186	1.276
8	393–486	1.259
9	1048–1356	1.309
10	2814–3645	1.296
11	7767–10530	1.367
	9477*	1.230
12	21395–29889	1.406
	27702*	1.303
13	59049	1
14	165775–177147	1.074

The interesting values for $K_3(n, 1)$ are for $n \leqslant 14$, since in most countries the football pool coupons contain no more than 14 matches [4]. In Table I best known upper and lower bounds (or exact values if these coincide) are shown for $K_3(n, 1)$, $n \leqslant 14$ [15, 19]. Improvements obtained in this note are marked by a star. The densities of codes attaining the upper bounds are also given. The density of a $(3, n, M)1$ code is

$$\mu = (2n+1)\, M/3^n.$$

For perfect codes $\mu = 1$, otherwise $\mu > 1$. It is known that the density of an optimal ternary code with covering radius one approaches 1 as n approaches infinity [6]. Of course, that does not prove anything for small n, but Table I gives a good indication which attempts to improve upper bounds might succeed.

2. New Upper Bounds

Simulated annealing (SA) [1, 9, 10] was first used in the construction of covering codes by Wille [20], and has since then succesfully been used many times in the search for good covering codes [11, 14, 15]. SA works very well in the search for small codes. However, with an increasing number of codewords the processing time required grows very fast, and the search for $(3, 7, 186)1$ codes discussed in [11] required many attempts and an extremely slow cooling rate.

To overcome this problem, we use a method that was first considered by Kamps and van Lint [8] and later generalized by Blokhuis and Lam [2]. Carnielli [3] and van Lint, Jr. [12] independently generalized it to arbitrary covering radii.

Let $\mathbf{A} = (\mathbf{I}; \mathbf{M}) = (a_1, ..., a_n)$ be an $r \times n$ matrix where $\mathbf{I}$ is the $r \times r$ identity matrix and $\mathbf{M}$ is an $r \times (n-r)$ matrix with entries from F_q. For $s \in F_q^r$ we define

$$S_{A,R}(s) = \left\{ s + \sum_{j=1}^{n} \alpha_j a_j \mid \alpha_j \in F_q,\ |\{j \mid \alpha_j \neq 0\}| \leqslant R \right\}.$$

and say that s *R-covers* $S_{A,R}(s)$ *using* $\mathbf{A}$. Consequently, $\mathbf{A} = \mathbf{I}$ corresponds to covering in the traditional sense. A subset S of F_q^r *R-covers* F_q^r *using* $\mathbf{A}$ if

$$F_q^r = \bigcup_{s \in S} S_{A,R}(s).$$

THEOREM 1 (Carnielli [3, Theorem 2.1], van Lint, Jr. [12, Theorem 1.4.4]). *If S R-covers F_q^r using an $r \times n$ matrix $\mathbf{A} = (\mathbf{I}; \mathbf{M})$, then $W = \{w \in F_q^n \mid \mathbf{A}w \in S\}$ covers F_q^n with radius R. $|W| = |S| \cdot q^{n-r}$.*

Now SA can be used to find the codewords (S) and the matrix $\mathbf{A}$ (i.e., the $\mathbf{M}$ part), after first fixing the parameters of the spaces and $|S|$. The energy function is $E = |F_q^r \backslash \bigcup_{s \in S} S_{A,R}(s)|$ (cf. [11]). If the value of this function reaches 0, a solution is found. Since we now have to find both codewords in the set S, and the matrix $\mathbf{M}$, it is not immediately clear how to use SA in the search for a covering. In [11] the approach is to change both the codewords in S and the matrix $\mathbf{M}$ during the annealing process. This method has turned out to perform well only when there are very few words (less than about 10) in S.

A closer analysis of this approach reveals that the code we construct using this method can be seen as a union of $|S|$ translates of a linear code whose parity check matrix is $\mathbf{A}^T$. In the sequel we call two matrices nonequivalent if their transposes are parity check matrices of nonequivalent codes. If there is a small number of nonequivalent matrices $\mathbf{A}$, all these matrices can be considered in finding a covering set S. The number of nonequivalent matrices of size $r \times n$ over F_q having no repeated columns and no columns of zero is denoted by $\Phi_q(n, r)$ (cf. [17]). Tables on $\Phi_2(n, r)$ were first calculated by Slepian [16] (where $\bar{S}_{nr} = \Phi_2(n, r)$ is used). Some values are also given by Sloane in [17].

The task of producing $\Phi_q(n, r)$, $n > r$, nonequivalent matrices having no repeated columns and no columns of zero is much more difficult than just enumerating them. Only for the simplest case (i.e., $\mathbf{M}$ has only one column) an easy explicit description of the classes can be given. $\Phi_q(n, n-1) = n - 2$; $\mathbf{M}$ is then a single column vector with 2 to $n - 1$ nonzero positions.

In our approach we have speeded up the construction of nonequivalent matrices by only considering matrices whose transposes are parity check matrices for codes with different weight distributions (we consider the *Hamming weight enumerator* [13, p. 146]; i.e., the weight of a codeword is the number of nonzero components). Unfortunately we do not get all nonequivalent matrices, but as can be see from the results to be presented, the approach is quite promising.

Having computed the nonequivalent matrices **M** it is possible to use SA in trying to find a set S for all these possibilities. With tens of thousands of matrices this method is very time-consuming. A considerable speed-up can be achieved by performing the first annealing with a fast cooling rate, after which matrices **M** that lead to bad coverings can be discarded. Repetitions of this procedure with slower and slower cooling rates and fewer and fewer matrices **M** hopefully lead to a covering.

2.1. 11 *Matches*

The densities in Table I indicate that the upper bounds for 11 and 12 matches can be improved. To find suitable parameters, we maximize $a < 100$ (not to get too many codewords in S) such that $a3^k < b$, where k is an integer and b is the best known upper bound. For 11 matches this is fulfilled for $a = 43$, $k = 5$. Following the procedure described earlier we have been able find a record-breaking code, and it turned out that covering codes could be found with a 6×11 matrix **A** and $|S| = 39$. We were even able to find some symmetrics, which led to the following simplification of the results. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

THEOREM 2. *The set S consisting of the following 13 codewords 1-covers F_3^5 using* **A**:

01112	10101	20011
01121	11010	21100
01211	10220	20220
02111	12002	22002
02222.		

In the following the word cover is used in the meaning "covered by S using $\mathbf{A}$."

LEMMA 1. *For any* $x_1, x_2, x_3, x_4, x_5 \in F_3$ *the following holds.*

(a) $(0, x_2, x_3, x_4, x_5)$ *is covered iff* $(0, x_2, x_4, x_3, x_5)$ *is covered.*

(b) $(1, x_2, x_3, x_4, x_5)$ *is covered iff* $(2, x_2, x_4, x_3, x_5)$ *is covered.*

(c) $(x_1, x_2, x_3, x_4, x_5)$ *is covered iff* $(x_1, x_5, x_4, x_3, x_2)$ *is covered.*

(d) $(x_1, x_2, x_3, x_4, x_5)$ *is covered iff* $(x_1, x_4, x_5, x_2, x_3)$ *is covered.*

Proof. We can see that $(1, x_2, x_3, x_4, x_5)((0, x_2, x_3, x_4, x_5))$ is a word in S or a column (or its multiple) of $\mathbf{A}$ iff $(2, x_2, x_4, x_3, x_5)$ $((0, x_2, x_4, x_3, x_5))$ also is. Equivalences (a) and (b) are easily proved using this fact. We get (c) by noting that if $(x_1, x_2, x_3, x_4, x_5)$ is a word in S or a column (or its multiple) of $\mathbf{A}$, so is $(x_1, x_5, x_4, x_3, x_2)$. Finally, (d) follows from the fact that if $(x_1, x_2, x_3, x_4, x_5)$ is a word in S or a column (or its multiple) of $\mathbf{A}$, so is $(x_1, x_4, x_5, x_2, x_3)$. Especially note that $2 \cdot (0, 1, 2, 2, 1) = (0, 2, 1, 1, 2)$. ∎

Some symmetries have been proved in Lemma 1. These will help the presentation of the proof of Theorem 2.

Proof of Theorem 2. We show that $(x_1, x_2, x_3, x_4, x_5)$ is covered for all $x_1, x_2, x_3, x_4, x_5 \in F_3$. Lemma 1(b) makes it possible to exclude the case $x_1 = 2$ immediately. We first consider the case $x_1 = 0$. By using words in S that have a 0 in the first coordinate and the $\mathbf{I}$ part of $\mathbf{A}$ the following cases are proved (in the parentheses the number of zeros, ones and twos in the last four coordinates are given): (0, 0, 4), (0, 1, 3), (1, 0, 3), (0, 2, 2), (0, 3, 1), (1, 2, 1), and (1, 3, 0). By further considering all columns of $\mathbf{A}$ that have a zero in the first position the following distributions of the values are also settled: (3, 0, 1), (3, 1, 0), (0, 4, 0), and (4, 0, 0). Due to Lemmas 1(a), (c) and (d), the following results are enough to complete this case: $(0, 2, 2, 2, 2) + (0, 1, 2, 2, 1) = (0, 0, 1, 1, 0)$, $(2, 0, 0, 1, 1) + (1, 0, 0, 0, 0) = (0, 0, 0, 1, 1)$, $(2, 1, 1, 0, 0) + 2 \cdot (2, 1, 0, 1, 0) = (0, 0, 1, 2, 0)$, $(1, 2, 0, 0, 2) + (2, 1, 0, 1, 0) = (0, 0, 0, 1, 2)$, $(2, 0, 2, 2, 0) + (1, 0, 0, 0, 0) = (0, 0, 2, 2, 0)$, $(2, 0, 0, 1, 1) + (1, 0, 0, 1, 1) = (0, 0, 0, 2, 2)$, $(0, 1, 1, 1, 2) + 2 \cdot (0, 1, 2, 2, 1) = (0, 0, 2, 2, 1)$, and $(2, 0, 2, 2, 0) + 2 \cdot (2, 0, 1, 0, 1) = (0, 0, 1, 2, 2)$.

We now turn to the case $x_1 = 1$. By considering the $\mathbf{I}$ part of $\mathbf{A}$ (and all words in S) the following distributions of the values in the last four coordinates are proved: (0, 0, 4), (1, 0, 3), (0, 3, 1), (3, 0, 1), (1, 3, 0), and (3, 1, 0). The distributions (0, 2, 2), (2, 0, 2), and (2, 2, 0) are solved by the words in S that are of form $(1, x_2, x_3, x_4, x_5)$ and the column (0, 1, 1, 1, 1) of $\mathbf{A}$. To complete the case $x_1 = 1$, we can restrict ourselves to the following instances (cf. Lemmas 1(c) and (d)): $(2, 1, 1, 0, 0) + 2 \cdot (1, 1, 1, 0, 0) = (1, 0, 0, 0, 0)$, $(1, 0, 2, 2, 0) + (0, 1, 2, 2, 1) = (1, 1, 1, 1, 1)$, $(0, 1, 2, 1, 1) +$

$(1,0,0,1,1) = (1,1,2,2,2)$, $(2,2,0,0,2) + (2,1,0,1,0) = (1,0,0,1,2)$, $(1,0,2,2,0)+2\cdot(0,0,1,0,0)=(1,0,1,2,0)$, $(0,1,1,1,2)+2\cdot(2,1,0,1,0) =(1,0,1,0,2)$, $(2,0,0,1,1)+(2,0,1,0,1)=(1,0,1,1,2)$, $(1,0,1,0,1)+2\cdot(0,0,0,1,0)=(1,0,1,2,1)$, $(0,2,1,1,1)+(1,1,1,0,0)=(1,0,2,1,1)$, $(1,1,0,1,0)+2\cdot(0,1,2,2,1)=(1,0,1,2,2)$, $(2,0,2,2,0)+2\cdot(1,0,0,1,1) =(1,0,2,1,2)$, and $(1,0,2,2,0)+(0,0,0,0,1)=(1,0,2,2,1)$. This completes the proof. ∎

Theorem 1 applied to the result of Theorem 2 gives a new upper bound for the football pool problem for 11 matches.

COROLLARY 1. $K_3(11,1) \leqslant 9477$.

2.2. 12 *Matches*

Immediately, $K_3(12,1) \leqslant 3 \cdot K_3(11,1) \leqslant 28431$, which improves on the old bound in Table I. Attempts to find better codes with parameters $|S|=12$, $r=5$, $n=12$ have not succeeded. However, the aforementioned trivial upper bound can be proved by the fact that if S consists of the codewords in Theorem 1, then $S' = S \oplus \{0,1,2\}$ 1-covers F_3^6 using

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

We have been able to find a record-breaking code using this matrix.

THEOREM 3. $K_3(12,1) \leqslant 27702$.

Proof. Let $r=6$ and $n=12$. The following 38 codewords 1-cover F_3^6 using (1):

000102	012110	100210	111022	200210	212122
001010	012221	100211	111111	200211	220112
002120	020201	100222	112200	201121	221000
010121	021002	101212	120120	202202	221001
011120	022121	102021	121000	210100	221002
012020	022212	110212	121001	212011	222220
012101	100012.				

Theorem 1 gives that there are $38 \cdot 3^6 = 27702$ codewords that 1-cover F_3^{12}. Thus $K_3(12,1) \leqslant 27702$. ∎

3. Discussion

The densities for the new codes are 1.230 and 1.303 for $n=11$ and $n=12$, respectively. The code of length 11 is then certainly quite good, since it has the smallest known density for $q=3$, $R=1$, and $6 \leqslant n \leqslant 12$. Attempts have without success been made to improve other upper bounds for these values of n. The case $n=9$ is worth mentioning, many coverings that leave one single word in F_3^6 uncovered have been found for $|S|=50$ using, e.g., the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}.$$

This gives that $K_3(9, 1) \leqslant 51 \cdot 3^3 = 1377$, and that there are $50 \cdot 3^3 = 1350$ codewords that cover all but 27 words in F_3^9.

We conclude this note by calling attention to the fact that the increasing performance of computers will in the future make more thorough attacks on the football pool problem possible, which probably will lead to further improvements.

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References

1. E. Aarts and J. Korst, "Simulating Annealing and Boltzmann Machines: A Stochastic Approach to Combinatorial Optimization and Neural Computing," Wiley, Chichester, 1989.
2. A. Blokhuis and C. W. H. Lam, More coverings by rook domains, *J. Combin. Theory Ser. A* **36** (1984), 240–244.
3. W. A. Carnielli, Hyper-rook domain inequalities, *Stud. Appl. Math.* **82** (1990), 59–69.
4. H. Fernandes and E. Rechtschaffen, The football pool problem for 7 and 8 matches, *J. Combin. Theory Ser. A* **35** (1983), 109–114.
5. H. Hämäläinen and S. Rankinen, Upper bounds for football pool problems and mixed covering codes, *J. Combin. Theory Ser. A* **56** (1991), 84–95.
6. G. A. Kabatyanskii and V. I. Panchenko, Unit sphere packings and coverings of the Hamming space, *Problemy Peredachi Informatsii* **24** (1988), 3–16. [in Russian]; English translation in *Probl. Inform. Transmission* **24** (1988), 261–272.

7. H. J. L. Kamps and J. H. van Lint, The football pool problem for 5 matches, *J. Combin. Theory* **3** (1967), 315–325.
8. H. J. L. Kamps and J. H. van Lint, A covering problem, *in* "Colloq. Math. Soc. János Bolyai; Hung. Combin. Theory and Appl.," pp. 679–685. Balantonfüred, Hungary, 1969.
9. S. Kirkpatrick, C. D. Gelatt, Jr., and M. P. Vecchi, Optimization by simulated annealing, *Science* **220** (1983), 671–680.
10. P. J. M. van Laarhoven and E. H. L. Aarts, "Simulated Annealing: Theory and Applications," Reidel, Dordrecht, 1987.
11. P. J. M. van Laarhoven, E. H. L. Aarts, J. H. van Lint, and L. T. Wille, New upper bounds for the football pool problem for 6, 7 and 8 matches, *J. Combin. Theory Ser. A* **52** (1989), 304–312.
12. J. H. van Lint, Jr., "Covering Radius Problems," M.Sc. thesis, Eindhoven University of Technology, The Netherlands, June 1988.
13. F. J. MacWilliams and N. J. A. Sloane, "The Theory of Error-Correcting Codes," North-Holland, Amsterdam, 1977.
14. P. R. J. Östergård, A new binary code of length 10 and covering radius 1, *IEEE Trans. Inform. Theory* **37** (1991), 179–180.
15. P. R. J. Östergård, Upper bounds for q-ary covering codes, *IEEE Trans. Inform. Theory* **37** (1991), 660–664; **37** (1991), 1738.
16. D. Slepian, Some further theory of group codes, *Bell System Tech. J.* **39** (1960), 1219–1252.
17. N. J. A. Sloane, A new approach to the covering radius of codes, *J. Combin. Theory Ser. A* **42** (1986), 61–86.
18. E. W. Weber, On the football pool problem for 6 matches: A new upper bound, *J. Combin. Theory Ser. A* **35** (1983), 106–108.
19. G. J. M. van Wee, Some new lower bounds for binary and ternary covering codes, *IEEE Trans. Inform. Theory* **39** (1993), 1422–1424.
20. L. T. Wille, The football pool problem for 6 matches: A new upper bound obtained by simulated annealing, *J. Combin. Theory Ser. A* **45** (1987), 171–177.