An Inequality for the Spectral Radius of an Interval Matrix

Mau-hsiang Shih* and Yung-yih Lur
Department of Mathematics
Chung Yuan University
Chung-Li, Taiwan 320

and

Chin-tzong Pang
Department of Information Management
Yuan-Ze Institute of Technology
Nei-Li, Taiwan 320

Submitted by Ludwig Elsner

ABSTRACT

For an \( n \times n \) interval matrix \( \mathcal{A} = (A_{ij}) \), we say that \( \mathcal{A} \) is majorized by the point matrix \( \mathcal{A} = (a_{ij}) \) if \( a_{ij} = |A_{ij}| \) when the \( j \)th column of \( \mathcal{A} \) has the property that there exists a power \( \mathcal{A}^m \) containing in the same \( j \)th column at least one interval not degenerated to a point interval, and \( a_{ij} = A_{ij} \) otherwise. Denoting the generalized spectral radius (in the sense of Daubechies and Lagarias) of \( \mathcal{A} \) by \( \rho(\mathcal{A}) \), and the usual spectral radius of \( \mathcal{A} \) by \( \rho(\mathcal{A}) \), it is proved that if \( \mathcal{A} \) is majorized by \( \mathcal{A} \) then \( \rho(\mathcal{A}) \leq \rho(\mathcal{A}) \). This inequality sheds light on the asymptotic stability theory of discrete-time linear interval systems. © 1998 Elsevier Science Inc.

1. INTRODUCTION

Let \( \mathcal{A} = (A_{ij}) \) be an \( n \times n \) (real) interval matrix, that is, each \( A_{ij} \) is a real compact interval \([a_{ij}, \bar{a}_{ij}]\), where \( a_{ij} \leq \bar{a}_{ij} \). As usual, \( \mathcal{A} \) is also considered

*This work was supported in part by National Science Council of the Republic of China.
E-mail: mhshih@poincare.cycu.edu.tw
as the bounded set of matrices

$$\mathcal{A} = \{ A = (a_{ij}) : a_{ij} \leq a_{ij} \leq \bar{a}_{ij}, \ i, j = 1, \ldots, n \}. $$

For $\mathcal{A} = (A_{ij}), \mathcal{B} = (B_{ij})$, the interval-matrix sum and product are formally defined by

$$\mathcal{A} + \mathcal{B} = (A_{ij} + B_{ij}),$$

$$\mathcal{A} \cdot \mathcal{B} = \left( \sum_s A_{is} B_{sj} \right),$$

respectively, where the binary operations on the set of intervals have their usual meanings (see Alefeld and Herzberger [1]). The power $\mathcal{A}^k$ are defined by

$$\mathcal{A}^k = \mathcal{A}^{k-1} \cdot \mathcal{A}, \quad k = 2, 3, \ldots$$

Let $\Sigma$ be a bounded set of $n \times n$ real matrices. For $m \geq 1$, $\Sigma_m$ is the set of all products of matrices in $\Sigma$ of length $m$, that is,

$$\Sigma_m = \{ A_1 A_2 \cdots A_m : A_i \in \Sigma, \ i = 1, \ldots, m \}.$$

Denoting by $\rho(A)$ the spectral radius and by $\| A \|$ an operator norm of a matrix, the joint spectral radius of $\Sigma$ (see Rota and Strang [9]), $\hat{\rho}(\Sigma)$, is defined by

$$\hat{\rho}(\Sigma) = \limsup_{m \to \infty} \left( \sup_{A \in \Sigma_m} \| A \| \right)^{1/m}.$$

The generalized spectral radius of $\Sigma$ (see Daubechies and Lagarias [3]), $\rho(\Sigma)$, is defined by

$$\rho(\Sigma) = \limsup_{m \to \infty} \left( \sup_{A \in \Sigma_m} \rho(A) \right)^{1/m}.$$

The generalized Gelfand spectral radius formula, that is, $\rho(\Sigma) = \hat{\rho}(\Sigma)$, was conjectured by Daubechies and Lagarias [3] and proved by Berger and Wang [2] using tools from ring theory and then by Elsner [4] using analytic-geomet-
ric tools (see also Shih, Wu, and Pang [10] for another proof by a dynamics method).

For a real interval \([a, \bar{a}]\), the width \(d([a, \bar{a}])\) and the absolute value \(|[a, \bar{a}]|\) are defined by

\[
d[a, \bar{a}] = \bar{a} - a, \quad |[a, \bar{a}]| = \max\{|a|, |\bar{a}|\},
\]

respectively. For an \(n \times n\) interval matrix \(A\), we define the real matrices

\[
d(A) = \left( d(A_{ij}) \right) \text{ and } |A| = \left( |A_{ij}| \right).
\]

For \(j = 1, \ldots, n\), we define

\[
M_j(A) = \{A^m : \text{the } j\text{th column of } A^m \text{ (} m \geq 1 \text{) contains at least one interval not degenerate to a point interval}\}.
\]

This notion was introduced by Mayer [6] in his discussion of the convergence of powers \(\{A^k\}\) to the null matrix. The condition \(M_j(A) \neq \emptyset\) can be examined in a simple way by graph-theoretic notions (Mayer [6]). Following Mayer, the directed graph \(G(A)\) of an \(n \times n\) interval matrix \(A\) is the directed graph of the real matrix \(|A|\). Mayer observed that \(M_j(A) \neq \emptyset\) if and only if there exists a directed path \(P_i, P_{i_1}, P_{i_2}, \ldots, P_{i_l}, P_j\) in \(G(A)\) which ends at \(P_j\) and which contains any two neighboring nodes \(P_a, P_\beta\) whose corresponding matrix entry \(A_{a\beta}\) is not a point interval [that is, \(d(A_{a\beta}) > 0\)].

We introduce the following:

**DEFINITION 1.** Let \(A = (A_{ij})\) be an \(n \times n\) interval matrix. We say that \(A\) is majorized by a point matrix \(\tilde{A} = (\tilde{a}_{ij})\) if

\[
a_{ij} = \begin{cases} |A_{ij}| & \text{if } M_j(A) \neq \emptyset, \\ A_{ij} & \text{if } M_j(A) = \emptyset. \end{cases}
\]
2. RESULT

We shall establish the following:

**THEOREM 1.** Let $\mathcal{A}$ be an $n \times n$ interval matrix which is majorized by $\mathcal{A}$. Then the inequality

$$
\rho(\mathcal{A}) \leq \rho(\mathcal{A})
$$

holds.

**Proof.** Divide the numbers $1, \ldots, n$ into two classes:

$$
j \in I_1 \text{ if } M_j(\mathcal{A}) \neq \emptyset, \quad j \in I_2 \text{ if } M_j(\mathcal{A}) = \emptyset.
$$

We distinguish three cases, according as $I_1 = \emptyset$, or $I_2 = \emptyset$, or $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$.

**Case 1.** $I_1 = \emptyset$. Then $\mathcal{A}$ is a point matrix and $\mathcal{A} = \tilde{\mathcal{A}}$; thus $\rho(\mathcal{A}) = \rho(\tilde{\mathcal{A}})$.

**Case 2.** $I_2 = \emptyset$. Then $\tilde{\mathcal{A}} = |\mathcal{A}|$. Let $A_1, \ldots, A_k \in \mathcal{A}$ be given; then

$$
|A_1 A_2 \cdots A_k| \leq |A_1| \cdots |A_k| \leq |\mathcal{A}|^k,
$$

so that by a comparison result for spectral radii (see, e.g., Varga [11, p. 47])

$$
\rho(A_1 \cdots A_k) \leq \rho(|\mathcal{A}|)^k.
$$

Therefore

$$
\rho(\mathcal{A}) \leq \rho(|\mathcal{A}|) = \rho(\tilde{\mathcal{A}}).
$$

**Case 3.** $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$. Let the cardinality of $I_2$ be $k$, $1 \leq k \leq n - 1$. Choose a permutation matrix $P$ so that

$$
M_j(P^{-1}\mathcal{A}P) = \emptyset \quad \text{for } j = 1, \ldots, k
$$

and

$$
M_j(P^{-1}\mathcal{A}P) \neq \emptyset \quad \text{for } j = k + 1, \ldots, n.
$$
We now demonstrate that each $rs$ entry of $P^{-1}\mathcal{A}P$ satisfies

$$
(P^{-1}\mathcal{A}P)_{rs} = 0 \quad \text{for} \quad k + 1 \leq r \leq n \text{ and } 1 \leq s \leq k. \quad (3)
$$

To this end suppose, by contradiction, that

$$
(P^{-1}\mathcal{A}P)_{rs} \neq 0 \quad \text{for some } k + 1 \leq r \leq n \text{ and some } 1 \leq s \leq k.
$$

Since $M_r(P^{-1}\mathcal{A}P) \neq \emptyset$, by the definition of $M_r(\cdot)$ there exists a positive integer $m$ such that the width

$$
d(P^{-1}\mathcal{A}^mP)_{ar} \neq 0 \quad \text{for some } 1 \leq a \leq n.
$$

Hence

$$
d(P^{-1}\mathcal{A}^{m+1}P)_{as} = d[(P^{-1}\mathcal{A}^mP)(P^{-1}\mathcal{A}P)]_{as} \neq 0,
$$

contradicting (2), and with this contradiction the assertion (3) follows. Therefore, according to (3), $P^{-1}\mathcal{A}P$ is represented in the block interval matrix form

$$
P^{-1}\mathcal{A}P = \begin{pmatrix}
A & \mathcal{B} \\
0 & \mathcal{C}
\end{pmatrix}, \quad (4)
$$

where $A$ is a point matrix. Let $P^{-1}\mathcal{A}P$ be majorized by $P^{-1}\mathcal{A}P$; then

$$
P^{-1}\mathcal{A}P = P^{-1}\mathcal{A}P = \begin{pmatrix}
A & |\mathcal{B}| \\
0 & |\mathcal{C}|
\end{pmatrix}. \quad (5)
$$

Therefore, it follows from (4) that

$$
\rho(\mathcal{A}) = \rho(P^{-1}\mathcal{A}P) = \max\{\rho(A), \rho(\mathcal{C})\}. \quad (6)
$$

Also, we conclude from (5) that

$$
\rho(\mathcal{A}) = \rho(P^{-1}\mathcal{A}P) = \max\{\rho(A), \rho(|\mathcal{C}|)\}. \quad (7)
$$

As the same discussion in case 2 yields $\rho(\mathcal{C}) \leq \rho(|\mathcal{C}|)$, the desired inequality (1) follows from (6) and (7). This completes the proof. \[\square\]
3. DISCUSSION

Let us explain why we consider this inequality to be worth presenting. For an $n \times n$ interval matrix, let us recall that a matrix $A \in \mathcal{A}$ is called a vertex matrix of $\mathcal{A}$ if for each pair $i, j \in \{1, \ldots, n\}$, either $a_{ij} = a_{jj}$ or $a_{ij} = a_{jj}$ holds. The set of all vertex matrices of $\mathcal{A}$ is denoted by $\text{ext} \mathcal{A}$. The Krein-Milman theorem (see, e.g., Heuser [5, p. 374]) yields the following identity.

**Lemma.**

$$\rho(\mathcal{A}) = \rho(\text{ext} \mathcal{A}).$$  \hspace{1cm} (8)

**Proof.** We first prove that

$$\rho(\mathcal{A}) < 1 \iff \rho(\text{ext} \mathcal{A}) < 1.$$  \hspace{1cm} (9)

The implication $\Rightarrow$ is immediate because $\rho(\text{ext} \mathcal{A}) \leq \rho(\mathcal{A})$. Assume now that $\rho(\text{ext} \mathcal{A}) < 1$. By the generalized Gelfand spectral-radius formula, $\hat{\rho}(\text{ext} \mathcal{A}) < 1$. Let

$${\text{ext} \mathcal{A}} = \{ A_1, \ldots, A_m \}.$$

According to Rota and Strang's theorem [9] (see also Elsner [4, Lemma 1])

$$\hat{\rho}(\text{ext} \mathcal{A}) = \inf_{\nu \text{ operator norm}} \sup_{B \in \text{ext} \mathcal{A}} \nu(B),$$

there exists an operator norm $\| \cdot \|$ and a constant $\alpha > 0$ such that

$$\| A_i \| \leq \alpha < 1 \quad \text{for} \quad i = 1, \ldots, m.$$

Since $\mathcal{A}$ is compact convex, according to the Krein-Milman theorem $\mathcal{A}$ is the convex hull of $\text{ext} \mathcal{A}$. Then for each $A \in \mathcal{A}$,

$$A = \sum_{i=1}^{m} \lambda_i A_i \quad \text{for some} \quad \lambda_i \geq 0 \quad \text{with} \quad \sum_{i=1}^{m} \lambda_i = 1.$$
Therefore
\[ \| A \| \leq \sum_{i=1}^{m} \lambda_i \| A_i \| \leq \alpha < 1 \quad \text{for all} \quad A \in \mathcal{A}, \]
and hence \( \rho(\mathcal{A}) < 1 \), so that \( \rho(\mathcal{A}) < 1 \). Since (9) shows that
\[ \rho(\mathcal{A}) < \gamma \iff \rho(\text{ext } \mathcal{A}) < \gamma, \]
the identity (8) is immediate.

Though the computation of \( \rho(\mathcal{A}) \) reduces, by the lemma, to a simpler computation of \( \rho(\text{ext } \mathcal{A}) \), we do know that even the computation of \( \rho(\text{ext } \mathcal{A}) \) is generally difficult. Since the computation of \( \rho(\mathcal{A}) \) is much simpler, the inequality (1) gives a simple sufficient condition to check \( \rho(\mathcal{A}) < 1 \). The following example illustrates this.

**Example 1.** Let
\[
\mathcal{A} = \begin{pmatrix} \frac{3}{5} & \left[ -\frac{39}{50}, 0 \right] \\ 0, \frac{1}{5} & \left[ -\frac{3}{5}, \frac{3}{5} \right] \end{pmatrix}.
\]
There are eight testing matrices of \( \text{ext } \mathcal{A} \); thus the computation of \( \rho(\text{ext } \mathcal{A}) \) is quite cumbersome. However,
\[
\tilde{\mathcal{A}} = \begin{pmatrix} \frac{3}{5} & \frac{39}{50} \\ \frac{1}{5} & \frac{3}{5} \end{pmatrix}
\]
and \( \rho(\tilde{\mathcal{A}}) = (3 + \sqrt{3.9})/5 < 1 \), so that \( \mathcal{A} \) is asymptotically stable (that is, there exists an operator norm \( \| \cdot \| \) and a positive number \( \alpha \) such that \( \| A \| \leq \alpha < 1 \) for all \( A \in \mathcal{A} \)) in view of the inequality (1). Notice that
\[
A = \begin{pmatrix} \frac{3}{5} & -\frac{39}{50} \\ 0 & -\frac{3}{5} \end{pmatrix} \in \text{ext } \mathcal{A},
\]
and \( \| A \|_1 > 1, \| A \|_2 > 1, \) and \( \| A \|_\infty > 1 \). Thus the asymptotic stability of \( \mathcal{A} \) cannot be concluded from computation of the \( l_1, l_2, \) and \( l_\infty \) norms of the matrices in \( \text{ext } \mathcal{A} \).
In view of the above discussion, we see that the inequality (1) is useful in the study of the discrete-time linear interval systems.

We now consider the discrete time-varying linear system

\[ x(k + 1) = A(k)x(k), \quad k = 0, 1, \ldots, \] (10)

where \( A(k) \in \mathcal{A} = \{(a_{ij}): a_{ij} \leq a_{ij} \leq \tilde{a}_{ij}, i, j = 1, \ldots, n\} \).

If \( \rho(\mathcal{A}) < 1 \), by the inequality (1) and the lemma, \( \rho(\text{ext } \mathcal{A}) < 1 \). Assume now that \( \rho(\text{ext } \mathcal{A}) < 1 \). Then from the proof of the lemma, there exists an operator norm \( \| \cdot \| \) and a constant \( 0 < \alpha < 1 \) such that

\[ \| A \| \leq \alpha \quad \text{for all} \quad A \in \mathcal{A}. \]

Let \( x(0) \in \mathbb{R}^n \) be given. Then

\[ \| x(k + 1) \| = \| A(k) \cdots A(0)x(0) \| \leq \alpha^{k+1}\| x(0) \| \to 0 \quad \text{as} \quad k \to \infty. \]

Therefore the system (10) is globally asymptotically stable.

We summarize as follows:

**Theorem 2.** If \( \rho(\text{ext } \mathcal{A}) < 1 \) or \( \rho(\mathcal{A}) < 1 \), then the system (10) is globally asymptotically stable.

The matrix product condition given in Myszkorowski [7] is more restrictive than \( \rho(\mathcal{A}) < 1 \), and \( \rho(\tilde{\mathcal{A}}) < 1 \) may be reached through a simpler criterion \( \rho(\widetilde{\mathcal{A}}) < 1 \). Another approach to the analysis of asymptotic stability of discrete-time linear interval systems can be found in Pérez, Dvcampo, and Abdallah [8].

4. **EQUALITIES**

We remark here that \( \rho(\mathcal{A}) = \rho(\tilde{\mathcal{A}}) \) does not hold in general, as the following example shows.
Example 2. Let

\[ A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & \left[0, \frac{1}{2}\right] & 0 \end{pmatrix}, \]

Then

\[ A^2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}. \]

For

\[ A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & a & 0 \end{pmatrix} \in A, \quad \text{where } 0 \leq a \leq \frac{1}{2}, \]

we have

\[ A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & -a & 0 \end{pmatrix} \]

and

\[ A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Accordingly, \( \rho(A) = 0 < \rho(A') = 2. \)

We note also that each of the following conditions is sufficient for \( \rho(A) = \rho(A'). \)

(i) There exists a permutation matrix \( P \) such that \( P^{-1}A'P \) is upper triangular (or lower triangular).

(ii) There exists \( A \in A \) such that \( |A| = A. \)

(iii) There exists a diagonal matrix \( P \) with diagonal entries 1 or \( -1 \) such that \( A' \in P^{-1}A'P. \)

It may be of interest to find necessary and sufficient conditions such that the equality \( \rho(A) = \rho(A') \) holds.
REFERENCES


*Received 5 December 1996, final manuscript accepted 24 April 1997*