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ORIGINAL ARTICLE

# Shape preserving rational cubic spline for positive and convex data

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**Abstract** In this paper, the problem of shape preserving $C^2$ rational cubic spline has been proposed. The shapes of the positive and convex data are under discussion of the proposed spline solutions. A $C^2$ rational cubic function with two families of free parameters has been introduced to attain the $C^2$ positive curves from positive data and $C^2$ convex curves from convex data. Simple data dependent constraints are derived on free parameters in the description of rational cubic function to obtain the desired shape of the data. The rational cubic schemes have unique representations.



## 1. Introduction

Shape preservation of a given data is an important topic in the field of data visualization. In data visualization techniques researchers convert any information into graphical views. These graphical views have great importance in many fields including engineering, military, transport, advertising, medicine, education, art, etc. Data that is used for the visualization has some hidden properties (such as positive or convex). It is observed, in Figs. 1 and 3, that a normal cubic spline interpolates the data points but may not preserve the inherent features of the positive data. Similarly, Figs. 5 and 7 reflect that the convexity is not preserved by an ordinary cubic spline while the data is convex. This is not desired in scientific computing.

In recent years, some work [1–10] has been published on shape preservation. Asim and Brodlie [1] discussed the problem of drawing a positive curve through positive data set. They used piecewise cubic Hermite interpolation to fit a positive curve. In any interval where the positivity is lost they added extra knots to cubic Hermite interpolant to obtain the desired curve. In [2], Brodlie and Butt discussed the problem of shape preservation of convex data and in [3]; Butt and Brodile discussed the problem of shape preservation of positive data. In

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**Table 1** A positive data.

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 0.0 | 1.0 | 1.7 | 1.8 |
| $y_i$ | 0.25 | 1.0 | 11.10 | 25 |

[2,3] they used the cubic Hermite interpolation to preserve the shape of convex and positive data. The algorithms developed in [2,3] work by inserting one or two extra knots, wherever necessary, to preserve the shape of the data. Duan et al. [4] discussed rational interpolation based on function values and also discussed constrained control of the interpolanting curves. They obtained conditions on function values for constraining the interpolating curves to lie above, below or between the given straight lines. In [4] the authors assumed suitable values of parameters to obtain $C^2$ continuous curve and the method works for equally spaced data. Fangxun et al. [5] developed methods for value control, inflection point control and convexity control with rational cubic spline. To control the shape of the curve they assumed certain value of function and obtained conditions at that value and to control derivative of the interpolating curve at some points. They assumed derivative value according to desire and imposed condition at that value. Degree of smoothness they achieved was $C^1$. Fiorot and Tabka [6] used $C^2$ cubic polynomial spline to preserve the shape of convex or monotone data. In [6], the values of derivative parameters are obtained by using three systems of linear equations. In [7], Floater proved that total positivity and rational convexity preservation are equivalent. Gal [8] divided the book in four chapters: firstly he talked about the shape-preserving approximation and interpolation of real functions of one real variable by real polynomials, secondly he discussed the shape-preserving approximation of real functions of several real variables by multivariate real polynomials; thirdly he discussed shape-preserving approximation of analytic functions of one complex variable by complex polynomials in the unit disk, and at the last shape-preserving approximation of analytic functions of several complex variables on the unit ball or the unit polydisk by polynomials of several complex variables has also been discussed.

Hussain et al. [9] discussed the problem of visualization of scientific data; a rational cubic function was used to achieve the goal for shaped data. They derived the conditions on free parameters in the description of rational cubic function to obtain desired shapes of the data and the degree of smoothness attained was $C^1$. Sarfraz et al. [10] constructed a $C^1$ interpolant to visualize the shape of 2D positive data. They derived the conditions on free parameters in the description of rational cubic function to visualize the shape of 2D positive data. Further they also extended their scheme to visualize the shape of 3D positive data.

This paper is also devoted to the subject of shape preservation of data. In this paper, the authors have developed a rational cubic spline with two free parameters in its description

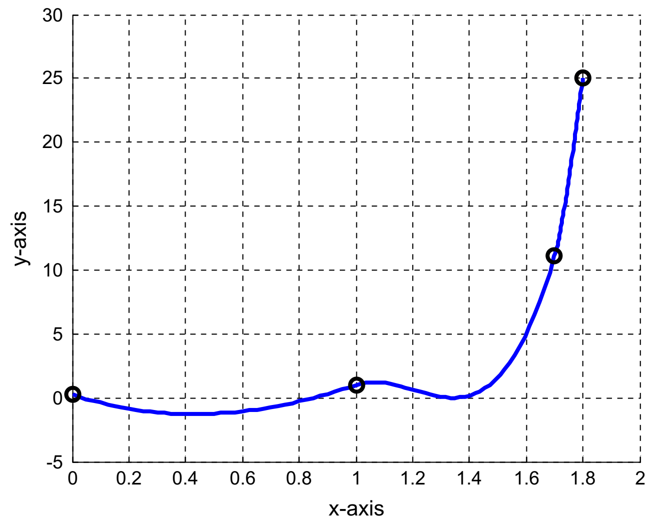


**Figure 1** Cubic Hermite function.

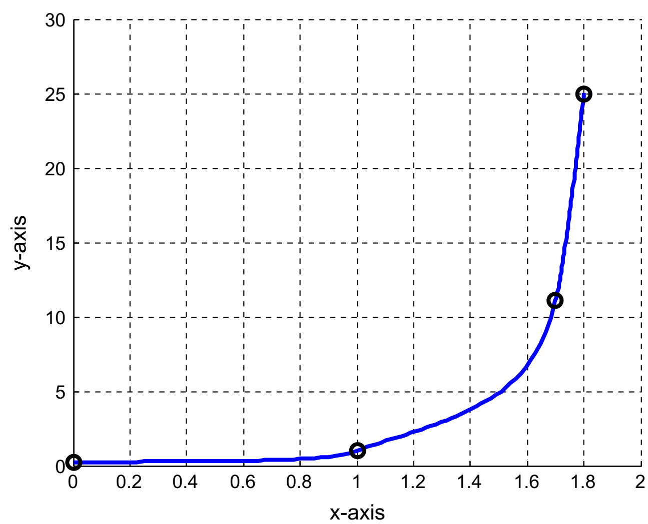


**Figure 2** $C^2$ positive rational cubic function with $\upsilon_i = 2.5$ and $\kappa_i = 0.05$.

to preserve the shape of positive and convex data. The proposed spline is $C^2$, i.e. its second ordered derivative exists and is continuous. The proposed schemes have various advantages including the followings:

- In [1–3], the authors developed the schemes to attain the desired shape of the data by inserting extra knots between any two knots while in this paper we preserve the shape of positive and convex data by imposing conditions on free parameters in the description of rational cubic function without inserting any extra knot.

**Table 2** A positive data.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | −1 | 4 | 8 | 16 | 20 | 24 | 28 | 32 | 36 | 40 | 44 |
| $y_i$ | 18 | 3 | 0.05 | 1 | 4 | 8 | 15 | 24 | 34 | 40 | 43 |

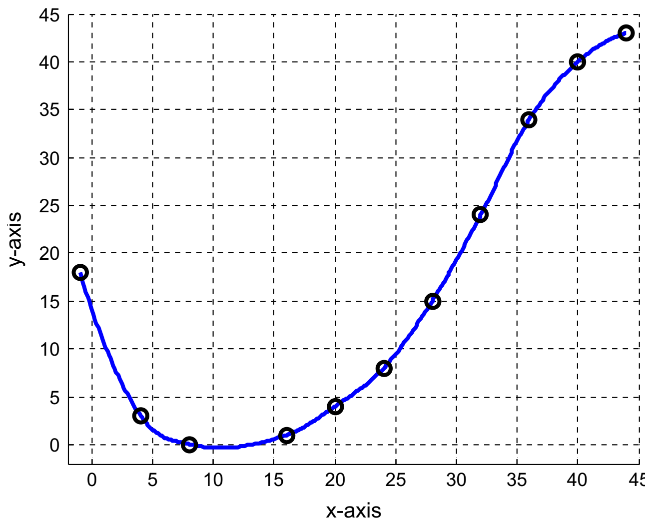


**Figure 3** Cubic Hermite function.

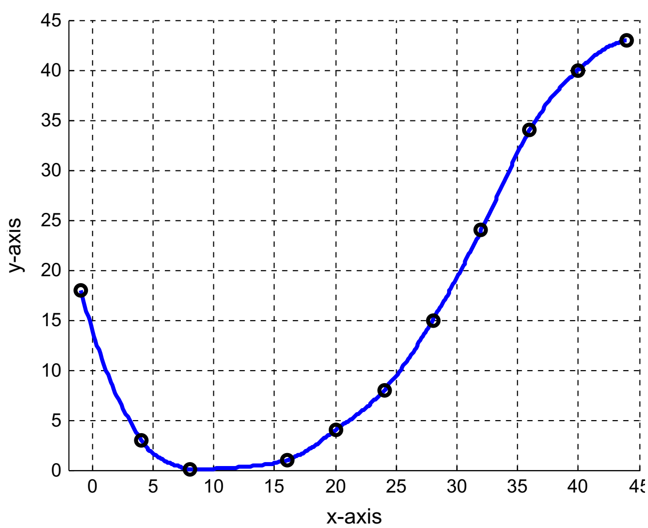


**Figure 4** $C^2$ positive rational cubic function with $\upsilon_i = 2.05$ and $\kappa_i = 0.5$.

- In [4] the authors developed schemes that work for equally spaced data while the schemes developed in this paper work for both equally and unequally spaced data.
- In [5], to control the shape of the data, certain function values and derivative values are assumed. In this paper, desired shape of the data is attained by imposing the data dependent constraints on the free parameters in the description of rational cubic function.
- In this paper, finding the values of the derivatives parameters, is computationally less expensive as compared to the method developed in [6].
- In [5,9,10], degree of smoothness attained is $C^1$ while in this paper degree of smoothness is$C^2$.

This paper is organized as follows. In Section 2, a $C^2$ rational cubic function is introduced with two free parameters in its description. In Section 3, a scheme is presented for shape preservation of positive data whereas Section 4 is dedicated to the presentation of convexity preserving scheme. Section 5 concludes the paper.

## 2. $C^2$ rational cubic function

Let $\{(x_i, f_i), i = 1, 2, 3, \ldots, n\}$ be given set of data points such that $x_1 < x_2 < \cdots < x_n$. In each interval $[x_i, x_{i+1}]$, the $C^2$ rational cubic function $S(x)$ is defined as:

$$S(x) \equiv S(x_i) = \frac{p_i(\theta)}{q_i(\theta)} = \frac{\sum_{i=0}^{3}\omega_i(1-\theta)^{3-i}\theta^i}{q_i(\theta)} \tag{1}$$

with

$$h_i = x_{i+1} - x_i, \quad \theta = \frac{(x - x_i)}{h_i}, \quad i = 1, 2, 3, \ldots, n-1,$$

and

$$\begin{aligned}
\omega_0 &= \mu_i f_i,\\
\omega_1 &= \mu_i h_i d_i + (2\mu_i + \upsilon_i) f_i,\\
\omega_2 &= -\upsilon_i h_i d_{i+1} + (\mu_i + 2\upsilon_i) f_{i+1},\\
\omega_3 &= \upsilon_i f_{i+1},\\
q_i(\theta) &= \mu_i(1-\theta)^2 + (\mu_i + \upsilon_i)\theta(1-\theta) + \upsilon_i\theta^2.
\end{aligned}$$

where $\mu_i$, $\upsilon_i$ are the shape parameters that are used to control the shape of the interpolating curve. Let $S^{(1)}(x)$ and $S^{(2)}(x)$ denote the first and the second ordered derivatives with respect to $x$ and $d_i$ denote derivative value at the knot $x_i$. Then, the $C^2$ splining constraints:

$$\left.\begin{array}{ll}
S(x_i) = f_i, & S(x_{i+1}) = f_{i+1}\\
S^{(1)}(x_i) = d_i, & S^{(1)}(x_{i+1}) = d_{i+1}\\
S^{(2)}(x_i+) = S^{(2)}(x_i-), & i = 2, 3, \ldots, n-1
\end{array}\right\} \tag{2}$$

produce, on the first derivative parameters $d_2, d_3, \cdots, d_{n-1}$, the following system of linear equations:

$$\begin{aligned}
&\mu_{i-1}\upsilon_{i-1}h_i d_{i-1} + [\upsilon_{i-1}(\mu_{i-1} + \upsilon_{i-1})h_i + \mu_i(\mu_i + \upsilon_i)h_{i-1}]d_i\\
&\quad + \mu_i\upsilon_i h_{i-1} d_{i+1} = h_{i-1}\mu_i(\mu_i + 2\upsilon_i)\Delta_i + h_i\upsilon_{i-1}(\upsilon_{i-1} + 2\mu_{i-1})\Delta_{i-1},
\end{aligned} \tag{3}$$

where $\Delta_i = (f_{i+1} - f_i)/h_i$, and the derivative parameters $d_1$, $d_n$ are determined by appropriate end conditions.

**Remark 1.** Since, the linear system of Eq. (3) is a strictly tri-diagonal for all $\mu_i, v_i > 0$, it has a unique solution for the derivative parameters $d_i's$. Moreover, it is efficient to apply LU-decomposition method to solve the system for the derivative parameters $d_i's$.

**Remark 2.** In each interval $[x_i, x_{i+1}]$, the piecewise rational cubic function $S(x) \in C^2[x_1, x_n]$ has shape parameters $\mu_i's$ and $\upsilon_i's$. It is observed that in each interval $[x_i, x_{i+1}]$, when $\mu_i = \upsilon_i = 1$, the rational spline reduces to the standard cubic spline.

**Table 3** Numerical results of Fig. 2.

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $d_i$ | 0.5231 | 2.7874 | 122.2116 | 154.5714 |
| $\upsilon_i$ | 2.5 | 2.5 | 2.5 | – |
| $\mu_i$ | 2.55 | 2.15 | 2.85 | – |

**Table 4** Numerical results of Fig. 4.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_i$ | −4.26 | −1.87 | −0.31 | 0.43 | 0.88 | 1.38 | 2.0 | 2.38 | 2.0 | 1.13 | 0.38 |
| $\upsilon_i$ | 2.05 | 2.05 | 2.05 | 2.05 | 2.05 | 2.05 | 2.05 | 2.05 | 2.05 | 2.05 | – |
| $\mu_i$ | 0.50 | 0.50 | 2.338 | 1.04 | 0.94 | 0.87 | 0.80 | 0.72 | 0.66 | 0.62 | – |

## 3. $C^2$ positive rational cubic function

In this section, we discuss the problem of shape preserving $C^2$ positive rational cubic spline. For the positive set of data $\{(x_i, f_i), i = 1, 2, 3, \ldots, n\}$; $x_1 < x_2 < \cdots < x_n$, and $f_1 > 0$, $f_2 > 0, \cdots f_n > 0$, it is required to construct a positive interpolant $S(x)$. The rational cubic function (1) is positive if both $p_i(\theta)$ and $q_i(\theta)$ are positive. Since $q_i(\theta) > 0$ for all $\mu_i, \upsilon_i > 0$. Now $p_i(\theta)$ can be expressed as follows:

$$p_i(\theta) = a_i\theta^3 + b_i\theta^2 + c_i\theta + e_i, \tag{4}$$

where

$$\begin{aligned}
a_i &= \{\mu_i(d_i - \Delta_i) + \upsilon_i(d_{i+1} - \Delta_i)\}h_i,\\
b_i &= \{\mu_i(\Delta_i - 2d_i) + \upsilon_i(2\Delta_i - d_{i+1})h_i,\\
c_i &= \mu_i(h_i d_i - f_i) + \upsilon_i f_i,\\
e_i &= \mu_i f_{i+1},\\
h_i &= x_{i+1} - x_i, \quad \Delta_i = \frac{f_{i+1} - f_i}{h_i}, \quad i = 1, 2, \ldots, n-1.
\end{aligned}$$

According to Butt and Brodlie [3], $p_i(\theta) > 0$ if and only if $(p_i'(0), p_i'(1)) \in R_1 \cup R_2$, where

$$R_1 = \left\{(a, b) :\ a > \frac{-3p_i(0)}{h_i},\ b < \frac{3p_i(1)}{h_i}\right\},$$

$$\begin{aligned}
R_2 = \{(a, b)&\\
:\ & 36 f_i f_{i+1}(a^2 + b^2 + ab - 3\Delta_i(a + b) + 3\Delta_i^2) + 4h_i(f_{i+1}a^3\\
& - f_i b^3) - h_i^2 a^2 b^2 + 3(f_{i+1}a - f_i b)(2h_i ab - 3f_{i+1}a + 3f_i b)\\
& > 0\},
\end{aligned}$$

$$p_i'(0) = \frac{\{\upsilon_i f_i + \mu_i(h_i d_i - f_i)\}}{h_i} \quad \text{and } p_i'(1) = \frac{\{\upsilon_i(f_{i+1} + h_i d_{i+1}) - \mu_i f_{i+1}\}}{h_i}.$$

Now $(p_i'(0), p_i'(1)) \in R_1 \cup R_2$, is true when $(p_i'(0), p_i'(1) \in R_1)$.where $p_i'(0) > \frac{-3p_i(0)}{h_i}, p_i'(1) < \frac{3p_i(1)}{h_i}$. This yields to the following constraints:

$$\mu_i > \frac{-\upsilon_i f_i}{2f_i + h_i d_i}, \quad \mu_i > \frac{(h_i d_{i+1} - 2f_i)\upsilon_i}{f_{i+1}} \tag{5}$$

All the above discussion can be summarized as:

**Theorem 1.** The $C^2$ rational cubic function (1) is positive in each interval $[x_i, x_{i+1}]$ if the shape parameters $\mu_i$, $\upsilon_i$ satisfy the following constraints:

$$\upsilon_i > 0, \tag{5a}$$

$$\mu_i = \kappa_i + \max\left\{0, \frac{-\upsilon_i f_i}{2f_i + h_i d_i}, \frac{(h_i d_{i+1} - 2f_i)\upsilon_i}{f_{i+1}}\right\}, \quad \kappa_i > 0. \tag{5b}$$

**Proof.** Since we assume throughout that $\mu_i, \upsilon_i > 0$, for all $i$. Therefore, (5a) and (5b) simply follow from (5). □

### 3.1. Demonstration

Consider positive data sets in Tables 1 and 2. Figs. 1 and 3 are produced by cubic Hermite spline which loose the shape of the data. The $C^2$ positive rational cubic spline curve, in Figs. 2 and 4, are generated by using Theorem 1. This guarantees the preserved shape. Tables 3 and 4 demonstrate the computed values from the proposed scheme of Figs. 2 and 4 respectively.

## 4. Convex rational cubic function

This section deals with the problem of shape preserving $C^2$ convex cubic function. Consider a set of convex data points $\{(x_i, f_i), i = 1, 2, 3, \ldots, n\}$, such that $\Delta_1 < \Delta_2 < \cdots < \Delta_{n-1}$. The necessary condition for a convex curve is that the derivative parameters must satisfy the following criterion:

$$d_1 < \Delta_1 < \cdots \Delta_{i-1} < d_i < \Delta_i \cdots < \Delta_{n-1} < d_n.$$

Now $S_i(x)$ is convex if and only if $S_i^{(2)}(x) \geqslant 0$. We have

$$S^{(2)}(x) = \frac{\sum_{i=1}^{6} \alpha_i \theta^{i-1}(1-\theta)^{6-i}}{(q_i(\theta))^3} \tag{6}$$

where

$$\begin{aligned}
\alpha_1 &= 2\mu_i^2\{(\mu_i + \upsilon_i)(\Delta_i - d_i) - \upsilon_i(d_{i+1} - \Delta_i)\}/h_i,\\
\alpha_2 &= 2\upsilon_i^2\{(2\mu_i + 5\upsilon_i)(\Delta_i - d_i) - 2\upsilon_i(d_{i+1} - \Delta_i)\}/h_i,\\
\alpha_3 &= 2\mu_i\{\mu_i(\mu_i + 7\upsilon_i)(\Delta_i - d_i) + \upsilon_i(d_{i+1} - \Delta_i)(-\mu_i + 3\upsilon_i)\}/h_i,\\
\alpha_4 &= 2\upsilon_i\{\mu_i(3\mu_i - \upsilon_i)(\Delta_i - d_i) + \upsilon_i(d_{i+1} - \Delta_i)(7\mu_i + \upsilon_i)\}/h_i,\\
\alpha_5 &= 2\mu_i^2\{(5\mu_i + 2\upsilon_i)(d_{i+1} - \Delta_i) - 2\upsilon_i(\Delta_i - d_i)\}/h_i,\\
\alpha_6 &= 2\upsilon_i^2\{(\mu_i + \upsilon_i)(d_{i+1} - \Delta_i) - \mu_i(\Delta_i - d_i)\}/h_i,
\end{aligned}$$

Now $S_i^{(2)}(x) \geqslant 0$ if all $\alpha_i's > 0, \quad i = 1, 2 \ldots 6$.

Now, $\alpha_i's > 0$, if

$$\upsilon_i > 0, \quad \mu_i > 0, \quad \mu_i > \frac{(d_{i+1} - d_i)\upsilon_i}{(\Delta_i - d_i)}, \quad \mu_i > \frac{(\Delta_i - d_{i+1})\upsilon_i}{(d_i - \Delta_i)}. \tag{7}$$

All the above discussion can be summarized as:

**Theorem 2.** The $C^2$ rational cubic function (1) is convex in each interval $[x_i, x_{i+1}]$ if the shape parameters $\mu_i$, $\upsilon_i$ satisfy the following constraints:

**Table 5** A convex data.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $x_i$ | −9 | −8 | −4 | 0 | 4 | 8 | 9 |
| $y_i$ | 7 | 5 | 3.5 | 3.25 | 3.5 | 5 | 7 |

**Table 6** A convex data.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 1 | 1.5 | 1.75 | 2 | 2.5 | 3 | 5 | 10 | 10.5 | 11 | 12 |
| $y_i$ | 10 | 7 | 5 | 2.5 | 1 | 0.6 | 0.4 | 1 | 3 | 5 | 9 |

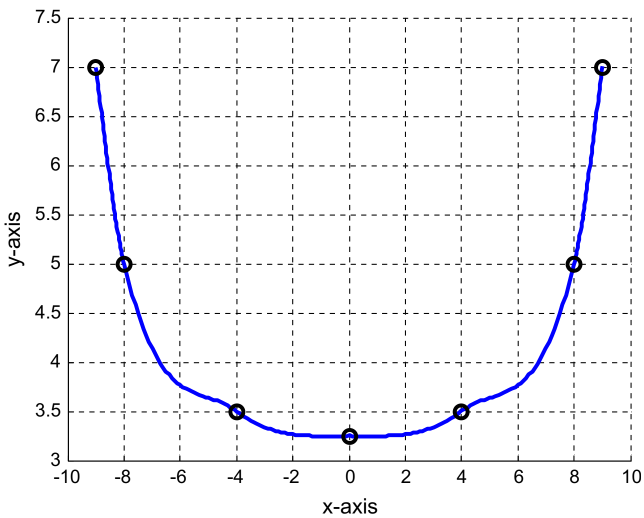


**Figure 5** Cubic Hermite function.

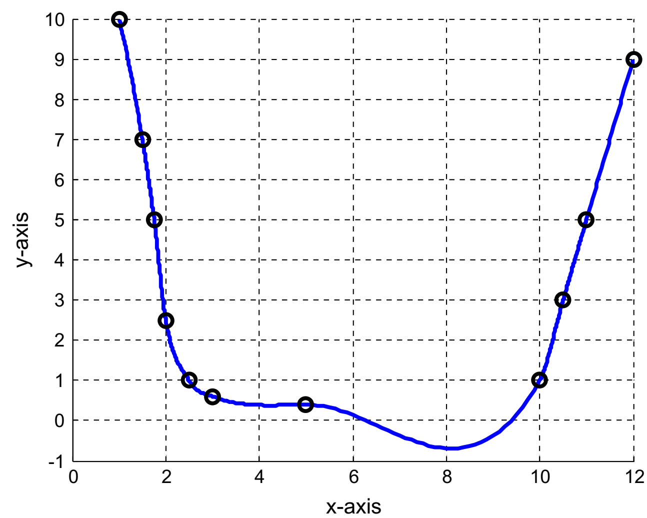


**Figure 7** Cubic Hermite function.

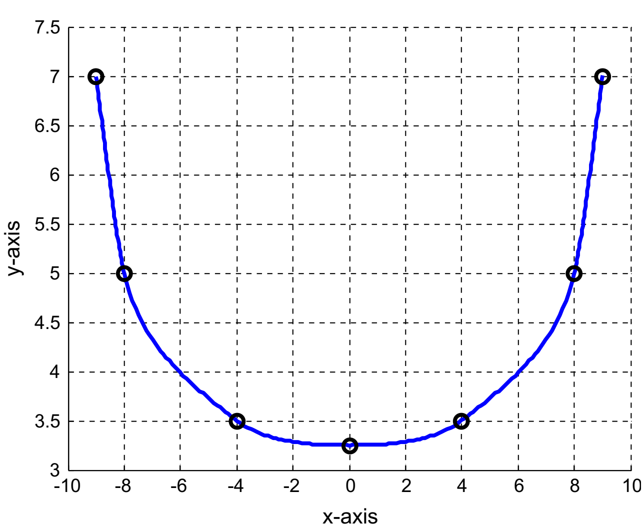


**Figure 6** $C^2$ convex rational cubic function with $\upsilon_i = 2.01$ and $\delta_i = 0.05$.

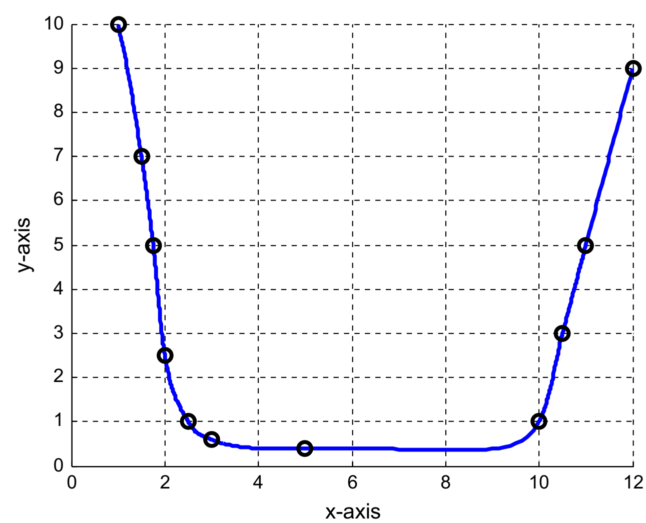


**Figure 8** $C^2$ convex rational cubic function with $\upsilon_i = 2.025$ and $\delta_i = 0.75$.

$$\upsilon_i > 0, \tag{8}$$

$$\mu_i = \delta_i + \max\left\{0, \frac{(d_{i+1} - d_i)\upsilon_i}{(\Delta_i - d_i)}, \frac{(\Delta_i - d_{i+1})\upsilon_i}{(d_i - \Delta_i)}\right\}, \quad \delta_i > 0. \tag{9}$$

**Proof.** Since we assume throughout that $\mu_i, \upsilon_i > 0$, for all $i$. Therefore, (8&9) simply follow from (7). □

### 4.1. Demonstration

Consider convex data set in Tables 5 and 6. Figs. 5 and 7 are produced by cubic spline which violates the shape of the data. It can be observed that both the figures have undesired oscillations. The $C^2$ convex rational cubic spline curve, in Figs. 6 and 8, are generated by using the proposed scheme as summarized in Theorem 2. This guarantees the preserved shape. Tables 7

**Table 7** Numerical results of Fig. 6.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $d_i$ | −2.3250 | −1.1875 | −0.2188 | 0 | 0.2188 | 1.1875 | 2.3250 |
| $\upsilon_i$ | 2.01 | 2.01 | 2.01 | 2.01 | 2.01 | 2.01 | – |
| $\mu_i$ | 2.05 | 3.55 | 2.05 | 2.05 | 3.55 | 6.25 | – |

**Table 8** Numerical results of Fig. 8.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_i$ | −4.67 | −7.00 | −9.00 | −6.50 | −1.90 | −0.45 | 0.01 | 2.06 | 4.00 | 4.00 | 4.00 |
| $\upsilon_i$ | 2.025 | 2.025 | 2.025 | 2.025 | 2.025 | 2.025 | 2.025 | 2.025 | 2.025 | 2.025 | – |
| $\mu_i$ | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 | 0.63 | 5.75 | 0.93 | 0.80 | 0.8222 | – |

and 8 demonstrate the computed values from the proposed scheme of Figs. 6 and 8 respectively.

## 5. Conclusion

To deal with smooth visualization of shaped data, $C^2$ rational cubic splines are developed. Two free parameters are introduced in its representations to preserve the shapes of positive and convex data. The shape constraints are restricted on free parameters to secure the shape preservation of the data. The developed schemes are applicable to such problems in which only data points are known. There is no need of additional information about derivatives because they are estimated directly from given data. It works for both equally and unequally spaced data. The order of continuity attained is $C^2$. In [1–3] additional knots are inserted between any two knots to attain desired shape of the data while in this paper we obtained desired shape without inserting extra knots. In this paper, for finding the values of derivative parameters, we obtained only one tridiagonal system of linear equations. While in [6], the authors obtained three systems of linear equations for finding the values of derivative parameters, which is computationally more expensive as compared to the schemes developed in this paper. The schemes developed in this paper are smoother and visually pleasing as compared to the schemes developed in [9,10].

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