On the dimension of $n$-point sets

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Abstract

We give an affirmative answer to a question raised by Khalid Bouhjar and Jan J. Dijkstra concerning whether or not every one-dimensional partial $n$-point set contains an arc by showing that a partial $n$-point set is one-dimensional if and only if it contains an arc.

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1. Introduction

The subject of $n$-point sets has been studied since 1914 with the result of S. Mazurkiewicz that there exists a subset of the plane which intersects every straight line in exactly two points. An $n$-point set is a subset of the plane which intersects every straight line in exactly $n$ points. A partial $n$-point set is a subset of the plane which intersects every straight line in at most $n$ points. In this paper we answer a question of Bouhjar and Dijkstra [1] concerning whether or not every one-dimensional partial $n$-point set contains an arc, thereby establishing a characterization of one-dimensional $n$-point sets. These results extend the results of David L. Fearnley and Jack W. Lamoreaux that every three point set is zero-dimensional [4].

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2. Every one-dimensional $n$-point set contains an arc

It has been shown by Kulesza [5] that every 2-point set is zero-dimensional. For $n > 3$ is has been shown by Bouhjar et al. [2] that $n$-point sets may contain arcs.

We will now show that every partial $n$-point set which is one-dimensional contains an arc. Throughout this paper all closed intervals stated are assumed to mean non-degenerate closed intervals unless otherwise stated.

As discussed in Engelking [3], for separable metric spaces all the usual definitions of dimension are equivalent. We will use small inductive dimension in this paper.

**Theorem 2.1.** Every non-zero-dimensional partial $n$-point set contains an arc.

**Proof.** Let $S$ be a partial $n$-point set which is not zero-dimensional. Let $p$ be a point of $S$ where $S$ is not zero-dimensional and let $O$ be an open set in the plane containing $p$ such that no open set containing $p$ is contained in $O$ and has a boundary which is disjoint from $S$. Choose horizontal line segments $U_p$ and $L_p$ above and below $p$, respectively, which are contained in $O$ which do not contain points of $S$ directly above or below $p$ so that all points between points of $U_p$ and $L_p$ are also contained in $O$. This is possible because there are no more than $n$ points of $S$ on the vertical line through $p$. Hence, we may choose horizontal lines closer to $p$ than any other point of $S$ on the vertical line containing $p$. Since there are also only finitely many points of $S$ on these horizontal lines, there are horizontal line segments whose projections onto the $x$-axis are closed intervals, which are contained in these horizontal lines, centered directly above and below $p$ which contain no points of $S$. Let $I$ be the interval of points in the intersection of the projections of $U_p$ and $L_p$ onto the $x$-axis. For each $z \in I$ we define $R(z)$ to be the vertical line $x = z$. We refer to the union of all the segments of points $R(x)$ for all $x \in I$ which are between $U_p$ and $L_p$ as $T$, which is a rectangle contained in $O$.

We denote in descending order by height $x(1), x(2), \ldots, x(j)$ to be the points of $S$ on $R(x)$. We define a point $x(k)$ to be avoidable if there is an arc entirely contained in $T$ which is a positive distance from the vertical line containing $p$ containing no points of $S$, containing a point on $T \cap R(x)$ below $x(k)$, which is between $x(k)$ and $x(k+1)$ if $x(k+1)$ exists, and a point on $T \cap R(x)$ above $x(k)$, which is between $x(k-1)$ and $x(k)$ if $x(k-1)$ exists. It follows that all vertical lines through points of $I$ on either the right or left side of $p$ contain points which are not avoidable (or else one could construct an arc avoiding all points on some vertical line on both sides of $p$, connect them at segments $U_p$ and $L_p$ on the top and bottom of $T$, and then the open set bounded within this path would be an open set with empty boundary in $S$, contradicting the assumptions regarding $p$ and $O$). So, we may choose a closed interval $H$ contained in $I$ so that for all $x \in H$, $R(x)$ contains unavoidable points of $S$, and $H$ is a positive distance from the $x$-coordinate of $p$.

For each $x \in H$ we let $x_F$ be the highest point of $S$ in $T$ on $R(x)$ so that $x_F$ is unavoidable. We define a one to one function $g : H \to S$ by $g(x) = x_F$. We wish to show that on some interval $g$ is continuous, and hence its image is an arc.

For each point $x \in H$ we choose horizontal line segments $U(x)$ and $L(x)$ directly above and below $x_F$ respectively, which do not intersect $S$, whose projections onto the $x$-coordinate are closed intervals centered at $x$ and are of equal radius $\delta_x$, which intersect
R(x) at points closer to x_F than to any other point of S on the line R(x). Then since we may choose such a δ_x for each point x ∈ H, by the Baire category theorem there is a set of points D which is dense in some non-degenerate closed interval K ⊂ H so that for some positive number γ, for each point x ∈ D, it is the case that δ_x > γ. We wish to show that g is continuous on the interior of K.

Suppose g is not continuous on the interior of K. Then we may choose a point z in the interior of K, and an ε > 0, so that for each δ > 0 there is some x ∈ K so that |x − z| < δ and the distance between the vertical coordinates of x_F and z_F is greater than ε. In the same manner as we have done previously, we may choose closed horizontal line segments U_z and L_z which have empty intersection with S, and are centered directly above and below z_F, respectively, whose projections onto the x-axis are contained in the interior of K, so that all points of U_z and L_z are closer to z_F than half of the minimum of ε and the distance from z_F to the nearest other point of S on the line R(z). We let J be the intersection of the projections of U_z and L_z onto the x-coordinate, and note that J is a closed interval centered at z. We choose x ∈ (D ∩ J) so that |x − z| < γ and the distance between the vertical coordinates of x_F and z_F is greater than ε. Now, if x_F is below L_z then one can see that z_F is avoidable by following U_z, then essentially following R(x) down, replacing segments by arcs to avoid any points on R(x) ∩ S that lie between U_z and L_z, and finally following L_z back to R(z). This involves a contradiction. If x_F lies above L_z then one can see that z_F is avoidable by following U_z, then essentially following R(x) down, replacing segments by arcs to avoid any points on R(x) ∩ S that lie between U_z and L_z, and finally following L_z back to R(z). Thus, either way we reach a contradiction. It follows that g is continuous on the interior of K and so S contains an arc. □

Every n-point set is a partial n-point set and every subset of the plane which contains an arc is at least one-dimensional. Hence, the following corollary is a direct result of this theorem.

Theorem 2.2. An n-point set is one-dimensional if and only if it contains an arc.

It is also of interest to note that there is no place that the fact that there are only n points of S on each line was used in the above proof, only that there were only finitely many points of S contained in each line. Hence, we may observe that the following theorem also follows directly.

Theorem 2.3. Let S be a subset of the plane which intersects every straight line in only a finite number of points. Then S is one-dimensional if and only if S contains an arc.

References