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Complex inversion and uniqueness theorems for a generalized Laplace transform

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ABSTRACT

In this paper a complex inversion formula for a generalized Laplace transform

$$F(x)=\frac{\Gamma(A)}{\Gamma(B)}\int_0^\infty (xt)^\beta {}_1F_1(A,B;-xt)f(t)\,dt$$

where $A=\beta+\eta+1$; $B=\alpha+\beta+\eta+1$; $\beta\geq 0$ and $\eta>0$ has been obtained and extended to a class of generalized functions. A uniqueness theorem has been established for it.

1. IN AN EARLIER PAPER [3] WE HAVE EXTENDED THE INTEGRAL TRANSFORM

$$F(x)=\frac{\Gamma(A)}{\Gamma(B)}\int_0^\infty (xt)^\beta {}_1F_1(A,B;-xt)f(t)\,dt \tag{1.1}$$

studied by Joshi [2], where ${}_1F_1$ denotes the confluent hypergeometric function, $A=\beta+\eta+1$, $B=\alpha+\beta+\eta+1$, $\beta\geq 0$ and $\eta>0$, to a class of generalized functions and have proved an analyticity theorem for it. (1.1) reduces to Laplace transform for $\alpha=\beta=0$. In this paper the complex inversion formula

$$\frac{f(t+)+f(t-)}{2}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha+\eta-s+1)t^{s-1}}{\Gamma(\beta+s)\Gamma(\eta-s+1)}\Phi(s)\,ds,$$

where $\Phi(s)=\int_0^\infty x^{s-1}F(x)\,dx$, valid under certain conditions on $f(t)$ and parameters involved, has been extended to a class of generalized functions and a uniqueness theorem has been established for it.

Let I stand for the open interval $(0,\infty)$. $D(I)$ is the space of smooth functions on I with compact support. $M_{a,b}(I)$ is the testing function space of all those complex-valued smooth functions $\phi(x)$ defined on $0<x<\infty$ for which $\mathrm{Sup}_{0<x<\infty}|e^{bx}x^{a+k}D_x^n x^{-\beta}\phi(x)|$ is finite for all $k=0,1,2,\ldots$; where $D_x^n \equiv d^n/dx^n$ and $b<0$. $D'(I)$ and $M'_{a,b}(I)$ are the topological dual of the spaces $D(I)$ and $M_{a,b}(I)$ respectively. σ_f is a real number (possibly $-\infty$) such that $f\in M'_{a,b}(I)$ for $a>\sigma_f$ and $f\notin M'_{a,b}(I)$ for $a<\sigma_f$.

The $M_{A,B}$-transform of $f\in M_{a,b}(I)$ denoted by $M_{A,B}(f)$ is defined by

$$F(s)=M_{A,B}(f)(s)$$
$$=\langle f(x), h(sx)\rangle$$

where

$$(1.2) \qquad h(sx)=\frac{\Gamma(A)}{\Gamma(B)}(sx)^{\beta}\,{}_1F_1(A,B;-sx)$$

and $s\in\Omega_f$.

The region Ω_f is defined as follows:

$$\Omega_f=\{s:\mathrm{Re}s>\sigma_f, s\neq 0, -\pi<\arg s<\pi\}.$$

2. WE FIRST PROVE A FEW LEMMAS

LEMMA 2.1

If $f\in M'_{a,b}(I)$, then

$$\int_0^\infty x^{s-1}\langle f(u), h(xu)\rangle\,\mathrm{d}x=\langle f(u), \int_0^\infty x^{s-1}h(xu)\,\mathrm{d}x\rangle$$

where $h(xu)$ is given in (1.2), and $\mathrm{Re}s>1$, $A=\beta+\eta+1$, $B=\alpha+\beta+\eta+1$, $\beta\geq 0$, $\eta>0$.

PROOF.

It is clear that

$$x^{s-1}\langle f(u), h(xu)\rangle=\langle f(u), x^{s-1}h(xu)\rangle.$$

Let $1_{0,\infty}(x)$ denote the function

$$1_{0,\infty}(x)\begin{cases}=0 & x\leq 0\\ =1 & 0<x<\infty\end{cases}$$

and the corresponding generalized function belonging to $M'_{a,b}(I)$, then using the definition of the product of generalized functions [Zemanian 5, p. 121], we have for $\mathrm{Re}s>1$

$$(2.1) \qquad \langle 1_{0,\infty}(x)f(u), x^{s-1}h(xu)\rangle$$

$$(2.2) \qquad =\langle 1_{0,\infty}(x), \langle f(u), x^{s-1}h(xu)\rangle\rangle$$

$$(2.3) \qquad =\int_0^\infty \langle f(u), x^{s-1}h(xu)\rangle\,\mathrm{d}x.$$

From [Slater 4, p. 59] it follows that

$$ {}_1F_1(a,b,-x)=\frac{\Gamma(b)}{\Gamma(b-a)}x^{-a}\{1+0|x|^{-1}\},\ x\to\infty, \mathrm{Re}\,b>\mathrm{Re}\,a>0 $$

and

$$ {}_1F_1(a,b,-x)=0(1),\ x\to 0. $$

It can be easily seen that for $\mathrm{Re}\,s>1$,

$$ \langle f(u), x^{s-1}h(xu)\rangle \in M_{a,b}(I) $$

and hence step (2.2) is justified and (2.3) follows from it as $1_{0,\infty}(x)$ is a regular generalized function.

Since the product of generalized functions is commutative (2.1) can be written as

$$ \langle 1_{0,\infty}(x)f(u), x^{s-1}h(xu)\rangle = \langle f(u), \langle 1_{0,\infty}(x), x^{s-1}h(xu)\rangle\rangle $$

$$ = \langle f(u), \int_0^\infty x^{s-1}h(xu)\,\mathrm{d}x\rangle. $$

Hence the lemma.

LEMMA 2.2

Let $\phi\in D(I), r$ be a fixed real number. If

$$ \Psi(s)=\int_0^\infty y^{s-1}\phi(y)\,\mathrm{d}y $$

where $s=c+iT$ with c fixed and

$$ a-\beta>\mathrm{Re}\,s>\max(\sigma_f,1),\ -\infty<T<\infty $$

and if $f\in M'_{a,b}(I)$, then

$$ (2.4)\qquad \frac{1}{2\pi}\int_{-r}^{r}\langle f(u),u^{-s}\rangle\,\Psi(s)\,\mathrm{d}T=\left\langle f(u), \frac{1}{2\pi}\int_{-r}^{r}u^{-s}\Psi(s)\,\mathrm{d}T\right\rangle $$

PROOF.

For $\phi(y)=0$, the proof is trivial.

Let $\phi(y)\neq 0$ and let

$$ (2.5)\qquad \langle f(u),u^{-s}\rangle=\lambda(s). $$

The left hand side of (2.5) is justified as $u^{-s}\in M_{a,b}(I)$ for $\mathrm{Re}\,s<a-\beta$. $\lambda(s)$ is seen to be analytic in $a-\beta>\mathrm{Re}\,s>\max(\sigma_f,1)$ and $\Psi(s)$ is also analytic for all finite values of s. Thus the left hand side of (2.4) is an integrand which is an analytic function over a finite region and hence converges uniformly. Now

$$ \left| e^{bu}u^{a+n}\frac{d^n}{du^n}u^{-\beta}\left\{\frac{1}{2\pi}\int_{-r}^{r}u^{-s}\Psi(s)\,\mathrm{d}T\right\}\right| $$

$$= \left| e^{bu} u^{a+n} \left\{ \frac{1}{2\pi} \int_{-r}^{r} \frac{d^n}{du^n} u^{-(\beta+s)} \Psi(s)\, dT \right\} \right|$$

$$\le \frac{1}{2\pi} \int_{-r}^{r} |e^{bu} u^{a-\beta-s}| |(-1)^n (\beta+s)(\beta+s+1)\dots(\beta+s+n-1)\Psi(s)\, dT|.$$

Since $\int_{-r}^{r} (\beta+s)(\beta+s+1)\dots(\beta+s+n-1)\Psi(s)\, dT$ is finite and

$$\operatorname*{Sup}_{0<u<\infty} |e^{bu} u^{a-\beta-s}| < \infty$$

for $a-\beta > \operatorname{Re} s > \max(\sigma_f, 1)$ and $b<0$.

We see that

$$\operatorname*{Sup}_{0<u<\infty} \left| e^{bu} u^{a+n} \frac{d^n}{du^n} u^{-\beta} \left\{ \frac{1}{2\pi} \int_{-r}^{r} u^{-s} \Psi(s)\, dT \right\} \right| < \infty.$$

This proves that

$$1/2\pi \int_{-r}^{r} u^{-s} \Psi(s)\, dT \in M_{a,b}(I).$$

Hence the right hand side of (2.4) is meaningful. Now to prove the equality let us partition the path of integration on the straight line from $c-ir$ to $c+ir$ into m sub-intervals each of length $2r/m$. Let $s_p = c+iT_p$ be a point in the pth interval. Let us set

$$V_m(u) = \sum_{p=1}^{m} u^{-s_p} \Psi(s_p) \frac{2r}{m}.$$

We can write

$$(2.6) \quad \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{-r}^{r} \langle f(u), u^{-s} \rangle \Psi(s)\, dT = \operatorname*{Lim}_{m\to\infty} \sum_{p=1}^{m} \frac{1}{2\pi} \langle f(u), u^{-s_p} \rangle \Psi(s_p) \frac{2r}{m} \\ = \operatorname*{Lim}_{m\to\infty} \left\langle f(u), \displaystyle\sum_{p=1}^{m} \frac{1}{2\pi} u^{-s_p} \Psi(s_p) \frac{2r}{m} \right\rangle. \end{cases}$$

If we can show that the sum within the last expression converges in $M_{a,b}(I)$ to $1/2\pi \int u^{-s} \Psi(s)\, dT$, the equality (2.4) will be proved.

Let us consider $A(u,m)$ where

$$(2.7) \quad \begin{cases} A(u,m) = e^{bu} u^{a+n} D_u^n u^{-\beta} [V_m(u) - \displaystyle\int_{-r}^{r} u-s\, \Psi(s)\, dT] \\ = e^{bu} u^{a+n} [\displaystyle\sum_{p=1}^{m} (-1)^n (s_p+\beta)(s_p+\beta+1) \\ \quad \dots (s_p+\beta+n-1) u^{-s_p-\beta-n} \Psi(x_p) \frac{2r}{m} \\ - \displaystyle\int_{-r}^{r} (-1)^n (s+\beta)(s+\beta+1)\dots(s+\beta+n-1) u^{-s-\beta-n} \Psi(s)\, dT]. \end{cases}$$

We have to show that $A(u,m)$ converges uniformly to zero on $0<u<\infty$ as $m\to\infty$.

For $b<0$, we see that

$$|e^{bu}u^{a-\beta-s}(s+\beta)(s+\beta+1)\dots(s+\beta+n-1)|$$

tends uniformly to zero on $-r\le T\le r$ as $u\to\infty$. Hence for given $\varepsilon>0$, there exists a $u'>0$ such that $u>u'>0$ and $-r\le T\le r$,

$$|e^{bu}u^{a-\beta-s}(s+\beta)(s+\beta-1)\dots(s+\beta+n-1)|<\frac{\varepsilon}{3}[\int_{-r}^{r}\Psi(s)\,\mathrm{d}T]^{-1}$$

[since $\int_{-r}^{r}|\Psi(s)\,\mathrm{d}T|$ is finite and nonzero because of $\phi(t)\neq 0$.]

It follows that

$$\text{(2.8)}\qquad \operatorname*{Sup}_{u>u'}\left|e^{bu}u^{a+n}\frac{d^n}{du^n}u^{-\beta}\{\int_{-r}^{r}u^{-s}\Psi(s)\,\mathrm{d}T\}\right|<\frac{\varepsilon}{3}.$$

Also for all m

$$\text{(2.9)}\qquad \operatorname*{Sup}_{u>u'}\left|e^{bu}u^{a+n}\frac{d^n}{du^n}u^{-\beta}V_m(u)\right|<\frac{\varepsilon}{3}[\int_{-r}^{r}|\Psi(s)\,\mathrm{d}T|]^{-1}\frac{2r}{m}\sum_{p=1}^{m}|\Psi(s_p)|.$$

Thus, there exists m_0 such that for $m>m_0$, the right hand side of (2.9) is bounded by $2\varepsilon/3$.

From (2.8) and (2.9) we have for $m>m_0$,

$$u>u',\ |A(u,m)|<\varepsilon.$$

Let us now consider the range $0<u\le u'$, with c fixed in $a-\beta>\operatorname{Re}s>\max(\sigma_f,1)$. We see that $u^{a-\beta-s}(s+\beta)(s+\beta+1)\dots(s+\beta+n-1)\Psi(s)$ is a uniformly continuous function of (u,T), $0<u\le u'$, $-r\le T\le r$. This together with (2.7) shows that there exists m_1 such that for all $m>m_1$, $|A(u,m)|<\varepsilon$ on $0<u<u'$ as well.

Thus when $m>\max(m_0m_1)$, we have $|A(u,m)|<\varepsilon$ uniformly on $0<u<\infty$.

Hence the lemma.

LEMMA 2.3

If (i) $\phi\in D(I)$

(ii) a,b,c and r be real numbers such that $a-b>c>\max(\sigma_f,1)$ and $b<0$, then

$$\frac{1}{\pi}\int_0^\infty\phi(y)\left(\frac{y}{u}\right)^{c-1}\frac{\sin r\log y/u}{u\log y/u}\,\mathrm{d}y\to\phi(u)$$

in $M_{a,b}(I)$ as $r\to\infty$.

PROOF.

Let

$$I=\frac{1}{\pi}\int_0^\infty\phi(y)\left(\frac{y}{u}\right)^{c-1}\frac{\sin r\log y/u}{u\log y/u}\,\mathrm{d}y.$$

Putting $\log y/u = t$ i.e., $y = u\,e^t$ in I, we have

$$I = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(ue^t) e^{(c-1)t} \frac{\sin rt}{ut} ue^t \, \mathrm{d}t$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(ue^t) e^{ct} \frac{\sin rt}{t} \, \mathrm{d}t.$$

Hence

$$[1 - \phi(u)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(ue^t) e^{ct} \frac{\sin rt}{t} \, \mathrm{d}t - \phi(u)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} [e^{ct} \phi(ue^t) - \phi(u)] \frac{\sin rt}{t} \, \mathrm{d}t$$

$$\left[\text{since } \int_{-\infty}^{\infty} \frac{\sin rt}{t} \, \mathrm{d}t = \pi\right].$$

Let

$$B_r(u) = e^{bu} u^{a+n} \frac{d^n}{du^n} u^{-\beta} \frac{1}{\pi} \int_{-\infty}^{\infty} [e^{ct} \phi(ue^t) - \phi(u)] \frac{\sin rt}{t} \, \mathrm{d}t.$$

Our lemma will be proved if we are able to show that $B_r(u) \to 0$ uniformly on $0 < u < \infty$ as $r \to \infty$, $n = 0, 1, 2$.

Taking the differential operator inside the integral sign, we have

$$B_r(u) = \frac{1}{\pi} e^{bu} u^{a+n} \int_{-\infty}^{\infty} [e^{ct} D_u^n u^{-\beta} \phi(ue^t) - D_u^n u^{-\beta} \phi(u)] \frac{\sin rt}{t} \, \mathrm{d}t$$

$$= \frac{1}{\pi} e^{bu} e^{a+n} \left[\int_{-\infty}^{\infty} + \int_{-\partial}^{\partial} + \int_{\partial}^{\infty} \right]$$

$$= I_1(u) + I_2(u) + I_3(u).$$

Here $I_1(u)$, $I_2(u)$ and $I_3(u)$ denote the quantities obtained by integrating over the intervals $-\infty < t < -\partial$, $-\partial < t < \partial$ and $\partial < t < \infty$, respectively, where $\partial > 0$.

Let us consider I_2 first and set

$$R(t, u) = e^{bu} u^{a+n} \left[\frac{e^{ct} D_u^n u^{-\beta} \phi(ue^t) - D_u^n u^{-\beta} \phi(u)}{t} \right].$$

Here $R(t, u)$ is a continuous function of (t, u) for all u and $t \neq 0$. Also,

$$\operatorname*{Lim}_{t \to 0} R(t, u) = \operatorname*{Lim}_{t \to 0} e^{bu} u^{a+n} D_t [e^{ct} D_u^n u^{-\beta} \phi(ue^t)]$$

(By L'Hospital's Rule).

Hence assigning the value

$$e^{bu} u^{a+n} D_t [e^{ct} D_u^n u^{-\beta} \phi(ue^t)]_{t=0}$$

to $R(0, u)$, we see that $R(t, u)$ is a continuous function of (t, u) in $-\partial < t < \infty$,

$0<u<\infty$ and since $\phi(u)$ is smooth, $R(t,u)$ is bounded, say by K. Hence for any $\varepsilon>0$, there is a ∂ so small that

$$|I_2(u)| = \left| \frac{1}{\pi} \int_{-\partial}^{\partial} R(t,u) \sin rt \, \mathrm{d}t \right| \leq \frac{1}{\pi} \int_{-\partial}^{\partial} |R(t,u)| \, \mathrm{d}T$$

or, $|I_2(u)| \leq K\, 2\partial/\pi < \varepsilon$, if ∂ is so fixed that $\partial < \pi\varepsilon/2K$.

Now let us consider $I_1(u)$.

$$\begin{aligned} I_1(u) &= \frac{1}{\pi} e^{bu} u^{a+n} \int_{-\infty}^{-\partial} [e^{ct} D_u^n u^{-\beta} \phi(ue^t) - D_u^n u^{-\beta} \phi(u)] \frac{\sin rt}{t} \, \mathrm{d}t \\ &= \frac{1}{\pi} e^{bu} u^{a+n} \int_{-\infty}^{-\partial} [e^{ct} D_u^n u^{-\beta} \phi(ue^t) \frac{\sin rt}{t} \, \mathrm{d}t \\ &= \frac{1}{\pi} e^{bu} u^{a+n} \int_{-\infty}^{-\partial} [D_u^n u^{-\beta} \phi(u)] \frac{\sin rt}{t} \, \mathrm{d}t \\ &= J_1(u) - J_2(u) \text{ (say).} \end{aligned}$$

We have

$$J_2(u) = \frac{1}{\pi} e^{bu} u^{a+n} \{D_u^n u^{-\beta} \phi(u)\} \int_{-\infty}^{-r\partial} \frac{\sin z}{z} \, \mathrm{d}z.$$

Now as $\{e^{bu} u^{a+n} D_u^n u^{-\beta} \phi(u)\}$ is bounded in $0<u<\infty$ and $\int_{-\infty}^{0} \sin z/z \, \mathrm{d}z$ is convergent, $J_2(u)$ tends uniformly to zero in $0<u<\infty$ as $r \to \infty$ (since $\int_{-\infty}^{-r} \sin z/z \, \mathrm{d}z = 0$).

On integration by parts

$$\begin{aligned} J_1(u) &= \frac{1}{\pi} e^{bu} u^{a+n} \left[\frac{-e^{ct} D_u^n u^{-\beta} \phi(ue^t)}{t} \frac{\cos rt}{r} \right]_{-\infty}^{-\partial} \\ &\quad + \frac{1}{\pi r} e^{bu} u^{a+n} \int_{-\infty}^{-\partial} \cos rt \, D_t \left\{ \frac{e^{ct}}{t} D_u^n u^{-\beta} \phi(ue^t) \right\} \mathrm{d}t \end{aligned}$$

since $\phi(u) \in D(I)$ and $c>1$, we have

$$\text{(2.10)} \quad \left\{ \begin{aligned} J_1(u) &= \frac{1}{\pi} e^{bu} u^{a+n} \left[\frac{-e^{-c\partial} D_u^n u^{-\beta} \phi(ue^{-\partial})}{-\partial} \frac{\cos rt}{r} \right] \\ &\quad + \lim_{t \to -\infty} \frac{1}{\pi} e^{bu} u^{a+n} \left[\frac{e^{ct} D_u^n u^{-\beta} \phi(ue^t)}{t} \frac{\cos rt}{r} \right] \\ &\quad + \frac{1}{\pi r} e^{bu} u^{a+n} \int_{-\infty}^{-\partial} \cos rt \, D_t \left\{ \frac{e^{ct}}{t} D_u^n u^{-\beta} \phi(ue^t) \right\} \mathrm{d}t \\ &= \frac{1}{\pi} e^{bu} u^{a+n} \left[\frac{e^{-c\partial} D_u^n u^{-\beta} \phi(ue^{-\partial})}{\partial} \frac{\cos r\partial}{r} \right] + 0 \\ &\quad + \frac{1}{\pi r} e^{bu} u^{a+n} \int_{-\infty}^{-\partial} \cos rt \, D_t \left\{ \frac{e^{ct}}{t} D_u^n u^{-\beta} \phi(ue^t) \right\} \mathrm{d}t. \end{aligned} \right.$$

The first term of (2.10) tends uniformly to zero in $0<u<\infty$ as $r\to\infty$, since ∂ and c (>1) are fixed and $e^{bu}u^{a+n}D_u^n u^{-\beta}\phi(ue^{-\partial})$ is a bounded function of u in $0<u<\infty$.

Also,

$$e^{bu}u^{a+n}D_t\left[\frac{e^{ct}}{t}D_u^n u^{-\beta}\phi(ue^t)\right]$$

$$=e^{bu}u^{a+n}\left(\frac{tce^{ct}-e^{ct}}{t^2}\right)D_u^n u^{-\beta}\phi(ue^t)$$

$$+e^{bu}u^{a+n}\frac{e^{ct}}{t}\frac{d}{dt}[D_u^n u^{-\beta}\phi(ue^t)].$$

Since each term is a bounded function of u and t, $0<u<\infty$, $-\infty<t<\infty$, hence the second term in the right hand side of (2.10) also goes to zero uniformly as $r\to\infty$.

Thus we see that $J_1(u)\to 0$ as $r\to\infty$, and hence $I_1(u)\to 0$ as $r\to\infty$.

In a similar manner we can prove that $I_3(u)$ converges uniformly to zero in $0<u<\infty$ as $r\to\infty$.

Combining these results we see that $\text{Lim}_{r\to\infty}|B_r(u)|<\varepsilon$, $0<u<\infty, \varepsilon>0$ being arbitrary small. Hence the Lemma.

3. COMPLEX INVERSION FORMULA

THEOREM 3.1

If (a) $f\in M'_{a,b}(I)$

(b) $F(x)$ is defined by

$$F(x)=\langle f(u),h(x,u)\rangle$$

where $h(xu)=\Gamma(A)/\Gamma(B)(xu)^{-\beta}\,{}_1F_1(A,B;-xu)$

(c) α,β,η,a,b are real numbers with

(i) $a-\beta>c>\max.\ (\sigma_f,1)$

(ii) $b<0$ and

(iii) $\beta\geq 0$, $\eta>0$, $0<\text{Re}(\beta+s)<\text{Re}(\beta+\eta+1)$, $s=c+IT$ then for any $\phi(y)\in D(I)$

$$\left\langle\frac{1}{2\pi i}\int_{c-ir}^{c+ir}\frac{\Gamma(\alpha+\eta-s+1)}{\Gamma(\beta+s)\Gamma(\eta-s+1)}\Phi(s)y^{s-1}\,\text{d}s,\phi(y)\right\rangle\to\langle f,\phi\rangle \text{ as } f\to\infty$$

where $\Phi(s)=\int_0^\infty x^{s-1}F(x)\,\text{d}x$.

PROOF.

The theorem will be proved by justifying the following steps:

$$\left\langle\frac{1}{2\pi i}\int_{c-ir}^{c+ir}\frac{\Gamma(\alpha+\eta-s+1)}{\Gamma(\beta+s)\Gamma(\eta-s+1)}\Phi(s)y^{s-1}\,\text{d}s,\phi(y)\right\rangle$$

$$(3.1)\quad \begin{cases} = \left\langle \dfrac{1}{2\pi i} \displaystyle\int_{c-ir}^{c+ir} M\Phi(s) y^{s-1}\,\mathrm{d}s, \phi(y) \right\rangle \\ \text{where, } M = \dfrac{\Gamma(\alpha+\eta-s+1)}{\Gamma(\beta+s)\Gamma(\eta-s+1)} \end{cases}$$

$$(3.2)\quad = \int_0^\infty \frac{1}{2\pi i} \int_{c-ir}^{c+ir} M\Phi(s) y^{s-1}\,\mathrm{d}s\,\phi(y)\,\mathrm{d}y$$

$$(3.3)\quad \begin{cases} = \dfrac{1}{2\pi} \displaystyle\int_{-r}^{r} M\Phi(s) \int_0^\infty y^{s-1}\phi(y)\,\mathrm{d}y\,\mathrm{d}T \\ (s = c+iT) \end{cases}$$

$$(3.4)\quad = \frac{1}{2\pi} \int_{-r}^{r} M\left\{ \int_0^\infty x^{s-1} F(x)\,\mathrm{d}x \right\} \int_0^\infty y^{s-1}\phi(y)\,\mathrm{d}y\,\mathrm{d}T$$

$$(3.5)\quad = \frac{1}{2\pi} \int_{-r}^{r} M\left\{ \int_0^\infty x^{s-1} \langle f(u), h(xu)\rangle\,\mathrm{d}x \right\} \int_0^\infty y^{s-1}\phi(y)\,\mathrm{d}y\,\mathrm{d}T$$

$$(3.6)\quad = \frac{1}{2\pi} \int_{-r}^{r} M\left\langle f(u), \int_0^\infty x^{s-1} h(xu)\,\mathrm{d}x \right\rangle \int_0^\infty y^{s-1}\phi(y)\,\mathrm{d}y\,\mathrm{d}T$$

$$(3.7)\quad = \frac{1}{2\pi} \int_{-r}^{r} M\langle f(u), u^{-s} M^{-1}\rangle \int_0^\infty y^{s-1}\phi(y)\,\mathrm{d}y\,\mathrm{d}T$$

$$(3.8)\quad = \left\langle f(u), \frac{1}{2\pi} \int_{-r}^{r} u^{-s} \int_0^\infty y^{s-1}\phi(y)\,\mathrm{d}y\,\mathrm{d}T \right\rangle$$

$$(3.9)\quad = \left\langle f(u), \frac{1}{2\pi} \int_0^\infty \phi(y) \int_{-r}^{r} u^{-s} y^{s-1}\,\mathrm{d}T\,\mathrm{d}y \right\rangle$$

$$(3.10)\quad \begin{cases} = \left\langle f(u), \dfrac{1}{\pi} \displaystyle\int_0^\infty \phi(y) \left(\dfrac{y}{u}\right)^{c-1} \dfrac{\sin r \log y/u}{u \log y/u}\,\mathrm{d}y \right\rangle \\ \to \langle f(u), \phi(u)\rangle \text{ as } r \to \infty. \end{cases}$$

Since the integral in (3.1) is a continuous function of y and $\phi\,(y)$ is a smooth function of compact support in $(0,\infty)$, (3.1) implies (3.2). As the integral in (3.2) is continuous on a closed and bounded domain or integration, we can change the order of integration in (3.2) to obtain (3.3), (3.4) and (3.5) are obvious. (3.6) is justified by Lemma 2.1.

Now from Erdélyi [1, p. 285] we have

$$\int_0^\infty x^{b-1}\,{}_1F_1(a, c, -t)\,\mathrm{d}t = \frac{\Gamma(b)\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} \qquad 0 < \operatorname{Re} b < \operatorname{Re} a.$$

Therefore,

$$\int_0^\infty x^{s-1} \frac{\Gamma(A)}{\Gamma(B)} (xu)^\beta\,{}_1F_1(A, B, -xu)\,\mathrm{d}x = u^{-s} \frac{\Gamma(\beta+s)\Gamma(\eta-s+1)}{\Gamma(\alpha+\eta-s+1)},$$

where $0 < \operatorname{Re}(\beta+s) < \operatorname{Re}(\beta+n+1)$. Hence (3.7) is a simplification of (3.6). (3.8) is obtained by using Lemma 2.2. As the integral in (3.8) converges uniformly we can change the order of integration to obtain (3.9). On simplification (3.9) reduces to (3.10). Lemma 2.3 shows that the integral within the brackets in (3.10) converges in $M_{a,b}(I)$ to $\phi(u)$ as $r \to \infty$. Hence the theorem.

4. UNIQUENESS THEOREM

Let $f, g \in M'_{a,b}(I)$ and

(i) $F(s) = (M_{A,B}f)(s)$, $s \in \Omega_f$

(ii) $G(s) = (M_{A,B}g)(s)$, $s \in \Omega_g$ and

(iii) $F(s) = G(s)$ for $s \in \Omega_f \cap \Omega_g$

then in the sense of equality in $D'(I)$, $f = g$.

The above weak version of uniqueness is an immediate consequence of the inversion theorem.

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