Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 163 (2011) 163–182
Journal of Approximation Theory
www.elsevier.com/locate/jat

Gamma-type operators and the Black–Scholes semigroup

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Received 28 April 2010; accepted 3 September 2010
Available online 15 September 2010

Communicated by Paul Nevai

Abstract

We study Gamma-type operators from the analytic and probabilistic viewpoint in the setting of weighted continuous function spaces and estimate the rate of convergence of their iterates towards their limiting semigroup, providing, in this way, a quantitative version of the classical Trotter approximation theorem. The semigroup itself has some interest, since it is generated by the Black–Scholes operator, frequently occurring in the theory of option pricing in mathematical finance.

Keywords: Positive linear operators; Strongly continuous semigroups; Markov processes; Rate of convergence; Black–Scholes operator

1. Introduction and notation

The aim of the present paper is to deepen the study of a particular sequence of positive linear operators, denoted by Q_n and introduced in [3,4] as perturbed Post–Widder operators, in the setting of weighted continuous function spaces.

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2010.09.001

In the above papers the authors show how the operators Q_n, besides their theoretical interest in approximation theory, may be successfully employed in concrete cases, namely in representing, in the spirit of the classical Trotter theorem (see, e.g., [6, Theorem 1.6.7, p. 67]) and according to formula (2.3), the semigroup $(S_m(t))_{t\geq 0}$ generated by a unidimensional Black–Scholes-type operator, which, in its simplest form, reads as

$$Lu(x) := \frac{\sigma^2}{2}x^2u''(x) + rxu'(x) - ru(x), \quad (x \geq 0).$$

As is widely known, the operator L is profoundly involved in the theory of option pricing in mathematical finance, describing, in fact, the cornerstone model for the whole theory. Here the volatility σ and the riskless interest rate r are assumed to be constant over time, which actually happens very seldom in real markets; for a rather complete survey on this subject we refer the reader to [18,30], for instance.

The existence of the semigroup $(S_m(t))_{t\geq 0}$ and its deep interplay with suitable Markov processes have been established in [5]. Subsequently, other results concerning both the operators Q_n and the limiting semigroup $(S_m(t))_{t\geq 0}$, with particular emphasis on shape-preserving properties and the asymptotic behaviour, have been obtained in [7].

It goes without saying that representation formula (2.3) plays a crucial role in this kind of approach, falling within a general scheme of investigation of the use of iterates of positive linear operators in the study of evolution problems, initiated by Karlin and Ziegler in [17] and continued and developed originally and extensively, in different contexts, by Altomare and his school: without attempting to be exhaustive in this respect, we confine ourselves to citing [6,3,5,7] and the references quoted therein.

Turning back to our paper, we proceed along the following lines, described in Sections 2–4.

In Section 2, after observing, as a preamble, how the operators Q_n may be rightly regarded as perturbed Gamma operators, we carry out a detailed analysis of their properties, proving their commutativity and computing explicitly the norm of each of them.

While performing such investigation, some (possibly) new results concerning the classical Gamma function $\Gamma(x)$ come to the fore quite naturally.

Particular attention is devoted to the determination of the moments of any possible order: our proof herein must be compared with the analogous one stated in [19] concerning the classical Gamma operators and based upon a different technique, involving Laguerre polynomials.

An alternative proof is presented as well, starting from a probabilistic viewpoint and making use of the so-called moment generating function.

A probabilistic interpretation in terms of Markov processes of the operators Q_n, of their iterates Q_n^k $(k \geq 2)$ and of the approximation formula (2.3) is given, too.

In Section 3, trying to answer to a natural question concerning the speed of convergence in the approximation formula (2.3), we manage to prove that $\|Q_n^{k(n)}f - S_m(t)f\|_m = O(\frac{1}{n})$ as $n \to +\infty$ for sufficiently regular functions f, establishing, in such a way, a quantitative version of the Trotter theorem which, as far as we know, seems to be new; however, we adopt some techniques used in [15] in the simpler framework of the classical Bernstein operators.

We also point out that some other results in this direction have been recently achieved in [9,10,14,21].

Finally, the last Section 4 deals with the asymptotic behaviour of $Q_n f(x)$ and, more generally, of $Q_n^k f(x)$ $(k \geq 2)$ as $x \to +\infty$: the main result ensures the existence of an asymptote at $+\infty$ and is strongly connected to [7, Theorem 3.1].

Passing to a concrete case, a specific application of all our main results to the classical Black–Scholes problem for European call options is fully treated as well.

The notation used throughout the paper is quite standard: if $k \geq 1$ is an integer and I is a real interval, $C^k(I)$ denotes the vector space of all real-valued functions on I that are k times continuously differentiable. The space of all real-valued continuous functions on I is, as usual, denoted by $C(I)$.

Occasionally we shall encounter the space $L^1(I)$ of all real-valued Lebesgue integrable functions on I.

For any real $\lambda \geq 0$, e_λ stands for the power function $e_\lambda(x) := x^\lambda$ $(x \geq 0)$, whereas $o(\cdot)$ and $O(\cdot)$ denote the classical Landau symbols. If x is a real number, $[x]$ is the integer part of x.

Other notation which is not encompassed above will be specified at each occurrence.

2. Preliminary properties

For every integer $m \geq 1$ let us define

$$E_m^0 := \left\{ f \in C([0,+\infty[) \mid \lim_{x\to+\infty} \frac{f(x)}{1+x^m} = 0 \right\},$$

which turns out to be a Banach space with respect to the weighted norm $\|f\|_m := \sup_{x\geq 0} \frac{|f(x)|}{1+x^m}$. For fixed parameters $\sigma > 0$ and $r \geq 0$, we shall consider *the Black–Scholes unidimensional operator*

$$Lu(x) := \frac{\sigma^2}{2} x^2 u''(x) + rxu'(x) - ru(x) \quad (x \geq 0), \tag{2.1}$$

with u belonging to the domain

$$D_m(L) := \left\{ u \in E_m^0 \cap C^2(]0,+\infty[) \mid \lim_{x\to 0^+} \left(\frac{\sigma^2}{2} x^2 u''(x) + rxu'(x) \right) = \lim_{x\to+\infty} \frac{1}{1+x^m} \left(\frac{\sigma^2}{2} x^2 u''(x) + rxu'(x) \right) = 0 \right\}.$$

In [5] $(L, D_m(L))$ was shown to be the infinitesimal generator of a positive strongly continuous semigroup $(S_m(t))_{t\geq 0}$ on E_m^0.

Subsequently, in [3] (see also [4]), aiming at representing explicitly $(S_m(t))_{t\geq 0}$ in the spirit of Altomare's theory, the authors introduced certain perturbed Post–Widder operators defined by

$$Q_n f(x) := \left(1 - \frac{r}{n\sigma^2}\right) \left(\frac{n^2\sigma^2}{(n\sigma^2 + r)x} \right)^n \frac{1}{\Gamma(n)} \cdot \int_0^{+\infty} e^{-\frac{n^2\sigma^2 y}{(n\sigma^2+r)x}} y^{n-1} f(y) \mathrm{d}y \tag{2.2}$$

for all $n \geq r/\sigma^2$, $f \in E_m^0$ and $x \geq 0$, and proved the following:

- the sequence $(Q_n)_{n \geq r/\sigma^2}$ is a positive approximation process on E_m^0;
- if $m \geq 2$, for any $f \in E_m^0$ and $t \geq 0$ one has

$$S_m(t) f = \lim_{n\to+\infty} Q_n^{k(n)} f \quad \text{in } E_m^0, \tag{2.3}$$

$(k(n))_{n\geq 1}$ being an arbitrary sequence of positive integers such that $\lim_{n\to+\infty} \frac{k(n)}{n} = \sigma^2 t$.

A closer insight into the foregoing operators and semigroup may be found in [7]. In order to continue and deepen such investigation, we first note that, in a slowly different fashion, (2.2) may be rewritten as

$$Q_n f(x) = \int_0^{+\infty} K_n(x, y) f(y) \mathrm{d}y \quad (n \geq r/\sigma^2, \ f \in E_m^0, x \geq 0), \tag{2.4}$$

so each Q_n may be rightly understood as perturbed Gamma-type operator with kernel

$$K_n(x, y) := \left(1 - \frac{r}{n\sigma^2}\right) \left(\frac{n^2\sigma^2}{(n\sigma^2 + r)x}\right)^n \cdot \frac{1}{\Gamma(n)} \mathrm{e}^{-\frac{n^2\sigma^2 y}{(n\sigma^2+r)x}} \cdot y^{n-1} \quad (x > 0, y \geq 0). \tag{2.5}$$

As a historical remark, we recall that the classical Gamma operators have been introduced by Müller in [23] and investigated in subsequent papers [20,24,25,29,19,1,2], for example.

The Gamma operators, in turn, may be approximated by using other operators: see, in this respect, [2,11].

As preparatory material, we also recall that, by means of quite elementary calculus, it is not difficult to show that for every real $\lambda \in [0, m[$ one has

$$Q_n e_\lambda = a_{n,\lambda} e_\lambda \quad \text{with } a_{n,\lambda} := \left(1 - \frac{r}{n\sigma^2}\right) \left(1 + \frac{r}{n\sigma^2}\right)^\lambda \cdot \frac{\Gamma(n + \lambda)}{n^\lambda \Gamma(n)}, \tag{2.6}$$

and, correspondingly, for any $t \geq 0$,

$$S_m(t) e_\lambda = e_\lambda \cdot \mathrm{e}^{(\lambda-1)(\frac{\lambda\sigma^2}{2}+r)t}, \tag{2.7}$$

as well as

$$\|S_m(t)\| = \mathrm{e}^{(m-1)(\frac{m\sigma^2}{2}+r)t} \tag{2.8}$$

(see [7, (4.1), (4.2), (2.5) and (2.4)]).

Moreover, since actually $Q_n e_\lambda$ may be computed with good reason for all $\lambda \geq 0$, the definition of $a_{n,\lambda}$ in (2.6) may be naturally extended to all $\lambda \geq 0$. Due to [7, formula (2.3)], the same is the case for the semigroup in (2.7), the expression of $S_m(t) e_\lambda$ remaining the same for all $\lambda \geq 0$ as well.

It will turn out to be useful in the sequel to have at our disposal an expression for the coefficients $a_{n,\lambda}$ that is easier to handle than (2.6); this is possible at least for $a_{n,k}$, k integer: indeed, one may show that

$$\begin{aligned} a_{n,k} &= 1 + \frac{(k-1)(2r + k\sigma^2)}{2n\sigma^2} + \frac{(k-2)(k-1)k(3k-1)}{24n^2} \\ &\quad + \frac{k(k-1)r^2 - 2kr^2 + rk\sigma^2(k-1)^2}{2n^2\sigma^4} + o\left(\frac{1}{n^2}\right). \end{aligned} \tag{2.9}$$

The proof is quite technical, though elementary, so we omit it for the sake of brevity.

Remark 2.1. This short remark is expressly devoted to some consequences of the representation formula (2.3), concerning the Gamma function $\Gamma(x)$. Considering the importance of such functions in pure and applied mathematics, we think that departing a little from the main purpose of the paper in order to deepen this aspect is, perhaps, an appropriate choice.

To start with, let $\lambda \geq 0$ be given. After choosing $m > \lambda$, formula (2.3) gives

$$S_m(t) e_\lambda = \lim_{n \to +\infty} Q_n^{k(n)} e_\lambda \quad \text{in } E_m^0.$$

In particular, for $t = 1$, from (2.6) and (2.7) we get

$$e^{(\lambda-1)(\frac{\lambda\sigma^2}{2}+r)} = \lim_{n\to+\infty} a_{n,\lambda}^{\tilde{k}(n)},$$

where $\frac{\tilde{k}(n)}{n} \to \sigma^2$ as $n \to +\infty$, i.e.,

$$e^{(\lambda-1)(\frac{\lambda\sigma^2}{2}+r)} = \lim_{n\to+\infty} \left(1 - \frac{r}{n\sigma^2}\right)^{\tilde{k}(n)} \cdot \left(1 + \frac{r}{n\sigma^2}\right)^{\lambda\tilde{k}(n)} \cdot \left(\frac{\Gamma(n+\lambda)}{n^\lambda \Gamma(n)}\right)^{\tilde{k}(n)}$$

$$= e^{(\lambda-1)r} \cdot \lim_{n\to+\infty} \left(\frac{\Gamma(n+\lambda)}{n^\lambda \Gamma(n)}\right)^{n\frac{\tilde{k}(n)}{n}},$$

and finally

$$\lim_{n\to+\infty} \left(\frac{\Gamma(n+\lambda)}{n^\lambda \Gamma(n)}\right)^{n} = e^{\frac{(\lambda-1)\lambda}{2}}, \tag{2.10}$$

whence obviously

$$\lim_{n\to+\infty} \frac{\Gamma(n+\lambda)}{n^\lambda \Gamma(n)} = 1. \tag{2.11}$$

While formula (2.11) is well-known as the Euler–Gauss formula (see, for instance [28, p. 58]) and reduces to the Wallis formula for $\lambda = 1/2$, (2.10) seems to be probably new.

If λ is a positive integer, something more may be said: in fact, for any $n \geq 1$, using Taylor's expansion we compute

$$n^2\left(\frac{\Gamma(n+\lambda)}{n^\lambda \Gamma(n)} - e^{\frac{(\lambda-1)\lambda}{2n}}\right) = n^2\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{\lambda-1}{n}\right) - e^{\frac{(\lambda-1)\lambda}{2n}}\right)$$

$$= n^2\left(1 + \frac{(\lambda-1)\lambda}{2n} + \frac{(\lambda-2)(\lambda-1)\lambda(3\lambda-1)}{24n^2} + o\left(\frac{1}{n^2}\right)\right.$$

$$\left. -1 - \frac{(\lambda-1)\lambda}{2n} - \frac{(\lambda-1)^2\lambda^2}{8n^2} + o\left(\frac{1}{n^2}\right)\right),$$

and so

$$\lim_{n\to+\infty} n^2\left(\frac{\Gamma(n+\lambda)}{n^\lambda \Gamma(n)} - e^{\frac{(\lambda-1)\lambda}{2n}}\right) = \frac{(1-\lambda)\lambda(2\lambda-1)}{12},$$

which again seems to be probably a new result.

Now let us pass to establishing the following propositions.

Proposition 2.2. *The operators Q_n commute, i.e., $Q_n Q_m = Q_m Q_n$ for any $n, m \geq r/\sigma^2$.*

Proof. Indeed, for fixed $n, m \geq r/\sigma^2$, $f \in E_m^0$ and $x \geq 0$, we have

$$Q_n(Q_m f)(x) = \int_0^{+\infty} K_n(x,y)(Q_m f)(y)\mathrm{d}y$$

$$= \int_0^{+\infty} K_n(x,y)\left(\int_0^{+\infty} K_m(y,z) f(z)\mathrm{d}z\right)\mathrm{d}y$$

$$= \int_0^{+\infty}\left(\int_0^{+\infty} K_n(x,y)K_m(y,z)\mathrm{d}y\right) f(z)\mathrm{d}z,$$

and therefore it is sufficient to prove that the composition of the corresponding kernels is commutative, i.e.,

$$\int_0^{+\infty} K_n(x,y)K_m(y,z)\mathrm{d}y = \int_0^{+\infty} K_m(x,y)K_n(y,z)\mathrm{d}y.$$

This is equivalent to showing that

$$\int_0^{+\infty} \frac{y^{n-1}}{x^n}\cdot\frac{z^{m-1}}{y^m}\mathrm{e}^{-\frac{n^2\sigma^2 y}{(n\sigma^2+r)x}-\frac{m^2\sigma^2 z}{(m\sigma^2+r)y}}\mathrm{d}y = \int_0^{+\infty} \frac{y^{m-1}}{x^m}\cdot\frac{z^{n-1}}{y^n}\mathrm{e}^{-\frac{m^2\sigma^2 y}{(m\sigma^2+r)x}-\frac{n^2\sigma^2 z}{(n\sigma^2+r)y}}\mathrm{d}y,$$

which may be readily achieved through the change of variable $y = \frac{xz}{t}$ in the first integral, and the proof is complete. □

Proposition 2.3. *For any $n \geq r/\sigma^2$ the norm of $Q_n : E_m^0 \to E_m^0$ is equal to $a_{n,m}$ (see (2.6)).*

Proof. Let us select $f \in E_m^0$, $\|f\|_m \leq 1$. Then $|f(x)| \leq 1 + x^m$ for all $x \geq 0$, i.e., $|f| \leq e_0 + e_m$. Since Q_n acts naturally on e_m (even though $e_m \notin E_m^0$), this implies that $|Q_n f| \leq Q_n e_0 + Q_n e_m = a_{n,0}\, e_0 + a_{n,m} e_m$, yielding

$$\frac{|Q_n f(x)|}{1+x^m} \leq \frac{a_{n,0} + a_{n,m}x^m}{1+x^m} \quad \text{for all } x \geq 0.$$

The function on the right hand side is increasing with respect to x, so $\sup\limits_{x\geq 0} \frac{|Q_n f(x)|}{1+x^m} \leq a_{n,m}$, i.e., $\|Q_n f\|_m \leq a_{n,m}$; we infer that $\|Q_n\| \leq a_{n,m}$. In contrast, let $\lambda \in [0, m[$ be a real number and recall that, by (2.6), $e_\lambda \in E_m^0$ with $Q_n\, e_\lambda = a_{n,\lambda}\, e_\lambda$. It immediately follows that $\|Q_n\| \geq a_{n,\lambda}$; letting $\lambda \to m$ gives $\|Q_n\| \geq a_{n,m}$ since $a_{n,\lambda}$ is continuous with respect to λ. The desired claim is therefore fully established. □

As a conclusion of this section, we calculate the moments of order k, i.e., $M_{n,k}(x) := Q_n\,(e_1 - xe_0)^k(x)(n \geq r/\sigma^2, x \geq 0)$.

For the classical Müller's Gamma operators, the related moments have been investigated in [19] with the help of some properties of Laguerre polynomials.

Obviously, since the power k runs between 0 and $m-1$, it is meaningful to assume $m \geq 3$ in our investigation.

In order to simplify the notation, from this point onward we set

$$\rho := \frac{r}{\sigma^2}. \tag{2.12}$$

As a first step, we note that for all $n \geq \rho$ and $x \geq 0$ we have

$$\begin{aligned} M_{n,0}(x) &= Q_n e_0(x) = 1 - \frac{\rho}{n}, \quad \text{and} \\ M_{n,1}(x) &= Q_n e_1(x) - x Q_n e_0(x) = \left(1-\frac{\rho}{n}\right)\frac{\rho}{n}x. \end{aligned} \tag{2.13}$$

Next, for a general $k = 1, \ldots, m-2$, differentiating we compute

$$\begin{aligned} M'_{n,k}(x) &= -\frac{n}{x}M_{n,k}(x) + \left(1-\frac{r}{n\sigma^2}\right)\left(\frac{n^2\sigma^2}{n\sigma^2+r}\right)^n \frac{1}{\Gamma(n)}x^{-n} \\ &\quad \times \int_0^{+\infty}\left(\frac{n^2\sigma^2(y-x)+n^2\sigma^2 x}{(n\sigma^2+r)x^2}\mathrm{e}^{-\frac{n^2\sigma^2 y}{(n\sigma^2+r)x}}y^{n-1}(y-x)^k\right. \end{aligned}$$

$$
- \mathrm{e}^{-\frac{n^2\sigma^2 y}{(n\sigma^2+r)x}} y^{n-1} k(y-x)^{k-1}\Bigg)\,\mathrm{d}y
$$

$$
= -\frac{n}{x} M_{n,k}(x) + \frac{n^2\sigma^2}{(n\sigma^2+r)x^2} M_{n,k+1}(x) + \frac{n^2\sigma^2}{(n\sigma^2+r)x} M_{n,k}(x) - k M_{n,k-1}(x),
$$

and from this we deduce the recurrence formula

$$
M_{n,k+1}(x) = \left(\frac{1}{n} + \frac{\rho}{n^2}\right) x^2 \left(M'_{n,k}(x) + k M_{n,k-1}(x)\right) + \frac{\rho}{n} x M_{n,k}(x), \tag{2.14}
$$

which holds true for $k = 1, \ldots, m-2$ and $x \geq 0$. In this way we are in a position to evaluate the moments of all (possible) orders k, starting from (2.13).

As a matter of fact, we may say something more: indeed, arguing by induction, on account of (2.13) and (2.14), one easily gets

$$
M_{n,k}(x) = d_{n,k} x^k \quad \text{for any } k = 0, \ldots, m-1 \text{ and } x \geq 0, \tag{2.15}
$$

where the coefficients $d_{n,k}$ fulfill a recurrence formula as follows:

$$
\begin{cases}
d_{n,0} = \dfrac{n-\rho}{n}, \\
d_{n,1} = \dfrac{\rho n - \rho^2}{n^2}, \\
d_{n,k+1} = \dfrac{(k+\rho)n + k\rho}{n^2} d_{n,k} + \dfrac{k(n+\rho)}{n^2} d_{n,k-1}, \quad k = 1, \ldots, m-2.
\end{cases} \tag{2.16}
$$

Incidentally, observe that $M_{n,k}(x) \geq 0$ since $n \geq \rho$ in any case.

Further information on the coefficients $d_{n,k}$ appearing in (2.15) will be necessary for our purposes; in fact, a standard induction-principle argument, making use of (2.16), provides the following identities:

$$
\begin{cases}
d_{n,2k-1} = \dfrac{c_{2k-1} n^{3k-2} + p(n)}{n^{4k-2}}, & k \geq 2, \\
d_{n,2k} = \dfrac{c_{2k} n^{3k} + q(n)}{n^{4k}}, & k \geq 1,
\end{cases} \tag{2.17}
$$

where $p(n) = o(n^{3k-2})$ and $q(n) = o(n^{3k})$ as $n \to +\infty$, and both $2k-1$ and $2k$ do not exceed $m-1$. Moreover we have explicitly

$$
c_j = \begin{cases}
\rho, \quad \text{if } j = 1, \\
2^{k-1}\,(k-1)!\rho + \displaystyle\sum_{i=0}^{k-2} (2k-3-2i)!!\, 2^i\, \frac{(k-1)!}{(k-i-1)!} (2k+\rho-2i-2), \\
\quad \text{if } j = 2k-1, k \geq 2, \\
(2k-1)!!, \quad \text{if } j = 2k, k \geq 1.
\end{cases}
$$

Finally, if we set $c_0 := 1$, then

$$
\lim_{n \to +\infty} n^{[\frac{j+1}{2}]} d_{n,j} = c_j, \quad \text{for all } j = 0, \ldots, m-1. \tag{2.18}
$$

For the proof, simply split to the cases of j “odd” or “even” and apply (2.17).

Remark 2.4. It is worthwhile noting that formula (2.15) and the calculus of the coefficients $d_{n,k}$ may be easily obtained through merely probabilistic arguments. More specifically, the moments $M_{n,k}$ can be written in the form

$$M_{n,k}(x) = \left(1 - \frac{\rho}{n}\right) E[(Y_n^x - x)^k] \quad \text{for any } k = 0, \ldots, m-1 \text{ and } x > 0, \tag{2.19}$$

where $E[\ \cdot\ ]$ denotes the mathematical expectation and Y_n^x is a random variable having *Gamma distribution* with density

$$g_{n,x}(y) = \left(\frac{n^2\sigma^2}{(n\sigma^2 + r)x}\right)^n \cdot \frac{1}{\Gamma(n)}\, \mathrm{e}^{-\frac{n^2\sigma^2}{(n\sigma^2+r)x} y} \cdot y^{n-1}, \quad y \in [0, +\infty[. \tag{2.20}$$

Recall that the *Laplace transform* of a random variable X is defined as

$$G(t) := E\left[\mathrm{e}^{tX}\right]$$

for all t for which this is finite; in a probabilistic setting $G(t)$ is also called the *moment generating function* (see, for instance, [8, Ch. 4]), because it can be used to generate the moments of order $k \geq 1$ of X through the following formula:

$$E[X^k] = \frac{\partial^k}{\partial t^k} G(t)|_{t=0}, \tag{2.21}$$

which holds true if $G(t)$ exists in some neighborhood of 0.

In our present framework, for fixed $n \geq \rho$ and $x > 0$, we are interested in computing the Laplace transform of the random variable $Y_n^x - x$, i.e., in computing

$$G_{n,x}(t) = E\left[\mathrm{e}^{t(Y_n^x - x)}\right] = \mathrm{e}^{-tx} E\left[\mathrm{e}^{tY_n^x}\right],$$

where $E\left[\mathrm{e}^{tY_n^x}\right]$ is the Laplace transform of the gamma distribution (2.20) given by

$$E\left[\mathrm{e}^{tY_n^x}\right] = \int_0^{+\infty} \mathrm{e}^{ty} g_{n,x}(y)\mathrm{d}y = \left(\frac{\frac{n^2\sigma^2}{(n\sigma^2+r)}}{\frac{n^2\sigma^2}{(n\sigma^2+r)} - tx}\right)^n, \quad t < \frac{n^2\sigma^2}{(n\sigma^2 + r)x}.$$

The moments $E[(Y_n^x - x)^k]$ of $Y_n^x - x$ of order $k = 0, \ldots, m-1$ can be evaluated by applying formula (2.21) with $G_{n,x}(t)$ instead of $G(t)$. Then (2.19) soon leads to (2.15) and (2.16).

Remark 2.5. It is worthwhile focusing the reader's attention upon the probabilistic interpretation of the approximation formula (2.3).

First note that the kernel function $K_n(x, y)$ given in (2.5) may be rewritten as

$$K_n(x, y) = \left(1 - \frac{\rho}{n}\right) g_{n,x}(y) \quad (x > 0, y \geq 0), \tag{2.22}$$

where $g_{n,x}$ is the density function (2.20).

For all $n \geq \rho$, let us consider a sequence of continuous random variables $Y_n(h), h = 0, 1, 2, \ldots$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with values in $[0, +\infty[$. Further, suppose that the one-step transition distribution function, i.e., the probability distribution of $Y_n(h) \in A$ for a fixed Borel set A, is given by

$$\mu_{n,x}(A) := \int_A g_{n,x}(y)\mathrm{d}y \quad (x > 0), \tag{2.23}$$

provided that $Y_n(h-1) = x$.

Next, choose a sequence $(\epsilon_n)_{n\geq 1}$ such that $\epsilon_n > 0$, $\lim_{n\to+\infty} \epsilon_n = 0$ and $\lim_{n\to+\infty} \frac{1}{n\epsilon_n} = \sigma^2$, and define the process $X_n := \{X_n(t); t \geq 0\}$ as

$$\begin{cases} X_n(0) = Y_n(0), \\ X_n(t) = Y_n(k(n)), \quad t > 0, \end{cases} \tag{2.24}$$

where we have set $k(n) := \left[\frac{t}{\epsilon_n}\right]$, $(k(n))_{n\geq 1}$ which is, in this way, a sequence of positive integers such that $\lim_{n\to+\infty} \frac{k(n)}{n} = \sigma^2 t$.

Referring to a concrete case, we are thinking of a particle which executes a *random walk* in the following way: if the particle is in the state $x \in [0, +\infty[$ at time $h\epsilon_n$ $(h = 0, 1, 2, \ldots)$, then it remains at x during the interval $[h\epsilon_n, (h+1)\epsilon_n[$, and, at time $(h+1)\epsilon_n$, "jumps", so the probability that it goes to any Borel set A is $\mu_{n,x}(A)$.

If a probability distribution on $[0, +\infty[$ for the initial position $Y_n(0)$ of the particle is fixed, then the sequence of continuous random variables $Y_n(h)$, $h = 0, 1, 2, \ldots,$ may be regarded as a discrete parameter Markov process $Y_n = \{Y_n(h)\}_{h\geq 0}$ with continuous state space $[0, +\infty[$.

Moreover, taking into account (2.22) and (2.23), we may rewrite (2.4) as

$$\begin{aligned} Q_n f(x) &= \left(1 - \frac{\rho}{n}\right) \int_0^{+\infty} f(y) g_{n,x}(y) \mathrm{d}y = \left(1 - \frac{\rho}{n}\right) \int_0^{+\infty} f(y) \mu_{n,x}(\mathrm{d}y) \\ &= \left(1 - \frac{\rho}{n}\right) E[f(Y_n(h)) | Y_n(h-1) = x]. \end{aligned}$$

In other words, the Markov process Y_n "corresponds" to the operator Q_n which may be rightly called the *one-step transition operator* (for more details concerning the connection between Markov processes and semigroups, the reader may refer, for instance, to [13, Ch. 4]).

As a consequence, the process X_n defined in (2.24) "corresponds" to the iterate $Q_n^{k(n)}$ of the operator Q_n, i.e., $Q_n^{k(n)}$ is the *transition operator after* $k(n)$ *steps*: indeed, one gets

$$\begin{aligned} Q_n^{k(n)} f(x) &= \left(1 - \frac{\rho}{n}\right)^{k(n)} E[f(Y_n(k(n))) | Y_n(0) = x] \\ &= \left(1 - \frac{\rho}{n}\right)^{k(n)} E[f(X_n(t)) | X_n(0) = x]. \end{aligned} \tag{2.25}$$

Now let $\Delta_{\epsilon_n} Y_n(h) = Y_n(h+1) - Y_n(h)$ $(h = 0, 1, 2 \ldots)$ be the increment in the process Y_n over a time interval of length ϵ_n. Referring to Remark 2.4, all the (possible) conditional moments of $\Delta_{\epsilon_n} Y_n(h)$ can be written as

$$E[(\Delta_{\epsilon_n} Y_n(h))^k | Y_n(h) = x] = E[(Y_n^x - x)^k]$$

for any $k \geq 1$ and $x > 0$, which soon yields, in particular and in view of (2.13) and (2.14),

$$\begin{aligned} &\lim_{\epsilon_n\to 0^+} \frac{E\left[\Delta_{\epsilon_n} Y_n(h) | Y_n(h) = x\right]}{\epsilon_n} = rx, \\ &\lim_{\epsilon_n\to 0^+} \frac{E\left[(\Delta_{\epsilon_n} Y_n(h))^2 | Y_n(h) = x\right]}{\epsilon_n} = \sigma^2 x^2, \\ &\lim_{\epsilon_n\to 0^+} \frac{E\left[(\Delta_{\epsilon_n} Y_n(h))^k | Y_n(h) = x\right]}{\epsilon_n} = 0, \quad k \geq 3. \end{aligned} \tag{2.26}$$

From (2.25) and [13, Theorem 2.6], it follows that formula (2.3) implies the existence of a Markov process $X = \{X(t); t \geq 0\}$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with state space $[0, +\infty[$, "corresponding" to the semigroup $(S_m(t))_{t\geq 0}$ and such that for any $t \geq 0$ the random variable $X_n(t)$ converges in distribution to the random variable $X(t)$ as $n \to +\infty$ or, equivalently,

$$\lim_{n\to+\infty} E[f(X_n(t))|X_n(0) = x] = E[f(X(t))|X(0) = x], \tag{2.27}$$

for any $f \in E_m^0$, $x \geq 0$ and $t > 0$.

The conditions (2.26) guarantee that X is a continuous path Markov process, i.e., an Itô diffusion with drift coefficient $a(x) := rx$ and diffusion coefficient $b(x) := \sigma x$ (see [16, Ch. 15, Section 1]).

The diffusion X can also be constructed as the unique (in law) solution of the stochastic differential equation

$$dX(t) = rX(t)dt + \sigma X(t)dW(t), \qquad X(0) = x \quad (t > 0, x > 0),$$

where $\{W(t); t \geq 0\}$ is a one-dimensional Brownian motion, whose sample paths are explicitly given (see [26, pp. 62–63]) by

$$X(t) = x \cdot \mathrm{e}^{(r-\frac{\sigma^2}{2})t+\sigma W(t)}, \quad t > 0.$$

This means that for any $f \in E_m^0$, $t > 0$ and $x > 0$, one has

$$E[f(X(t))|X(0) = x] = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f\left(x \cdot \mathrm{e}^{(r-\frac{\sigma^2}{2})t+\sigma y}\right) \mathrm{e}^{\frac{-y^2}{2t}} \mathrm{d}y.$$

Combining formulas (2.3), (2.25) and (2.27), we obtain the integral representation (2.3) of the semigroup $(S_m(t))_{t\geq 0}$ appearing in [7].

3. The rate of convergence

The main aim of the present section is to provide an estimate of the rate of convergence of the iterates of the operators Q_n towards the strongly continuous semigroup $(S_m(t))_{t\geq 0}$ generated by the differential operator $(L, D(L_m))$ defined according to (2.1).

In other words, starting from the representation formula (2.3), we are able to give an estimate of the magnitude of $\|(Q_n^{k(n)} - S_m(t))f\|_m$ when f belongs to a suitable "large" subspace of E_m^0.

As far as our methods of investigation allow, we shall conclude that the speed of convergence in (2.3) is rather slow; however, our result must be compared with analogous ones stated in different frameworks (see, e.g., [9,10,14,15,21]) and all looking for some kind of quantitative version of the classical Trotter approximation theory.

Let us start with the following density result which, beyond our specific purposes, has perhaps interest on its own.

Proposition 3.1. *If $m \geq 2$, the space*

$$\mathcal{U} := \{f \in C^4([0, +\infty[) \,|\, \|f^i\|_\infty < +\infty \text{ for } i = 1, \ldots, 4\} \tag{3.1}$$

is dense in $(E_m^0, \|\cdot\|_m)$.

Proof. First of all note that $\mathcal{U} \subset E_m^0$: indeed, if $f \in \mathcal{U}$, then

$$f(x) = f(0) + \int_0^x f'(t)\mathrm{d}t \quad (x \geq 0),$$

which soon entails

$$\frac{|f(x)|}{1+x^m} \le \frac{|f(0)|}{1+x^m} + \frac{x\|f'\|_\infty}{1+x^m} \quad (x \ge 0),$$

and therefore $\lim_{x\to+\infty} \frac{f(x)}{1+x^m} = 0$, i.e., $f \in E_m^0$, as announced. Now let us set

$$\mathcal{T} := \{f \in C^4([0,+\infty[) \mid \text{there exists a neighborhood } V \text{ of } +\infty \text{ such that } f_{|V} \equiv 0\},$$

and denote by $C_0([0,+\infty[)$ the Banach space of all real-valued continuous functions on $[0,+\infty[$ vanishing at $+\infty$, endowed with the sup-norm $\|\cdot\|_\infty$.

Obviously, $\mathcal{T}$ is a subalgebra of $C_0([0,+\infty[)$ which strongly separates the points of $[0,+\infty[$ and, consequently, it is dense in $(C_0([0,+\infty[), \|\cdot\|_\infty)$ by virtue of Stone's generalization of the Weierstrass theorem (see, e.g., [6, Theorem 4.4.4, p. 241]).

On the other hand, $\mathcal{T}$ is easily checked to be invariant under the isometry

$$f \in (C_0([0,+\infty[), \|\cdot\|_\infty) \mapsto f\cdot(1+e_m) \in (E_0^m, \|\cdot\|_m),$$

and this allows us to conclude that $\mathcal{T}$ is dense in $(E_0^m, \|\cdot\|_m)$, too.

But then the same happens *a fortiori* for $\mathcal{U}$, since clearly $\mathcal{T} \subset \mathcal{U}$, and the proof is now complete. □

Remark 3.2. As a consequence, the space $\mathcal{V} := span(\{e_0, e_1, \ldots, e_{m-1}\} \cup \mathcal{U})$ is dense in E_m^0 $(m \ge 2)$.

We are now in a position to state our main result of this section.

Theorem 3.3. *Let us assume $m > 4$. Then for any $f \in \mathcal{U}$, $t > 0$ and any sequence of positive integers such that $\frac{k(n)}{n} \to \sigma^2 t$ as $n \to +\infty$, one has*

$$\|(Q_n^{k(n)} - S_m(t))f\|_m = O\left(\frac{1}{n}\right) \quad \textit{as } n \to +\infty,$$

i.e., there exists a constant $C = C(f,t,\sigma,r)$ such that

$$\|(Q_n^{k(n)} - S_m(t))f\|_m \le \frac{C}{n} \quad \textit{for } n \textit{ large enough.} \tag{3.2}$$

Proof. The proof is rather technical and follows basically the same lines as the proof of a similar result for the classical Bernstein operators, stated in [15]: here, some non-trivial additional difficulties arise, due to the more sophisticated form of the operators under consideration.

Because of its length, we split the proof into different steps.

Let us fix, once and for all, $f \in \mathcal{U}$ and $t > 0$ and set $k(n) := [n\sigma^2 t]$ $(n \ge 1)$: as the reader will quickly realize, this last choice will simplify some facts, without any loss of generality herein.

For any $n \ge \rho$ such that $n\sigma^2 t \ge 2$, as a first step, we decompose the expression as follows:

$$\begin{aligned}\|(Q_n^{[n\sigma^2 t]} - S_m(t))f\|_m \le & \left\|\left(Q_n^{[n\sigma^2 t]} - S_m\left(\frac{[n\sigma^2 t]}{n\sigma^2}\right)\right)f\right\|_m \\ & + \left\|\left(S_m\left(\frac{[n\sigma^2 t]}{n\sigma^2}\right) - S_m(t)\right)f\right\|_m \\ := & \ I_1 + I_2,\end{aligned} \tag{3.3}$$

and I_1 and I_2 have to be estimated separately.

For I_2 it is fairly easy to proceed: in fact, since $f \in \mathcal{U}$, we have easily $f \in D_m(L)$ and, by a standard semigroup argument, we get

$$\begin{aligned} I_2 = \left\| \int_{\frac{[n\sigma^2 t]}{n\sigma^2}}^{t} S_m(s) L f \, \mathrm{d}s \right\|_m &\le \int_{\frac{[n\sigma^2 t]}{n\sigma^2}}^{t} \|S_m(s)\| \cdot \|Lf\|_m \mathrm{d}s \\ &\le \left(t - \frac{[n\sigma^2 t]}{n\sigma^2} \right) \sup_{\frac{[n\sigma^2 t]}{n\sigma^2} \le s \le t} \|S_m(s)\| \cdot \|Lf\|_m \\ &\le \frac{1}{n\sigma^2} \|S_m(t)\| \cdot \|Lf\|_m \\ &= \frac{1}{n\sigma^2} \mathrm{e}^{(m-1)(\frac{m\sigma^2}{2}+r)t} \cdot \|Lf\|_m, \end{aligned} \tag{3.4}$$

where we have applied formula (2.8), yielding the monotonicity of $\|S_m(t)\|$ with respect to t.

Now let's pass to considering I_1; first observe that, by Proposition 2.2 and the representation formula (2.3), each $S_m(t)$ commutes with Q_n; this circumstance, together with Proposition 2.3 and again (2.8), leads to

$$\begin{aligned} I_1 &= \left\| \left(Q_n^{[n\sigma^2 t]} - S_m^{[n\sigma^2 t]} \left(\frac{1}{n\sigma^2} \right) \right) f \right\|_m \\ &= \left\| \sum_{i=0}^{[n\sigma^2 t]-1} \left(Q_n^{[n\sigma^2 t]-1-i} S_m^i \left(\frac{1}{n\sigma^2} \right) \right) \left(Q_n - S_m \left(\frac{1}{n\sigma^2} \right) \right) f \right\|_m \\ &\le \sum_{i=0}^{[n\sigma^2 t]-1} \|Q_n\|^{[n\sigma^2 t]-1-i} \cdot \left\| S_m \left(\frac{i}{n\sigma^2} \right) \right\| \cdot \left\| \left(Q_n - S_m \left(\frac{1}{n\sigma^2} \right) \right) f \right\|_m \\ &= \sum_{i=0}^{[n\sigma^2 t]-1} a_{n,m}^{[n\sigma^2 t]-1-i} \cdot \mathrm{e}^{(m-1)(\frac{m\sigma^2}{2}+r)\frac{i}{n\sigma^2}} \cdot \left\| \left(Q_n - S_m \left(\frac{1}{n\sigma^2} \right) \right) f \right\|_m, \end{aligned} \tag{3.5}$$

where $a_{n,m}$ is explicitly known according to (2.6). Using $1 + t \le \mathrm{e}^t$ $(t \in \mathbb{R})$ yields

$$a_{n,m} \le \mathrm{e}^{\frac{(m-1)\rho}{n} + \frac{(m-1)m}{2n}},$$

which entails a uniform (with respect to i) upper bound of the form

$$a_{n,m}^{[n\sigma^2 t]-1-i} \cdot \mathrm{e}^{(m-1)(\frac{m\sigma^2}{2}+r)\frac{i}{n\sigma^2}} \le \mathrm{e}^{\frac{([n\sigma^2 t]-1)(m-1)}{n}(\rho+\frac{m}{2})}, \quad i = 0, \ldots, [n\sigma^2 t] - 1.$$

So we may estimate (3.5) and, summing up, we have just proved that

$$\begin{aligned} I_1 &\le \frac{1}{n} \cdot \frac{[n\sigma^2 t]}{n} \mathrm{e}^{\frac{([n\sigma^2 t]-1)(m-1)}{n}(\rho+\frac{m}{2})} \cdot \left\| n^2 \left(Q_n - S_m \left(\frac{1}{n\sigma^2} \right) \right) f \right\|_m \\ &\le \frac{\sigma^2 t}{n} \cdot \mathrm{e}^{\sigma^2 t(m-1)(\rho+\frac{m}{2})} \cdot \left\| n^2 \left(Q_n - S_m \left(\frac{1}{n\sigma^2} \right) \right) f \right\|_m. \end{aligned} \tag{3.6}$$

It is now obvious that we need a nice upper bound for $\left\| n^2 \left(Q_n - S_m \left(\frac{1}{n\sigma^2} \right) \right) f \right\|_m$, too, and it is to this goal that the remaining part of the proof is mainly addressed.

As before, first we decompose the expression further as follows:

$$\left\| n^2\left(Q_n - S_m\left(\frac{1}{n\sigma^2}\right)\right) f\right\|_m \leq \left\| n^2\left(Q_n f - f - \frac{1}{n\sigma^2}Lf\right)\right\|_m + \left\| n^2\left(S_m\left(\frac{1}{n\sigma^2}\right) f - f - \frac{1}{n\sigma^2}Lf\right)\right\|_m, \tag{3.7}$$

and then try to estimate the two terms on the right hand side separately.

As far as the first term is concerned, fix $x \geq 0$; then, by Taylor's formula, one readily finds

$$\left| f(t) - f(x) - (t-x)f'(x) - \frac{(t-x)^2}{2}f''(x) - \frac{(t-x)^3}{6}f'''(x)\right| \leq \frac{(t-x)^4}{24}\|f'^{v}\|_\infty,$$

for every $t \geq 0$, and consequently

$$\left| Q_n f(x) - M_{n,0}(x)f(x) - M_{n,1}(x)f'(x) - M_{n,2}(x)\frac{f''(x)}{2} - M_{n,3}(x)\frac{f'''(x)}{6}\right| \leq M_{n,4}(x)\frac{\|f'^{v}\|_\infty}{24},$$

where the moments $M_{n,i}(x)$ are given by (2.15).

Hence, in view of (2.15) and (2.16), it is not a difficult task to show that

$$\begin{aligned}\frac{n^2}{1+x^m}&\left| Q_n f(x) - f(x) - \frac{1}{n\sigma^2}Lf(x)\right| \\ &\leq \frac{n^2}{1+x^m}\left|\frac{\rho^2}{n^2}xf'(x) - \left(\frac{\rho+\rho^2}{n^2} - \frac{\rho^2+\rho^3}{n^3} - \frac{\rho^3}{n^4}\right)x^2\frac{f''(x)}{2}\right| \\ &\quad + n^2 d_{n,3}\frac{x^3}{1+x^m}\frac{\|f'''\|_\infty}{6} + n^2 d_{n,4}\frac{x^4}{1+x^m}\frac{\|f'^{v}\|_\infty}{24},\end{aligned} \tag{3.8}$$

and here the term on the right hand side is bounded from above by a constant $M = M(f, \sigma, r)$. Indeed, $f \in \mathcal{U}$ and, by (2.18), the sequences $(n^2 d_{n,3})_{n\geq\rho}$ and $(n^2 d_{n,4})_{n\geq\rho}$ are bounded, the same being true for $\frac{x^i}{1+x^m}$ $(i = 3, 4)$, because $m > 4$ by assumption.

Since x was arbitrarily chosen, taking the supremum with respect to x in (3.8) leads to the following final estimate:

$$\left\| n^2\left(Q_n f - f - \frac{1}{n\sigma^2}Lf\right)\right\|_m \leq M. \tag{3.9}$$

Lastly, when dealing with the second term on the right hand side in (3.7), by virtue of a general result of semigroup theory (see, e.g., [27, proof of Lemma 2.8, p. 7]), we already know that

$$\|S_m(t)f - f - tLf\|_m \leq \frac{t^2}{2}\|S_m(t)\| \cdot \|L^2 f\|_m \quad \text{for every } t \geq 0, \tag{3.10}$$

since $f \in D_m(L^2)$ as well.

For $t := \frac{1}{n\sigma^2}$, by (2.8) we soon deduce

$$\begin{aligned}\left\| n^2\left(S_m\left(\frac{1}{n\sigma^2}\right) f - f - \frac{1}{n\sigma^2}Lf\right)\right\|_m &\leq \frac{1}{2\sigma^4}e^{(m-1)(\frac{m\sigma^2}{2}+r)\frac{1}{n\sigma^2}} \cdot \|L^2 f\|_m \\ &\leq \frac{1}{2\sigma^4}e^{(m-1)(\rho+\frac{m}{2})} \cdot \|L^2 f\|_m.\end{aligned} \tag{3.11}$$

In conclusion, combining (3.3), (3.4), (3.6), (3.7), (3.9) and (3.11) completes the job and we finally obtain

$$\begin{aligned}\|(Q_n^{[n\sigma^2 t]} - S_m(t))f\|_m \leq\ & \frac{C}{n} \\ :=\ & \frac{1}{n}\left[\sigma^2 t \cdot \mathrm{e}^{\sigma^2 t(m-1)(\rho+\frac{m}{2})}\left(M + \frac{1}{2\sigma^4}\mathrm{e}^{(m-1)(\rho+\frac{m}{2})} \cdot \|L^2 f\|_m\right)\right. \\ & \left. + df1\sigma^2 \mathrm{e}^{(m-1)(\frac{m\sigma^2}{2}+r)t} \cdot \|Lf\|_m\right], \qquad (3.12)\end{aligned}$$

as desired. □

Remark 3.4. As a careful inspection of the proof shows, the assumptions $m > 4$ and $f \in \mathcal{U}$ play an essential role. Indeed, they both guarantee $f \in D_m(L^2)$, which is crucial in our approach, specifically in estimate (3.10).

Remark 3.5. We point out that the terms $\|Lf\|_m$ and $\|L^2 f\|_m$ appearing in (3.12) may be estimated in turn. In fact, a straightforward computation allows us to write

$$\begin{aligned}\|Lf\|_m &\leq \frac{\sigma^2}{2}\left(\frac{2}{m-2}\right)^{2/m} \cdot \frac{m-2}{m}\|f''\|_\infty + r\|f'\|_\infty + r\|f\|_m, \\ \|L^2 f\|_m &\leq \sum_{i=1}^{4} N_i \|f^i\|_\infty + r\|Lf\|_m,\end{aligned} \qquad (3.13)$$

where $N_i = N_i(\sigma, r)$ $(i = 1, 2, 3, 4)$.

In addition, the second estimate could be arranged in a better way, applying Landau's inequality to the derivatives (see, e.g., [22, p. 138]).

Since this does not meaningfully improve (3.12), we skip any further consideration in this respect.

In the next corollary we see how the class of functions for which (3.2) holds true may be considerably expanded.

Corollary 3.6. *As in Remark 3.2, let us set* $\mathcal{V} = \mathrm{span}(\{e_0, e_1, \ldots, e_{m-1}\} \cup \mathcal{U})$. *Then, under the assumption that* $m > 4$, *the estimate (3.2) still holds true for any* $f \in \mathcal{V}$.

Proof. Indeed, for a selected $p = 0, \ldots, m-1$, by a direct computation (apply (2.6), (2.9) and expand in Taylor series the relevant exponential (2.7)) one finds

$$\lim_{n\to+\infty} n^2 \left\|\left(Q_n - S_m\left(\frac{1}{n\sigma^2}\right)\right)e_p\right\|_m = \left|\frac{(p-1)p(1-2p)}{12} - \frac{(p+1)\rho^2}{2}\right| \cdot \|e_p\|_m,$$

and therefore $n^2\left\|\left(Q_n - S_m\left(\frac{1}{n\sigma^2}\right)\right)e_p\right\|_m$ is bounded from above by a constant, say K, for all $n \geq \rho$.

Arguing as in the proof of Theorem 3.3, after substituting e_p in place of f in (3.3), (3.4) and (3.6), we immediately get $\left\|\left(Q_n^{[n\sigma^2 t]} - S_m(t)\right)e_p\right\|_m \leq C/n$; here

$$C = \sigma^2 t \cdot \mathrm{e}^{\sigma^2 t(m-1)(\rho+\frac{m}{2})} \cdot K + \frac{1}{\sigma^2}\mathrm{e}^{(m-1)(\frac{m\sigma^2}{2}+r)t} \cdot \|Le_p\|_m,$$

where, in turn, $e_p \in D_m(L)$ and $\|Le_p\|_m = (p-1)\left(\frac{p\sigma^2}{2}+r\right)\cdot\left(\frac{p}{m-p}\right)^{p/m}\cdot\frac{(m-p)}{m}$.

The result is now fully established. □

4. Asymptotic behaviour

This section supplies additional information about the operators Q_n, namely about the asymptotic behaviour of $Q_n f(x)$ and, more generally, of the iterates $Q_n^k f(x)$ as $x \to +\infty$.

In this way we are able to pursue some results which are an improvement on those described in [7, Section 3], merely as regards the asymptotic behaviour of the semigroup $(S_m(t))_{t\geq 0}$ as $t \to +\infty$.

Let us start with the following result, which must be compared with Theorem 3.1 of [7].

Theorem 4.1. *If $f \in E_m^0$ $(m \geq 2)$ has the asymptote $y = ax + b$ as $x \to +\infty$, then each $Q_n f$ has the asymptote $y = \left(1 - \frac{\rho^2}{n^2}\right)ax + \left(1 - \frac{\rho}{n}\right)b$ as $x \to +\infty$. Consequently, for every integer $k \geq 2$, the iterate $Q_n^k f$ has the asymptote $y = \left(1 - \frac{\rho^2}{n^2}\right)^k ax + \left(1 - \frac{\rho}{n}\right)^k b$ as $x \to +\infty$.*

Proof. A change of variable in (2.4) is needed here, in order to map $Q_n f$ into a form easier to handle in this framework: namely, the reader will find no difficulty in showing that for all $n \geq \rho$ and $x \geq 0$,

$$Q_n f(x) = \left(1 - \frac{\rho}{n}\right)\frac{1}{\Gamma(n)} \cdot \int_0^{+\infty} \mathrm{e}^{-s} s^{n-1} f\left(\frac{(n\sigma^2+r)x}{n^2\sigma^2}s\right)\mathrm{d}s,$$

and, clearly,

$$\frac{Q_n f(x)}{1+x} = \left(1 - \frac{\rho}{n}\right)\frac{1}{\Gamma(n)} \cdot \int_0^{+\infty} \mathrm{e}^{-s} s^{n-1} \frac{f\left(\frac{(n\sigma^2+r)x}{n^2\sigma^2}s\right)}{1+\frac{(n\sigma^2+r)x}{n^2\sigma^2}s} \cdot \frac{1+\frac{(n\sigma^2+r)x}{n^2\sigma^2}s}{1+x}\mathrm{d}s. \tag{4.1}$$

By assumption, the function $y \mapsto \frac{f(y)}{1+y}$ is continuous on $[0, +\infty[$ and tends to $|a|$ as $y \to +\infty$, and therefore it is bounded on $[0, +\infty[$ by a constant $M > 0$; furthermore for any $x \geq 0$ we plainly have

$$\frac{1+\frac{(n\sigma^2+r)x}{n^2\sigma^2}s}{1+x} \leq max\left\{1, \frac{(n\sigma^2+r)x}{n^2\sigma^2}s\right\} \quad \text{for all } s \geq 0,$$

so the modulus of the integrand in (4.1) is bounded from above by ψ, where $\psi(s) := \mathrm{e}^{-s}s^{n-1} \cdot M \cdot max\left\{1, \frac{(n\sigma^2+r)x}{n^2\sigma^2}s\right\}$ $(s \geq 0)$. Since $\psi \in L^1([0, +\infty[)$, we may apply Lebesgue's dominated convergence, leading to

$$\begin{aligned}\lim_{x\to+\infty}\frac{Q_n f(x)}{1+x} &= \left(1 - \frac{\rho}{n}\right)\frac{1}{\Gamma(n)} \cdot \int_0^{+\infty} \mathrm{e}^{-s}s^{n-1} a\,\frac{(n\sigma^2+r)s}{n^2\sigma^2}\mathrm{d}s \\ &= \left(1 - \frac{\rho}{n}\right)\left(\frac{n\sigma^2+r}{n^2\sigma^2}\right)\frac{\Gamma(n+1)}{\Gamma(n)}a \\ &= \left(1 - \frac{\rho^2}{n^2}\right)a,\end{aligned}$$

or, equivalently, to

$$\lim_{x\to+\infty}\frac{Q_n f(x)}{1+x}=\left(1-\frac{\rho^2}{n^2}\right)a.$$

On the other hand, for any $x\geq 0$,

$$\begin{aligned}
&Q_n f(x)-\left(1-\frac{\rho^2}{n^2}\right)ax\\
&=\left(1-\frac{\rho}{n}\right)\left[\frac{1}{\Gamma(n)}\cdot\int_0^{+\infty}\mathrm{e}^{-s}s^{n-1}\left(f\left(\frac{(n\sigma^2+r)x}{n^2\sigma^2}s\right)-a\frac{(n\sigma^2+r)x}{n^2\sigma^2}s\right)\mathrm{d}s\right.\\
&\quad\left.+\frac{1}{\Gamma(n)}\cdot\int_0^{+\infty}\mathrm{e}^{-s}s^{n-1}a\,\frac{(n\sigma^2+r)x}{n^2\sigma^2}s\mathrm{d}s-\left(1+\frac{\rho}{n}\right)ax\right]\\
&=\left(1-\frac{\rho}{n}\right)\frac{1}{\Gamma(n)}\cdot\int_0^{+\infty}\mathrm{e}^{-s}s^{n-1}\left(f\left(\frac{(n\sigma^2+r)x}{n^2\sigma^2}s\right)-a\frac{(n\sigma^2+r)x}{n^2\sigma^2}s\right)\mathrm{d}s. \qquad (4.2)
\end{aligned}$$

Arguing as before, let us observe that the function $y\mapsto|f(y)-ay|$, being continuous on $[0,+\infty[$ and converging to $|b|$ as $y\to+\infty$, is bounded. Now again, by virtue of Lebesgue's theorem, we soon get

$$\begin{aligned}
\lim_{x\to+\infty}\left(Q_n f(x)-\left(1-\frac{\rho^2}{n^2}\right)ax\right)&=\lim_{x\to+\infty}\left(1-\frac{\rho}{n}\right)\frac{1}{\Gamma(n)}\cdot\int_0^{+\infty}\mathrm{e}^{-s}s^{n-1}b\mathrm{d}s\\
&=\left(1-\frac{\rho}{n}\right)b,
\end{aligned}$$

which concludes the proof of the first part of the statement. The second part (concerning the iterate $Q_n^k f$) is straightforward and needs no further comment. □

Remark 4.2. Let $\phi(x):=(x-K)^+(x\geq 0, K>0)$. Then (see [7, Section 6]) ϕ is a positive, increasing and convex function in E_m^0 $(m\geq 2)$, having $y=x-K$ as asymptote as $x\to+\infty$.

According to the previous theorem (with $a=1$ and $b=-K$), $Q_n\phi$ has the asymptote $y=\left(1-\frac{\rho^2}{n^2}\right)x-\left(1-\frac{\rho}{n}\right)K$ $(x\to+\infty)$ and, more generally, for any integer $u\geq 1$, the iterate $Q_n^u\phi$ has the asymptote

$$y=\left(1-\frac{\rho^2}{n^2}\right)^u x-\left(1-\frac{\rho}{n}\right)^u K. \qquad (4.3)$$

Moreover, from $e_1-Ke_0\leq\phi\leq e_1$, by virtue of (2.6) we soon deduce

$$a_{n,1}^u e_1-Ka_{n,0}^u e_0\leq Q_n^u\phi\leq a_{n,1}^u e_1,$$

whence

$$\left(\left(1-\frac{\rho^2}{n^2}\right)^u e_1-K\left(1-\frac{\rho}{n}\right)^u e_0\right)^+\leq Q_n^u\phi\leq\left(1-\frac{\rho^2}{n^2}\right)^u e_1, \qquad (4.4)$$

because $\phi\geq 0$ anyway.

It is worthwhile noticing that if $u=k(n)$ and $\frac{k(n)}{n}\to\sigma^2 t$ as $n\to+\infty$, passing to the limit as $n\to+\infty$ in (4.3) and (4.4), we get (ii) and (iii) of Corollary 6.1 in [7], i.e., each $S_m(t)f$ has the asymptote $y=x-K\mathrm{e}^{-rt}$ as $x\to+\infty$ and

$$(e_1-K\mathrm{e}^{-rt}e_0)^+\leq S_m(t)\phi\leq e_1\quad\text{for all } t\geq 0.$$

The next proposition resembles Proposition 6.1 of [7], dealing with the iterates $Q_n^u\phi$ instead of the limiting semigroup $(S_m(t))_{t\geq 0}$.

Recall that, in general, $Q_n f(0) = \left(1-\frac{\rho}{n}\right) f(0)$, so $Q_n\phi(0) = 0$.

Here, as in [7, Proposition 6.1], we set $h(p) := (p-1)p^{p/1-p}$ $(p > 1)$.

Proposition 4.3. *If the positive constant K appearing in ϕ fulfills $K \geq h(p)$ for some $p \in]1, +\infty[$, then*

$$0 \leq Q_n^u\phi \leq a_{n,p}^u e_p \quad (n \geq \rho), \tag{4.5}$$

for any integer $u \geq 1$, where $a_{n,p}$ is defined according to (2.6) even for $p > m$. It follows that $\frac{\mathrm{d}}{\mathrm{d}x} Q_n^u\phi(x)|_{x=0} = 0$. If $K \geq 1$, (4.5) remains true for all $p > 1$.

Proof. The proof runs in the same way as in [7, Proposition 6.1]. However, we impose no upper bound for p, since $Q_n e_p$ may be naturally computed even if $p > m$ (see the discussion after (2.8)). □

Remark 4.4. The situation described so far has an interesting geometric interpretation.

If $t \geq 0$ is fixed and $(k(n))_{n\geq 1}$ is an arbitrary sequence of positive integers satisfying $\frac{k(n)}{n} \to \sigma^2 t$ as $n \to +\infty$, after putting

$$\begin{aligned} q_n^{k(n)}(x) &:= \left(1-\frac{\rho^2}{n^2}\right)^{k(n)} x - \left(1-\frac{\rho}{n}\right)^{k(n)} K \quad (x \geq 0, n \geq \rho), \\ s_t(x) &:= x - K\mathrm{e}^{-rt} \quad (x \geq 0), \end{aligned} \tag{4.6}$$

we readily see that $\lim_{n\to+\infty} q_n^{k(n)}(x) = s_t(x)$ for all $x \geq 0$, i.e., not surprisingly, as $n \to +\infty$ the asymptote of the iterate $Q_n^{k(n)}\phi$ tends to the asymptote of $S_m(t)\phi$, in agreement with the representation formula (2.3) of the semigroup itself.

The above situation may be portrayed as in the following Fig. 1, where we have taken into account that $Q_n^{k(n)}\phi(0) = S_m(t)\phi(0) = 0$ with derivatives equal to 0 (see [7, Corollary 6.1 and Proposition 6.1]).

In Section 3 we have shown that the rate of convergence in (2.3) is of order $1/n$ for $m > 4$ and for regular functions in $\mathcal{V}$.

In terms of the classical semigroup theory (see, e.g., [12,27]), this means that *the strong solution of the Black–Scholes problem*

$$\begin{cases} u_t(x,t) = \dfrac{1}{2}\sigma^2 x^2 u_{xx}(x,t) + rxu_x(x,t) - ru(x,t) & (x \geq 0, t > 0), \\ u(x,0) = f(x) & (x \geq 0, f \in \mathcal{V} \subset D_m(L)), \end{cases} \tag{4.7}$$

which, as is well-known, is given by

$$u(x,t) := S_m(t)f(x) \quad (x, t \geq 0),$$

may be written down in terms of iterates of the Gamma-type operators Q_n with order of approximation $1/n$ (in $\|\cdot\|_m$).

Since $\phi \notin D_m(L)$ (even if $\phi \in E_m^0$), the function $v(x,t) := S_m(t)\phi(x)$ $(x, t \geq 0)$ is *a mild solution* of (4.7) and, anyway, neither Theorem 3.3 nor Corollary 3.6 may be applied in this case.

We point out that choosing ϕ as the initial datum in (4.7) is quite standard in the Black–Scholes model and has some interest in many concrete cases (see, for instance, [18,30]); therefore it is our

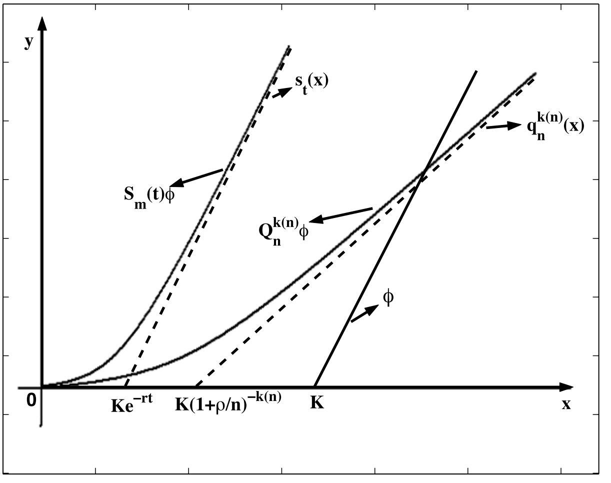

Fig. 1. The asymptotes $q_n^{k(n)}(x)$ and $s_t(x)$ of $Q_n^{k(n)}\phi$ and $S_m(t)\phi$, respectively, at a fixed time t.

intention to supply some information concerning the rate of approximation in (2.3) for $f = \phi$, too, exactly as already done for functions in $\mathcal{V}$.

We think that an estimate of the magnitude of $\|q_n^{k(n)} - s_t\|_m$ would suit nicely in this direction, since, according to the definition (4.6), the two terms $q_n^{k(n)}$ and s_t present well the features of $Q_n^{k(n)}\phi(x)$ and $S_m(t)\phi(x)$ at least for x large enough.

What we find out perfectly matches (3.2), as proved in the next proposition, where, for simplicity and without loss of generality, we may and do assume $k(n) := [n\sigma^2 t]$ $(t > 0, n \geq 1)$.

Proposition 4.5. *If $m \geq 2$, then for all $t > 0$ one has*

$$\|q_n^{[n\sigma^2 t]} - s_t\|_m = O\left(\frac{1}{n}\right) \quad \text{as } n \to +\infty,$$

i.e., there exists a constant $C = C(t, \sigma, r)$ such that

$$\|q_n^{[n\sigma^2 t]} - s_t\|_m \leq \frac{C}{n} \tag{4.8}$$

for every $n \geq \sup\left\{\rho, \frac{2}{\sigma^2 t}\right\}$.

Proof. Fix $t > 0$; then for all $n \geq \sup\left\{\rho, \frac{2}{\sigma^2 t}\right\}$ we simply have

$$\begin{aligned}\|q_n^{[n\sigma^2 t]} - s_t\|_m &\leq \|e_1\|_m \cdot \left|1 - \left(1 - \frac{\rho^2}{n^2}\right)^{[n\sigma^2 t]}\right| + K\|e_0\|_m \cdot \left|\mathrm{e}^{-rt} - \left(1 - \frac{\rho}{n}\right)^{[n\sigma^2 t]}\right| \\ &\leq [n\sigma^2 t]\frac{\rho^2}{n^2} + K \cdot \left|\left(\mathrm{e}^{\frac{-rt}{[n\sigma^2 t]}}\right)^{[n\sigma^2 t]} - \left(1 - \frac{\rho}{n}\right)^{[n\sigma^2 t]}\right| \\ &\leq \frac{\rho r t}{n} + K[n\sigma^2 t] \cdot \left|\mathrm{e}^{\frac{-rt}{[n\sigma^2 t]}} - 1 + \frac{\rho}{n}\right|,\end{aligned} \tag{4.9}$$

where we have used the obvious inequality $|a^n - b^n| \leq n|a - b|$, $0 \leq a, b < 1$.

Moreover, applying $e^{-t} - 1 + t \le \frac{t^2}{2}$ $(t > 0)$ and $n\sigma^2 t \ge 2$ (which is true by assumption) implies

$$[n\sigma^2 t] \cdot \left| e^{\frac{-rt}{[n\sigma^2 t]}} - 1 + \frac{\rho}{n} \right| \le [n\sigma^2 t] \cdot \left| e^{\frac{-rt}{[n\sigma^2 t]}} - 1 + \frac{rt}{[n\sigma^2 t]} \right| + [n\sigma^2 t] \cdot \left| \frac{rt}{[n\sigma^2 t]} - \frac{rt}{n\sigma^2 t} \right|$$
$$\le [n\sigma^2 t] \frac{1}{2} \cdot \frac{r^2 t^2}{[n\sigma^2 t]^2} + [n\sigma^2 t] rt \cdot \frac{1}{[n\sigma^2 t] n\sigma^2 t}$$
$$= \frac{1}{2} \cdot \frac{r^2 t^2}{[n\sigma^2 t]} + \frac{\rho}{n} \le \frac{1}{2} \cdot \frac{r^2 t^2}{n\sigma^2 t - 1} + \frac{\rho}{n} \le \frac{\rho r t^2}{nt} + \frac{\rho}{n} = \frac{\rho(rt+1)}{n}.$$

Substituting into (4.9) yields

$$\|q_n^{[n\sigma^2 t]} - s_t\|_m \le \frac{\rho r t}{n} + K \frac{\rho(rt+1)}{n},$$

whence we have (4.8) with $C := \rho\,(r(1+K)t + K)$. □

Remark 4.6. We point out that, if $f \in E_m^0$ $(m \ge 2)$ has the asymptote $y = ax + b$ as $x \to +\infty$, then after rewriting $q_n^{k(n)}(x)$ and $s_t(x)$ appearing in (4.6) in an obvious way according to Theorem 4.1 and to Theorem 3.1 of [7], the estimate (4.8) remains basically the same up to a slight change in the expression for the constant C, which becomes equal to $\rho\,(r(|a| + |b|)t + |b|)$.

Actually, we could have proved (4.8) in this general framework and then applied it to the case $f := \phi$; due to the importance of the function $\phi(x) = (x - K)^+$, often occurring in the theory of option pricing, and in connection with the spirit of the present paper, we have preferred to focus our attention right away upon this special case, for which, as already said, the result concerning the rate of convergence fails to hold.

Remark 4.7. It would be interesting to have a closer look into the asymptotic behaviour both of $Q_n f(x)$ and of $S_m(t) f(x)$ as $x \to +\infty$, starting from some assumptions concerning the growth rate at $+\infty$ of f.

At the same time, the limit of the overiterates of $Q_n f$, i.e., the limit of $Q_n^{k(n)} f$ as $n \to +\infty$ under the assumption that $k(n)/n \to +\infty$, perhaps deserves to be considered and studied.

All the foregoing aspects, however, leading further afield, will be addressed in a forthcoming paper.

References

[1] J. Adell, J. de la Cal, Using stochastic processes for studying Bernstein-type operators, Rend. Circ. Mat. Palermo 33 (Suppl. II) (2005) 17–33.

[2] J. Adell, J. de la Cal, On a Bernstein-type operator associated with the inverse Polya–Eggenberger distribution, Rend. Circ. Mat. Palermo 33 (Suppl. II) (2005) 143–154.

[3] F. Altomare, R. Amiar, Approximation by positive operators of the C_0-semigroups associated with one-dimensional diffusion equations, Part II, Numer. Funct. Anal. Optim. 26 (2005) 17–33.

[4] F. Altomare, R. Amiar, Corrigendum to: approximation by positive operators of the C_0-semigroups associated with one-dimensional diffusion equations, Part II, Numer. Funct. Anal. Optim. 26 (2005) 17–33;
F. Altomare, R. Amiar, Corrigendum to: Approximation by positive operators of the C_0-semigroups associated with one-dimensional diffusion equations, Part II, Numer. Funct. Anal. Optim. 27 (2006) 497–498.

[5] F. Altomare, A. Attalienti, Degenerate evolution equations in weighted continuous function spaces, Markov processes and the Black–Scholes equation, Part II, Results Math. 42 (2002) 212–228.

[6] F. Altomare, M. Campiti, Korovkin-type Approximation Theory and its Applications, W. de Gruyter, Berlin, New York, 1994.
[7] A. Attalienti, I. Rasa, Shape-preserving properties and asymptotic behaviour of the semigroup generated by the Black–Scholes operator, Czechoslovak Math. J. 58 (133) (2008) 457–467.
[8] P. Billingsley, Probability and Measure, 2nd ed., John Wiley & Sons, New York, 1986.
[9] M. Campiti, C. Tacelli, Rate of convergence in Trotter's approximation theorem, Constr. Approx. 28 (3) (2008) 333–341;
M. Campiti, C. Tacelli, Erratum to: Rate of convergence in Trotter's approximation theorem, Constr. Approx. 31 (3) (2010) 459–462.
[10] M. Campiti, C. Tacelli, Approximation processes for resolvent operators, Calcolo 45 (2008) 235–245.
[11] J. de la Cal, F. Luquin, A note on limiting properties of some Bernstein-type operators, J. Approx. Theory 68 (1992) 322–329.
[12] K. Engel, R. Nagel, One-parameter Semigroups for Linear Evolution Equations, Springer-Verlag, Berlin, 2000.
[13] S.N. Ethier, T.G. Kurtz, Markov Processes. Characterization and Convergence, J. Wiley & Sons, New York, 1986.
[14] H. Gonska, M. Heilmann, I. Rasa, Convergence of iterates of genuine and ultraspherical Durrmeyer operators to the limiting semigroup: C^2-estimates, J. Approx. Theory 160 (1–2) (2009) 243–255.
[15] H. Gonska, I. Rasa, The limiting semigroup of the Bernstein iterates: degree of convergence, Acta Math. Hungar. 111 (2006) 119–130.
[16] S. Karlin, H.M. Taylor, A Second Course in Stochastic Processes, Academic Press, Inc., New York, 1981.
[17] S. Karlin, Z. Ziegler, Iteration of positive approximation operators, J. Approx. Theory 3 (1970) 310–339.
[18] Y.K. Kwok, Mathematical Models of Financial Derivatives, Springer-Verlag, Berlin, 1998.
[19] A. Lupaş, D.H. Mache, V. Maier, M.V. Müller, Certain results involving Gamma operators, in: M.V. Müller, et al. (Eds.), New Developments in Approximation Theory. 2nd International Dortmund Meeting (IDoMAT)'98, Germany, February 23–27, 1998, in: ISNM, Int. Ser. Numer. Math., vol. 132, Birkäuser, Basel, 1999, pp. 199–214.
[20] A. Lupaş, M.V. Müller, Approximationseigenschaften der Gammaoperatoren, Math. Z. 98 (1967) 208–226.
[21] E.M. Mangino, I. Rasa, A quantitative version of Trotter's approximation theorem, J. Approx. Theory 146 (2) (2007) 149–156.
[22] D.S. Mitrinović, Analytic Inequalities, Springer-Verlag, Berlin, 1970.
[23] M.V. Müller, Die Folge der Gamma operatoren, Dissertation, Stuttgart, 1967.
[24] M.V. Müller, Punktweise und gleichmäßige Approximation durch Gammaoperatoren, Math. Z. 103 (1968) 227–238.
[25] M.V. Müller, Einige Approximationseigenschaften der Gammaoperatoren, Mathematica 10 (33) (1968) 303–310.
[26] B. Øksendal, Stochastic Differential Equations, 4th ed., Springer-Verlag, Berlin, 1995.
[27] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, New York, 1995.
[28] E.C. Titchmarsh, The Theory of Functions, Oxford Science Publications, 1997.
[29] V. Totik, The gamma operators in L^p spaces, Publ. Math. Debrecen 32 (1985) 43–55.
[30] P. Wilmott, S. Howison, J. Dewynne, The Mathematics of Financial Derivatives, Cambridge University Press, 1995.