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A generalized Lefschetz number for local Nielsen fixed point theory

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Abstract

Let X be a connected, finite dimensional, locally compact polyhedron. Let $f: U \to X$ be a compactly fixed map defined on an open, connected subset U of X, and let H be any normal subgroup of $\pi_1(X)$. We seek information about $N_H(f)$, the local H-Nielsen number of f. It is a lower bound for $\min\{|\text{Fix } g| : g \simeq f\}$, where the homotopies must be admissible.

Let $N_H(f;\ \tilde{f}, \tilde{\imath})$ denote the well-known sum $\Sigma_{\alpha \in W} i(N_H^\alpha)[\alpha]$, where $i(N_H^\alpha)$ is the local fixed point index of an H-Nielsen class, $[\alpha]$ is the Reidemeister orbit associated with that class and W is a set of representatives of the Reidemeister orbits. Then $N_H(f)$ is the number of terms of $N_H(f;\ \tilde{f}, \tilde{\imath})$ with nonzero coefficient. We call $N_H(f;\ \tilde{f}, \tilde{\imath})$ a Nielsen–Reidemeister chain, and we prove that for certain subsets of U, $N_H(f;\ \tilde{f}, \tilde{\imath})$ splits into the sum of the Nielsen–Reidemeister chains for the subsets.

We define the local generalized H-Lefschetz number $L_H(f;\ \tilde{f}, \tilde{\imath})$ in terms of a globally defined trace. We prove that, for X a connected, triangulable n-manifold with $n \geqslant 3$, $L_H(f; \tilde{f}, \tilde{\imath}) = N_H(f;\ \tilde{f}, \tilde{\imath})$. Thus, $L_H(f;\ \tilde{f}, \tilde{\imath})$ can provide a means to compute $N_H(f)$. Also, for $H = 1$, a generalization of the converse of the Lefschetz fixed point theorem holds.

Key words: Local Nielsen number; Lefschetz number; Generalized Lefschetz number; Topological fixed point theory; Nielsen fixed point theory

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1. Introduction

The generalized Lefschetz number is a globally defined trace that provides information about the Nielsen number of a map. (See [3,9].) In [4], Fadell and Husseini introduce local Nielsen theory and use covering spaces to study it. We combine these topics by defining a local setting for which a *local* generalized H-Lefschetz number can be defined. We prove that, under certain hypotheses, the local generalized H-Lefschetz number provides information about local H-Nielsen theory.

Let X be a connected, finite dimensional, locally compact polyhedron. Let $f: U \to X$ be a map defined on an open, connected subset U of X. The fixed point set of f is $\text{Fix } f = \{x \in U: f(x) = x\}$, and we consider only maps f with Fix f compact. Local Nielsen fixed point theory involves the estimation of min f, the minimum number of fixed points of any map homotopic to f via an admissible homotopy. The local Nielsen number of f is a lower bound for min f. For H a normal subgroup of $\pi_1(X)$, the local H-Nielsen number of f is often easier to calculate than the local Nielsen number of f. (See [9,12].) The local H-Nielsen number of f is a lower bound for the local Nielsen number of f and therefore provides less information. Local H-Nielsen theory is defined for *any* $H \lhd \pi_1(X)$. This is different from the usual H-Nielsen theory (with $U = X$) for which H must be invariant under $f_{\#}$.

As described above, our initial data is $U \subseteq X$ with $f: U \to X$ a compactly fixed map and $H \lhd \pi_1(X)$. In Section 2, we obtain information about the local H-Nielsen number of f from these initial data by constructing a local setting as follows. We begin by choosing a compact subset K of U with $\text{Fix } f \subseteq \text{int } K$ and such that the H-Nielsen equivalences in K are the same as those in U. The regular covering space for X, $\tilde{X}$, is determined by H. A regular covering space for K, $\tilde{K}$, is chosen for which there exist lifts of $f \mid K$ and of the inclusion to the covering spaces. This collection of covering spaces and lifts is called a local setting.

Once a local setting is constructed, the Nielsen–Reidemeister chain for f is defined. It is the formal sum of distinct Reidemeister orbits with each coefficient equal to the index of the associated local H-Nielsen class (determined by coincidence classes of the lifts). This familiar sum is denoted by $N_H(f;\, \tilde{f}, \tilde{\imath})$, and the number of terms with nonzero coefficients is the local H-Nielsen number of f. We show that $N_H(f;\, \tilde{f}, \tilde{\imath})$ is essentially independent of the choices made in the local setting. We then consider the splitting of $N_H(f;\, \tilde{f}, \tilde{\imath})$ when K is replaced by a finite number of disjoint compact connected subsets of U. Here we require Fix f to be contained in the union of the interiors of the subsets.

In Section 3, we define the local generalized H-Lefschetz number of f, denoted by $L_H(f;\, \tilde{f}, \tilde{\imath})$, to be the alternating sum of a trace-like function that is defined on simplicial chains. It provides a method for studying fixed points using a globally defined trace. To define $L_H(f;\, \tilde{f}, \tilde{\imath})$, we assign X, $\tilde{X}$, K, and $\tilde{K}$ compatible triangulations so that K is a subcomplex of X and the projection maps are simplicial maps that preserve orientation. By subdividing K and $\tilde{K}$ and taking simplicial approximations to f and the lifts, we are able to discuss the algebra needed to define $L_H(f;\ \tilde{f},\ \tilde{\imath})$. We prove that $L_H(f;\ \tilde{f},\ \tilde{\imath})$ is essentially independent of the choices made in its definition, and we prove an additivity property.

We are then able to connect our two main objects of study, the Nielsen–Reidemeister chain and the local H-Lefschetz number. We prove that, if X is a connected, triangulable n-manifold with $n \geqslant 3$, then we have $L_H(f;\ \tilde{f},\ \tilde{\imath}) = N_H(f;\ \tilde{f},\ \tilde{\imath})$. Thus if $L_H(f;\ \tilde{f},\ \tilde{\imath})$ can be written in reduced form with each Reidemeister orbit occurring once (not always an easy task), the local H-Nielsen number of f can be calculated in terms of global trace. This result also implies that, for $H=1$, $L_1(f;\ \tilde{f},\ \tilde{\imath})$ is related to the local obstruction defined in [4]. This local obstruction is the obstruction to min f being zero. This relationship provides a generalized version of the converse of the Lefschetz fixed point theorem. Examples of the computation of $L_H(f;\ \tilde{f},\ \tilde{\imath})$ are contained in [8].

Many of the results presented here are parts of the authors' dissertations, [5] and [7]. In [5], Fares first requires that $\tilde{K} \subseteq \tilde{X}$. With that restriction, not all sets of initial data $\{U,\ X,\ f,\ H\}$ have lifts of f and of the inclusion. Thus a local setting need not exist and the local H-Lefschetz number is not always defined. In [7], Hart chooses $\tilde{K}$ not necessarily contained in $\tilde{X}$, which forces consideration of coincidence points of lifts rather than fixed points. But with these changes the local generalized H-Lefschetz number can be defined for *all* sets of initial data.

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2. Local H-Nielsen fixed point theory and covering spaces

2.1. Preliminaries

Let X be a connected, finite dimensional, locally compact polyhedron. For U an open, connected subset of X, we define $\partial U = \overline{U} \cap \overline{X - U}$. Let $f: U \to X$ be a map with Fix $f = \{x \in U:\ f(x) = x\}$, the set of fixed points of f. We consider only maps for which Fix f is compact, and we say f is compactly fixed.

Let $x_0 \in U$ be a base point for X, and let H be a normal subgroup of $\pi_1(X,\ x_0)$. Let x be any point in U. As in [12], we define a normal subgroup H_x of $\pi_1(X,\ x)$ corresponding to H. Let λ be a path from x_0 to x. Then λ induces an isomorphism from $\pi_1(X,\ x_0)$ to $\pi_1(X,\ x)$. For every loop class $[\sigma] \in \pi_1(X,\ x_0)$, σ is sent to $\lambda^{-1} * \sigma * \lambda \in \pi_1(X,\ x)$. Let H_x be the image of H under this isomorphism. Then H_x is a normal subgroup of $\pi_1(X,\ x)$. Note that H_x is the same for any choice of path λ.

Local H-Nielsen theory involves partitioning the fixed points of f into equivalence classes. (See [9,12].) Two fixed points, x and y, are in the same equivalence class if and only if there exists a path ω in U from x to y for which the loop $(f \circ \omega) * \omega^{-1}$ is in a loop class in H_x. The equivalence classes of fixed points are local H-Nielsen classes for f. Each class is assigned an integer called the local index of the class. (See [6].)

A class of fixed points of f with nonzero index is called essential, because it cannot be removed by a deformation of f without introducing new fixed points. A homotopy $h: U \times I \to X$ is admissible if $\bigcup_{t \in I} \text{Fix } h_t$ is a compact subset of U. There is the expected one-to-one correspondence between the essential classes of f and the essential classes of g, whenever $g \simeq f$ via an admissible homotopy. The local H-Nielsen number of f, denoted by $N_H(f)$, is the number of essential local H-Nielsen classes. Thus $N_H(f)$ is invariant under admissible homotopy. We have $N_H(f) \leqslant \min\{|\text{Fix } g|: g \simeq f\}$, where the homotopies must be admissible.

A subset K of U is an (f, U)-subset of X if it is a compact, connected subset of U with Fix $f \subseteq \text{int } K$. For K an (f, U)-subset of X, two fixed points $x, y \in K$ are in the same local $(K; H)$-Nielsen class for f if and only if there exists a path ω in K from x to y for which the loop $(f \circ \omega) * \omega^{-1}$ is in a loop class in H_x. The local $(K; H)$-Nielsen number of f, denoted by $N_{(K;H)}(f)$, is the number of local $(K; H)$-Nielsen classes with index different from zero.

References for Nielsen fixed point theory (when $U = X$) are [1,10,11]. Local Nielsen fixed point theory is introduced in [4].

2.2. *(H, f)-admissible covering spaces for an (f, U)-subset of X*

Let $\tilde{X}$ be a regular covering space for X with $\tilde{\pi}_X$ the group of covering transformations for the covering projection $\tilde{p}_X: \tilde{X} \to X$. Let $i: K \hookrightarrow X$ be the inclusion map.

Definition 2.2.1 ((H, f)-admissible cover for K). Let $f: U \to X$ be compactly fixed, with $H \lhd \pi_1(X)$. Let K be an (f, U)-subset of X, and let $\tilde{X}$ be the regular covering space for X for which $\tilde{\pi}_X \cong \pi_1(X)/H$. A regular covering space $\tilde{K}$ of K with covering projection $\tilde{p}_K$ is (H, f)-admissible if there exist maps $\tilde{\imath}$ and $\tilde{f}$ for which $\tilde{p}_X \tilde{f} = f \tilde{p}_K$ and $\tilde{p}_X \tilde{\imath} = i \tilde{p}_K$.

The maps $\tilde{f}$ and $\tilde{\imath}$ are said to be lifts of f and i, respectively.

Remarks. (1) Let $J = \tilde{p}_{K\#}(\pi_1(\tilde{K}))$. The group of covering transformations for $\tilde{p}_K$ is $\tilde{\pi}_K = \pi_1(K)/J$. Note that $\tilde{K}$ is (H, f)-admissible if and only if $i_\#(J)$ and $f_\#(J)$ are contained in H.

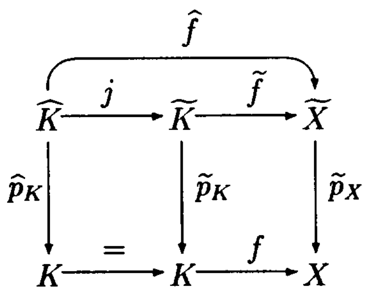
(2) Let $\hat{K}$ be the universal covering space of K with covering projection $\hat{p}_K$. Then $J = 1$, and $\hat{K}$ is an (H, f)-admissible cover for every normal subgroup H of $\pi_1(X)$ and every $f: U \to X$ for which Fix $f \subseteq \text{int } K$. Thus the set of (H, f)-admissible covers for a given f and a given H is always nonempty. This is an improvement over the local settings defined in [5], where there is not always a lift of f.

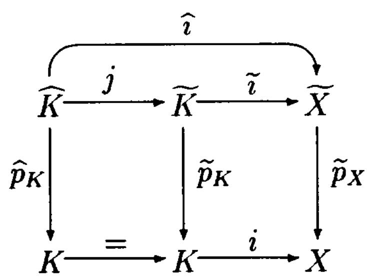
(3) Let $\hat{X}$ be the universal covering space for X. Then $H=1$. Let $\tilde{K}$ be any $(1,f)$-admissible cover for K. For any normal subgroup B of $\pi_1(X)$, $\tilde{K}$ is also a (B, f)-admissible covering space for K.

Base points and choices of lifts. As before, $\hat{K}$ and $\hat{X}$ denote the universal covers of K and X, respectively. Let $\tilde{K}$ be an (H, f)-admissible cover, and choose a base point x_0 in K. Let $\tilde{x}_0$ and $\hat{x}_0$ be base points for $\tilde{K}$ and $\hat{K}$, respectively, with $\tilde{p}_K(\tilde{x}_0)=x_0$ and $\hat{p}_K(\hat{x}_0)=x_0$. Let $\hat{\imath}:\hat{K}\to\tilde{X}$ and $\tilde{\imath}:\tilde{K}\to\tilde{X}$ be lifts of the inclusion $i:K\hookrightarrow X$ with $\tilde{\imath}(\tilde{x}_0)=\hat{\imath}(\hat{x}_0)\in\tilde{X}$.

Let $j:\hat{K}\to\tilde{K}$ be the covering projection satisfying $j(\hat{x}_0)=\tilde{x}_0$. Then $\hat{p}_K=\tilde{p}_K j$. Choose lifts $\hat{f}:\hat{K}\to\tilde{X}$ and $\tilde{f}:\tilde{K}\to\tilde{X}$ of f such that $\hat{f}(\hat{x}_0)=\tilde{f}(\tilde{x}_0)$. Then $\hat{f}(\hat{x}_0)=\tilde{f}j(\hat{x}_0)$, and $\hat{f}=\tilde{f}j$. Similarly, $\hat{\imath}=\tilde{\imath}j$.

These choices of lifts determine the following commutative diagrams.





Covering groups and induced homomorphisms. Let $\hat{\pi}_K$, $\tilde{\pi}_K$, $\hat{\pi}_X$ and $\tilde{\pi}_X$ be the covering groups for $\hat{p}_K$, $\tilde{p}_K$, $\hat{p}_X$ and $\tilde{p}_X$, respectively. Then $\tilde{\pi}_K=\hat{\pi}_K/\tilde{p}_{K\#}(\pi_1(\tilde{K}))$, and $\tilde{\pi}_X=\hat{\pi}_X/\tilde{p}_{X\#}(\pi_1(\tilde{X}))$. Here $H=\tilde{p}_{X\#}(\pi_1(\tilde{X}))$.

Each lift of f to $\tilde{K}$ may be written uniquely as $\alpha\tilde{f}$ for some $\alpha\in\tilde{\pi}_X$. Analogous statements are true for $\hat{f}$, $\tilde{\imath}$, and $\hat{\imath}$. The map $\hat{f}$ induces a homomorphism $\hat{\phi}:\hat{\pi}_K\to\tilde{\pi}_X$ given by $\hat{f}(\sigma\hat{y})=\hat{\phi}(\sigma)\hat{f}(\hat{y})$ for all $\sigma\in\hat{\pi}_K$ and all $\hat{y}\in\hat{K}$. Similarly, $\tilde{f}$ induces a homomorphism $\tilde{\phi}:\tilde{\pi}_K\to\tilde{\pi}_X$ given by $\tilde{f}(\tau\tilde{y})=\tilde{\phi}(\tau)\tilde{f}(\tilde{y})$ for all $\tau\in\tilde{\pi}_K$ and all $\tilde{y}\in\tilde{K}$. The map j induces a homomorphism $\psi:\hat{\pi}_K\to\tilde{\pi}_K$ given by $j(\sigma\hat{y})=\psi(\sigma)j(\hat{y})$ for all $\sigma\in\hat{\pi}_K$ and all $\hat{y}\in\hat{K}$. The map j is a lift of the identity map on K. Thus ψ is the same as the canonical quotient map from $\hat{\pi}_K$ to $\tilde{\pi}_K$.

The lifts $\hat{\imath}$ and $\tilde{\imath}$ of the inclusion $i:K\hookrightarrow X$ induce homomorphisms $\hat{\xi}:\hat{\pi}_K\to\tilde{\pi}_X$ and $\tilde{\xi}:\tilde{\pi}_K\to\tilde{\pi}_X$, respectively. These homomorphisms are determined, as expected, by the following formulae. For all $\sigma\in\hat{\pi}_K$ and $\hat{y}\in\hat{K}$, $\hat{\imath}(\sigma\hat{y})=\hat{\xi}(\sigma)\hat{\imath}(\hat{y})$. For all $\tau\in\tilde{\pi}_K$ and $\tilde{y}\in\tilde{K}$, $\tilde{\imath}(\tau\tilde{y})=\tilde{\xi}(\tau)\tilde{\imath}(\tilde{y})$. It can be shown that $\hat{\phi}=\tilde{\phi}\psi$ and $\hat{\xi}=\tilde{\xi}\psi$.

2.3. Reidemeister orbits and coincidence classes

The Reidemeister action. The Reidemeister action of $\tilde{\pi}_K$ on $\tilde{\pi}_X$ is given by the following group action. For any $\tau\in\tilde{\pi}_K$ and any $\alpha\in\tilde{\pi}_X$,

$$\tau\cdot\alpha=\tilde{\xi}(\tau)\alpha\tilde{\phi}(\tau^{-1}).$$

Let $[\alpha]$ denote the orbit of α under the Reidemeister action, and let $R(\tilde{\phi}, \tilde{\xi})$ denote the set of Reidemeister orbits.

In the usual Nielsen theory (with $U = X$), the Nielsen number is determined by considering Fix $\alpha\tilde{f}$ for each $\alpha \in \tilde{\pi}_X$. We do not require $\tilde{K}$ to be a subset of $\tilde{X}$, so we must consider sets of coincidences rather than sets of fixed points. Let

$$\text{Coin}(\alpha\tilde{f}, \tilde{\imath}) = \{\tilde{x} \in \tilde{K} \colon \alpha\tilde{f}(\tilde{x}) = \tilde{\imath}(\tilde{x})\}.$$

The following proposition is proven for $H = 1$ and $\tilde{K} = \hat{K}$ in [4]. The proof for H any normal subgroup of $\pi_1(X, x_0)$ and $\tilde{K}$ any (H, f)-admissible covering space is similar.

Proposition 2.3.1. *Let $\tilde{K}$ be an (H, f)-admissible covering space for K. For $\alpha \in \tilde{\pi}_X$, the set $\tilde{p}_K(\text{Coin}(\alpha\tilde{f}, \tilde{\imath}))$ either is a local $(K;\ H)$-Nielsen class for f or is the empty set. Let β be in $\tilde{\pi}_X$. If $\text{Coin}(\alpha\tilde{f}, \tilde{\imath}) \neq \emptyset$ and $\text{Coin}(\beta\tilde{f}, \tilde{\imath}) \neq \emptyset$, we have*

$$\tilde{p}_K(\text{Coin}(\alpha\tilde{f}, \tilde{\imath})) = \tilde{p}_K(\text{Coin}(\beta\tilde{f}, \tilde{\imath}))$$

if and only if α and β are in the same Reidemeister orbit.

2.4. Nielsen–Reidemeister chains

Definition 2.4.1 (*Set of Reidemeister representatives*). Let $R(\tilde{\phi}, \tilde{\xi})$ be the set of Reidemeister orbits. A set of Reidemeister representatives is a subset of $\tilde{\pi}_X$ containing exactly one element of each orbit in $R(\tilde{\phi}, \tilde{\xi})$.

If $\text{Coin}(\alpha\tilde{f}, \tilde{\imath})$ is nonempty, let $N^{\alpha}_{(K;H)}$ denote the local $(K;\ H)$-Nielsen class that is equal to $\tilde{p}_K(\text{Coin}(\alpha\tilde{f}, \tilde{\imath}))$.

In Definition 2.4.2, we introduce notation for a formal sum that is a well-known part of Nielsen fixed point theory.

Definition 2.4.2 (*The $(H, K, \tilde{f}, \tilde{\imath})$-NR chain for f*). Let W be a set of Reidemeister representatives. The $(H, K, \tilde{f}, \tilde{\imath})$-NR (Nielsen–Reidemeister) chain for f is

$$N_{(K;H)}(f;\ \tilde{f}, \tilde{\imath}) = \sum_{\alpha \in W} i(N^{\alpha}_{(K;H)})[\alpha].$$

Here $i(N^{\alpha}_{(K;H)})$ is the local index of the $(K;\ H)$-Nielsen class $N^{\alpha}_{(K;H)}$, and $[\alpha]$ is the Reidemeister orbit containing α.

Note that $N_{(K;H)}(f;\ \tilde{f}, \tilde{\imath})$ is an element of $\mathbb{Z}(R(\tilde{\phi}, \tilde{\xi}))$, the free Abelian group generated by $R(\tilde{\phi}, \tilde{\xi})$. The local $(K;\ H)$-Nielsen number of f, $N_{(K;H)}(f)$, is equal to the number of terms in $N_{(K;H)}(f;\ \tilde{f}, \tilde{\imath})$ with coefficient different from zero. The local Lefschetz number of f, $\lambda(f)$, is independent of K and H and is the sum of the coefficients in $N_{(K;H)}(f;\ \tilde{f}, \tilde{\imath})$. (See [6].)

Next we consider the effect on an NR chain of replacing $\tilde{f}$ with $\gamma\tilde{f}$ for some $\gamma \in \tilde{\pi}_X$. Note that $\gamma\tilde{f}$ induces the homomorphism $\gamma\tilde{\phi}(\cdot)\gamma^{-1}: \tilde{\pi}_K \to \tilde{\pi}_X$. For any $\alpha \in \tilde{\pi}_X$, let $[\alpha]_{\tilde{f}}$ and $[\alpha]_{\gamma\tilde{f}}$ be the orbits of α under the Reidemeister actions induced by $\tilde{f}$ and $\gamma\tilde{f}$, respectively. Note that if W is a set of Reidemeister representatives for $R(\tilde{\phi}, \tilde{\xi})$, then $W\gamma^{-1}$ is a set of Reidemeister representatives for $R(\gamma\tilde{\phi}(\cdot)\gamma^{-1}, \tilde{\xi})$.

Proposition 2.4.3. *Let W be a set of Reidemeister representatives for $R(\tilde{\phi}, \tilde{\xi})$. For $\alpha \in \tilde{\pi}_X$, we have $N_{(K;H)}(f; \tilde{f}, \tilde{\imath})$ equal to $N_{(K;H)}(f; \gamma\tilde{f}, \tilde{\imath})$ in the following sense. For any $\alpha \in W$, the coefficient of $[\alpha]_{\tilde{f}}$ in $N_{(K;H)}(f; \tilde{f}, \tilde{\imath})$ equals the coefficient of $[\alpha\gamma^{-1}]_{\gamma\tilde{f}}$ in $N_{(K;H)}(f; \gamma\tilde{f}, \tilde{\imath})$. A similar statement is true when $\tilde{\imath}$ is replaced by $\gamma\tilde{\imath}$.*

The proposition follows when we note that

$$\begin{aligned} N_{(K;H)}(f; \gamma\tilde{f}, \tilde{\imath}) &= \sum_{\theta \in W\gamma^{-1}} i\Big(\tilde{p}_K\big(\operatorname{Coin}(\theta\gamma\tilde{f}, \tilde{\imath})\big)\Big)[\theta]_{\gamma\tilde{f}} \\ &= \sum_{\alpha \in W} i\big(N^{\alpha}_{(K;H)}\big)\big[\alpha\gamma^{-1}\big]_{\gamma\tilde{f}}. \end{aligned}$$

Independence from the choice of (H, f)-admissible cover for K. Given a normal subgroup H of $\pi_1(X)$, the covering space $\tilde{X}$ is determined up to homeomorphism. We prove that once the choices of H and K are made, the resulting local $(K; H)$-Nielsen fixed point theory is independent of the choice of (H, f)-admissible cover for K. To do this, we compare an (H, f)-admissible cover $\tilde{K}$ with the universal cover $\hat{K}$.

The geometric approach. Recall that $j: \hat{K} \to \tilde{K}$ is a covering map with $\hat{f} = \tilde{f}j$, $\hat{\imath} = \tilde{\imath}j$ and $\hat{p}_K = \tilde{p}_K j$.

Proposition 2.4.4. *For all $\alpha \in \tilde{\pi}_X$,*

$$\tilde{p}_K\big(\operatorname{Coin}(\alpha\tilde{f}, \tilde{\imath})\big) = \hat{p}_K\big(\operatorname{Coin}(\alpha\hat{f}, \hat{\imath})\big).$$

Proof. It suffices to prove that $\operatorname{Coin}(\alpha\tilde{f}, \tilde{\imath}) = j(\operatorname{Coin}(\alpha\hat{f}, \hat{\imath}))$.

Let $\tilde{y} \in \operatorname{Coin}(\alpha\tilde{f}, \tilde{\imath})$, and let $\hat{y}$ be any point in $j^{-1}(\tilde{y})$. Then

$$\begin{aligned} \alpha\hat{f}(\hat{y}) &= \alpha\tilde{f}j(\hat{y}) = \alpha\tilde{f}(\tilde{y}) \\ &= \tilde{\imath}(\tilde{y}) = \tilde{\imath}j(\hat{y}) \\ &= \hat{\imath}(\hat{y}). \end{aligned}$$

Therefore $j^{-1}(\operatorname{Coin}(\alpha\tilde{f}, \tilde{\imath})) \subseteq \operatorname{Coin}(\alpha\hat{f}, \hat{\imath})$.

Let $\hat{z} \in \operatorname{Coin}(\alpha\hat{f}, \hat{\imath})$. Then

$$\begin{aligned} \alpha\tilde{f}j(\hat{z}) &= \alpha\hat{f}(\hat{z}) = \hat{\imath}(\hat{z}) \\ &= \tilde{\imath}j(\hat{z}). \end{aligned}$$

Therefore $j(\hat{z}) \in \operatorname{Coin}(\alpha\tilde{f}, \tilde{\imath})$ and $j(\operatorname{Coin}(\alpha\hat{f}, \hat{\imath})) = \operatorname{Coin}(\alpha\tilde{f}, \tilde{\imath})$. □

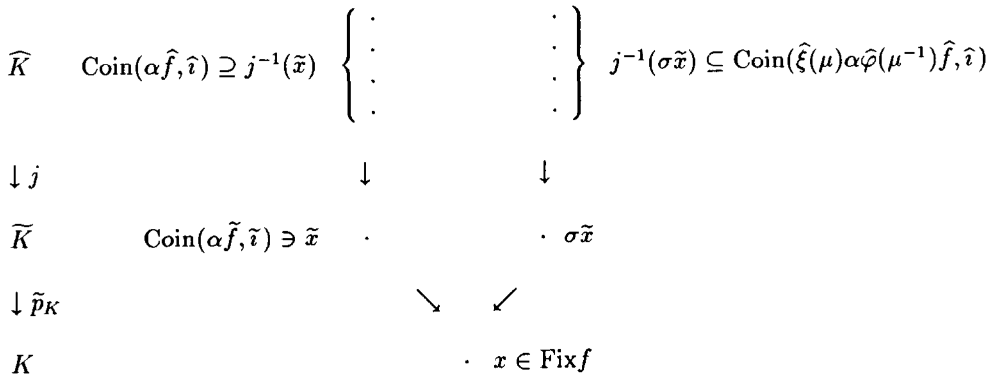


Fig. 1. Fibers.

For $x \in \operatorname{Fix} f$ and $\tilde{x}$ in $\tilde{p}_K^{-1}(x)$, all points in $j^{-1}(\tilde{x})$ are in the same coincidence class. For $\sigma \in \tilde{\pi}_K$ and for all $\mu \in \psi^{-1}(\sigma)$, we have the diagram given in Fig. 1. The previous proposition implies that no information is gained or lost if an (H, f)-admissible cover $\tilde{K}$ is replaced by the universal cover $\hat{K}$. Thus any (H, f)-admissible cover for K may be used to compute the local $(K; H)$-Nielsen number of f.

The algebraic approach. For $\sigma \in \hat{\pi}_K$, $\tau \in \tilde{\pi}_K$ and $\alpha \in \tilde{\pi}_X$, the Reidemeister actions are given by

$$\sigma \cdot \alpha = \hat{\xi}(\sigma)\alpha\hat{\phi}(\sigma^{-1}) \quad \text{and} \quad \tau \cdot \alpha = \tilde{\xi}(\tau)\alpha\tilde{\phi}(\tau^{-1}).$$

Let $\bar{\sigma}$ be the coset of $J = \tilde{p}_{K\#}(\pi_1(\tilde{K}))$ in $\hat{\pi}_K$ that contains σ. Note that

$$\sigma \cdot \alpha = \tilde{\xi}(\bar{\sigma})\alpha\tilde{\phi}(\bar{\sigma}^{-1}).$$

Thus if σ, $\mu \in \hat{\pi}_K$ with $\bar{\sigma} = \bar{\mu} \in \tilde{\pi}_K$, we have $\sigma \cdot \alpha = \mu \cdot \alpha$. Thus $R(\hat{\phi}, \hat{\xi}) = R(\tilde{\phi}, \tilde{\xi})$. By Proposition 2.4.4, the index of the local H-Nielsen class for f associated with a given $\alpha \in \tilde{\pi}_X$ for the setting involving $\tilde{K}$ is the same as the index of the local H-Nielsen class for f associated with α for the setting involving $\hat{K}$. Thus

$$N_{(K;H)}(f;\, \tilde{f}, \tilde{\imath}) = N_{(K;H)}(f;\, \hat{f}, \hat{\imath}).$$

For K any (f, U)-subset of X and H any normal subgroup of $\pi_1(X)$, the local $(K; H)$-Nielsen theory for f is independent of the choice of cover $\tilde{K}$ for K, provided $\tilde{K}$ is (H, f)-admissible. This repeats the result from the geometric approach.

Different choices of (f, U)-subsets K can produce different local $(K; H)$-Nielsen numbers $N_{(K;H)}(f)$. But in Section 2.5 we prove that, for K sufficiently large, any (f, U)-subset M of X containing K has $N_{(M;H)}(f) = N_{(K;H)}(f)$. We will call such a sufficiently large subset K a stable subset of X.

2.5. *The stability of NR-chains*

We consider (f, U)-subsets K of X that are large enough to contain all the information about local Nielsen classes that U contains. That is, we consider K for which $N_{(K;H)}(f) = N_H(f)$. We call such an (f, U)-subset stable and prove that every (f, U)-subset is contained in a stable (f, U)-subset. Recall that $N_H(f)$ is a lower bound for the size of the fixed point set of any $g: U \to X$ homotopic to f via an admissible homotopy on U.

Definition 2.5.1 (*Stable subset of X*). An (f, U)-subset K of X is stable if, for all $\delta \in \tilde{\pi}_X$, $N^{\delta}_{(K;H)}$ is equal to a local H-Nielsen class for f on U. This forces $N_{(K;H)}(f) = N_H(f)$.

A sufficient condition for an (f, U)-subset K to be stable is that the homomorphism induced by the inclusion of $U - K$ into X take $\pi_1(U - K)$ into H.

Proposition 2.5.2. *For every (f, U)-subset K of X, there exists S, a stable (f, U)-subset of X, with $K \subseteq S$.*

Proof. Let n be the number of local H-Nielsen classes for f on U. Let m be the number of local $(K; H)$-Nielsen classes. We have $0 \leqslant n \leqslant m < \infty$. Note that each local $(K; H)$-Nielsen class is contained in exactly one local H-Nielsen class (for f on U) and intersects no other local H-Nielsen class. If $n = m$, we are done. If $n < m$, then there is at least one local H-Nielsen class that is the union of more than one distinct local $(K; H)$-Nielsen classes. Choose $m - n$ paths λ in U, each with $[(f \circ \lambda) * \lambda^{-1}] \in H$ and $\lambda(t) \cap \text{Fix } f = \emptyset$ for all $t \in (0, 1)$, that connect the appropriate local $(K; H)$-Nielsen classes and group them into local H-Nielsen classes.

Let S be the union of K and these $m - n$ paths. Then S is connected and compact with $\text{Fix } f \subseteq \text{int } S$ and $S \subseteq U$. The local $(S; H)$-Nielsen classes are exactly the local H-Nielsen classes for f on U. Thus S is a stable (f, U)-subset of X. □

For K a stable subset of X and for $\alpha \in \tilde{\pi}_X$, let N^{α}_H denote the local H-Nielsen class previously known as $N^{\alpha}_{(K;H)}$. The $(H, K, \tilde{f}, \tilde{\imath})$-NR chain is independent of K as long as K is a stable (f, U)-subset of X. We use the notation $N_H(f; \tilde{f}, \tilde{\imath})$ for the $(H, \tilde{f}, \tilde{\imath})$-NR chain when K is stable.

Proposition 2.5.3. *In our study of local Nielsen fixed point theory, it suffices to consider only those stable (f, U)-subsets K for which $K = \overline{\text{int } K}$. Note that if X is an n-manifold, we have $K = \overline{\text{int } K}$ if and only if K is an n-submanifold of X.*

Proof. Given K an (f, U)-subset of X, we prove that there exists an (f, U)-subset P of X with $K \subseteq P$ and $P = \overline{\text{int } P}$. Thus we prove that we may restrict our study to those stable subsets K of X with $K = \overline{\text{int } K}$.

We have $K - \overline{\text{int } K} \subseteq \partial K$. Thus $\text{Fix } f \subseteq \overline{\text{int } K}$.

For each $x \in K$, let $\varepsilon_x = d(x, X - U)$. Let W_x be an open ball containing x of radius $\varepsilon_x/2$. Choose a finite subcover from this open cover of K, and let W be the union of the open sets in the subcover. Let $P = \overline{W}$. Then P is a compact, connected subset of U with $\text{Fix } f \subseteq \text{int } P$. Thus P is a stable (f, U)-subset of X with $P = \overline{\text{int } P}$. □

Definition 2.5.4 (*Local setting for f*). Let $f: U \to X$ be a map with Fix f compact. A local setting LS(f) for f consists of $\{X, K, f, \tilde{X}, \tilde{K}, \tilde{f}, \tilde{\imath}\}$ as in Definition 2.2.1 with $K = \overline{\text{int } K}$ a stable (f, U)-subset of X and $\tilde{K}$ an (H, f)-admissible covering space for K.

In [5], a local setting is defined with the requirement that $\tilde{K}$ be a component of $\tilde{p}_X^{-1}(K) \subseteq \tilde{X}$. Here, as in [7], we require only that $\tilde{K}$ be an (H, f)-admissible covering space for K. The advantage of this generalization is that for any f and any normal subgroup H, a local setting as in Definition 2.5.4 exists. When $\tilde{K}$ is required to be a component of $\tilde{p}_X^{-1}(K)$, there might not be lifts of f and of i to $\tilde{K}$. Then covering spaces cannot be used to study the local H-Nielsen theory of f. Thus our definition of a local setting provides an important improvement over that of [5].

Cases in which we must consider $\tilde{K} \not\subseteq \tilde{X}$ include the following.

(1) Assume H is a proper, nontrivial, normal subgroup of $\pi_1(X)$. Let K be such that $i_\# \pi_1(K) = H$. Then we have $i_\# \pi_1(K) = \tilde{p}_{X\#} \pi_1(\tilde{X})$. Then if $\tilde{K}$ must be a component of $\tilde{p}_X^{-1}(K)$, we have $\tilde{K}$ homeomorphic to K. Thus if $\pi_1(K)$ is not invariant under $f_\#$ there will be no lift of f and no local setting.

This situation occurs for the torus $X = T^2$. Let $\{a, b\}$ be the usual generators of $\pi_1(T^2)$, and let $H = \langle a \rangle$. Let K be an annulus that winds once around a loop in the loop class a. Choose $\tilde{T}^2$ to be $S^1 \times \mathbb{R}$ with $\pi_1(\tilde{T}^2) = \langle \tilde{a} \rangle$. For $f: K \to T^2$ such that $f_\#(a) = b$, there is no lift of f to $\tilde{K}$.

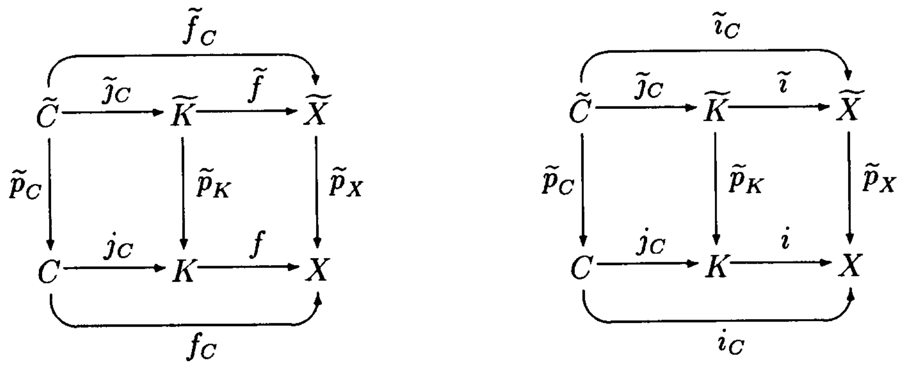
(2) Let K be contractible in X and $\tilde{X}$ be simply connected. If $H = 1$ and $f_\#(\pi_1(K)) \neq 1$, there is no lift of f. See [8] for an example of this involving K a solid torus that is contractible in a lens space X.

2.6. *The splitting of the (H, K, $\tilde{f}$, $\tilde{\imath}$)-NR chains*

Let C be a connected, compact subset of K with $\text{Fix } f \cap \partial C = \emptyset$. The function $f: K \to X$ restricts to $f_C: C \to X$. The local $(K; H)$-Nielsen theory of f is related to the local $(C; H)$-Nielsen theory of f_C. Note that we do not require C to contain all fixed points of f. Choose a point $c \in C$ as a base point for X, for K and for C. Let $j_C: C \hookrightarrow K$ and $i_C: C \hookrightarrow X$ be inclusion maps. Let $\tilde{K}$ be an (H, f)-admissible cover for K. Let $\tilde{C}$ be any (D, j_C)-admissible covering space for C with D a normal subgroup of $\pi_1(K)$ such that $\pi_1(K)/D = \text{cov}(\tilde{p}_K)$. Choose as a base point for $\tilde{C}$ a point $\tilde{c}$ in the fiber above c. Let $\tilde{\jmath}_C$ be a lift of j, and let $\tilde{\jmath}_C(\tilde{c})$ be the base point for $\tilde{K}$.

Let $\tilde{p}_C : \tilde{C} \to C$ be the covering projection. Let $\tilde{\imath}_C = \tilde{\imath}\tilde{\jmath}_C$ and $\tilde{f}_C = \tilde{f}\tilde{\jmath}_C$. The maps $\tilde{\imath}_C$ and $\tilde{f}_C$ are lifts of the inclusion i_C and of f_C, respectively. Thus the space $\tilde{C}$ is an (H, f_C)-admissible cover for C.

The following diagrams commute.



Let $\tilde{\pi}_C = \text{cov}(\tilde{p}_C)$. The maps $\tilde{\imath}_C$, $\tilde{\jmath}_C$ and $\tilde{f}_C$ induce homomorphisms $\tilde{\xi}_C : \tilde{\pi}_C \to \tilde{\pi}_X$, $\tilde{\eta}_C : \tilde{\pi}_C \to \tilde{\pi}_K$ and $\tilde{\phi}_C : \tilde{\pi}_C \to \tilde{\pi}_X$, respectively. Note that $\tilde{\xi}_C = \tilde{\xi}\tilde{\eta}_C$ and $\tilde{\phi}_C = \tilde{\phi}\tilde{\eta}_C$.

We now compare local $(K;\, H)$-Nielsen classes for f with local $(C;\, H)$-Nielsen classes for f_C. Note that $\tilde{\jmath}_C : \tilde{C} \to \tilde{K}$ and $\tilde{\imath} : \tilde{K} \to \tilde{X}$ are not necessarily inclusion maps and that $\tilde{\xi}_C : \tilde{\pi}_C \to \tilde{\pi}_X$ and $\tilde{\xi} : \tilde{\pi}_K \to \tilde{\pi}_X$ are not necessarily monomorphisms.

Proposition 2.6.1. *Let Σ_C be a set of representatives of the right cosets of* im $\tilde{\eta}_C$ *in $\tilde{\pi}_K$. For $\delta \in \tilde{\pi}_X$ and $\sigma \in \tilde{\pi}_K$, let $\sigma \cdot \delta = \tilde{\xi}(\sigma)\delta\tilde{\phi}(\sigma^{-1})$. Then*

$$N^{\delta}_{(K;H)} \cap C = \bigcup_{\sigma \in \Sigma_C} N^{\sigma\cdot\delta}_{(C;H)}.$$

Remark. There may be $\sigma, \tau \in \Sigma_C$ representing different cosets such that $N^{\sigma\cdot\delta}_{(C;H)} = N^{\tau\cdot\delta}_{(C;H)}$. Thus the union is not necessarily a union of distinct local $(C;\, H)$-Nielsen classes. We explore this more after the proof of the proposition.

Proof of Proposition 2.6.1. We prove that

$$N^{\delta}_{(K;H)} \cap C \subseteq \bigcup_{\sigma \in \Sigma_C} N^{\sigma\cdot\delta}_{(C;H)}.$$

Let $x \in N^{\delta}_{(K;H)} \cap C$. Then there exists $\tilde{x} \in \text{Coin}(\delta\tilde{f}, \tilde{\imath}) \subseteq \tilde{K}$ such that $\tilde{p}_K(\tilde{x}) = x$. Let $\tilde{x}_C \in \tilde{p}_C^{-1}(x) \subseteq \tilde{C}$. There exists $\mu \in \tilde{\pi}_K$ such that $\tilde{\jmath}_C(\tilde{x}_C) = \mu\tilde{x}$. We have

$$\tilde{f}(\mu\tilde{x}) = \tilde{\phi}(\mu)\delta^{-1}\tilde{\imath}(\tilde{x}) = \tilde{\phi}(\mu)\delta^{-1}\tilde{\xi}(\mu^{-1})\tilde{\imath}(\mu\tilde{x}),$$

and hence

$$\tilde{\xi}(\mu)\delta\tilde{\phi}(\mu^{-1})\tilde{f}_C(\tilde{x}_C) = \tilde{\imath}_C(\tilde{x}_C).$$

Therefore $x \in N^{\mu\cdot\delta}_{(C;H)}$.

For some $\theta \in \tilde{\pi}_C$ and some $\sigma \in \Sigma_C$, $\mu = \tilde{\eta}_C(\theta)\sigma$. We have

$$\tilde{\xi}(\mu)\delta\tilde{\phi}(\mu^{-1})\tilde{f}_C = \tilde{\xi}_C(\theta)\big(\tilde{\xi}(\sigma)\delta\tilde{\phi}(\sigma^{-1})\big)\tilde{\phi}_C(\theta^{-1})\tilde{f}_C.$$

By Proposition 2.3.1, $x \in N^{\sigma\cdot\delta}_{(C;H)} = N^{\mu\cdot\delta}_{(C;H)}$ with $\sigma \in \Sigma_C$.

Next we prove that for all $\sigma \in \tilde{\pi}_K$ we have

$$N^{\sigma\cdot\delta}_{(C;H)} \subseteq N^{\delta}_{(K;H)}.$$

Let $y \in N^{\sigma\cdot\delta}_{(C;H)}$. Then $y \in C$, and there exists $\tilde{y} \in \operatorname{Coin}(\tilde{\xi}(\sigma)\delta\tilde{\phi}(\sigma^{-1})\tilde{f}_C, \tilde{\imath}_C) \subseteq \tilde{C}$ such that $\tilde{p}_C(\tilde{y}) = y$. We have

$$\tilde{j}_C(\tilde{y}) \in \operatorname{Coin}\big(\tilde{\xi}(\sigma)\delta\tilde{\phi}(\sigma^{-1})\tilde{f}, \tilde{\imath}\big) \subseteq \tilde{K}.$$

Thus

$$y = j_C\tilde{p}_C(\tilde{y}) = \tilde{p}_K\tilde{j}_C(\tilde{y}) \in N^{\sigma\cdot\delta}_{(K;H)} = N^{\delta}_{(K;H)}. \qquad \square$$

Next we consider how to express $N^{\delta}_{(K;H)} \cap C$ as a union of distinct classes for any $\delta \in \tilde{\pi}_X$. Let $\tau, \sigma \in \tilde{\pi}_K$, with $(\tilde{\pi}_K)_\delta$ the isotropy subgroup for δ under the Reidemeister action of $\tilde{\pi}_K$ on $\tilde{\pi}_X$. It can be shown that, for $\operatorname{Coin}((\sigma\cdot\delta)\tilde{f}_C, \tilde{\imath}_C)$ and $\operatorname{Coin}((\tau\cdot\delta)\tilde{f}_C, \tilde{\imath}_C)$ both nonempty,

$$N^{\tau\cdot\delta}_{(C;H)} = N^{\sigma\cdot\delta}_{(C;H)} \quad \Leftrightarrow \quad \sigma \in (\operatorname{im}\tilde{\eta}_C)\tau(\tilde{\pi}_K)_\delta.$$

For any $\delta \in \tilde{\pi}_X$, the set Σ_C can be partitioned into subsets of the form $\Sigma_C \cap (\operatorname{im}\tilde{\eta}_C)\tau(\tilde{\pi}_K)_\delta$ with $\tau \in \Sigma_C$. Let $\mathscr{A}_C(\delta) \subseteq \Sigma_C$ be a set containing one element from each of the above subsets of Σ_C, and let W be a set of Reidemeister representatives for $R(\tilde{\phi}, \tilde{\xi})$. For any $\delta \in \tilde{\pi}_X$, it can be shown that the set $N^{\delta}_{(K;H)} \cap C$ can be expressed as the union of *distinct* local $(C; H)$-Nielsen classes for f_C as follows:

$$N^{\delta}_{(K;H)} \cap C = \bigcup_{\sigma\in\mathscr{A}_C(\delta)} N^{\sigma\cdot\delta}_{(C;H)}.$$

The indices of these classes satisfy

$$i\big(N^{\delta}_{(K;H)} \cap C\big) = \sum_{\sigma\in\mathscr{A}_C(\delta)} i\big(N^{\sigma\cdot\delta}_{(C;H)}\big),$$

and the union

$$W_C := \bigcup_{\delta\in W}\ \bigcup_{\sigma\in\mathscr{A}_C(\delta)} \sigma\cdot\delta$$

is a set of Reidemeister representatives for $R(\tilde{\phi}_C, \tilde{\xi}_C)$.

Let $\{C_i\}_{1,\ldots,n}$ be a family of disjoint, compact, connected subsets of K with $\operatorname{Fix} f \subseteq \bigcup_{i=1}^{n} C_i$ and $\operatorname{Fix} f \cap \partial C_i = \emptyset$ for all i. We use a simplified notation so that, for example, f_i, $\tilde{\phi}_i$ and $\tilde{\eta}_i$ denote f_{C_i}, $\tilde{\phi}_{C_i}$ and $\tilde{\eta}_{C_i}$, respectively. In addition, for each i let Σ_i be a set of representatives of the right cosets of $\operatorname{im}\tilde{\eta}_i$ in $\tilde{\pi}_K$. As above, the set Σ_i can be partitioned into subsets of the form $\Sigma_i \cap (\operatorname{im}\tilde{\eta}_i)\tau(\tilde{\pi}_K)_\delta$ with $\tau \in \Sigma_i$. Let $\mathscr{A}_i(\delta) \subseteq \Sigma_i$ be a set containing one element from each of these subsets of Σ_i. Let $C_{i*} : R(\tilde{\phi}_i, \tilde{\xi}_i) \to R(\tilde{\phi}, \tilde{\xi})$ be the function given by $C_{i*}([\gamma]_{\tilde{f}_i}) = [\gamma]_{\tilde{f}}$ for any $\gamma \in \tilde{\pi}_X$. We extend C_{i*} linearly to $C_{i*} : \mathbb{Z}(R(\tilde{\phi}_i, \tilde{\xi}_i)) \to \mathbb{Z}(R(\tilde{\phi}, \tilde{\xi}))$.

By applying the additivity of the local index once again, we have the following theorem.

Theorem 2.6.2. *Let* $\{C_i\}_{i=1,\dots,n}$ *be a family of subsets of* K *as above. For any* $\delta \in \tilde{\pi}_X$, *the local index of* $N^{\delta}_{(K;H)}$ *splits as follows.*

$$i\left(N^{\delta}_{(K;H)}\right) = \sum_{\sigma \in \mathscr{A}_1(\delta)} i\left(N^{\sigma\cdot\delta}_{(C_1;H)}\right) + \cdots + \sum_{\sigma \in \mathscr{A}_n(\delta)} i\left(N^{\sigma\cdot\delta}_{(C_n;H)}\right).$$

Thus the $(H, K, \tilde{f}, \tilde{\imath})$*-NR chain for* f *also splits. For* W_i *a set of Reidemeister representatives for* $R(\tilde{\phi}_i, \tilde{\xi}_i)$ *and for* W *a set of Reidemeister representatives for* $R(\tilde{\phi}, \tilde{\xi})$,

$$\begin{aligned} &N_{(K;H)}(f;\, \tilde{f}, \tilde{\imath}) \\ &\quad = \sum_{\delta \in W} i\left(N^{\delta}_{(K;H)}\right)[\delta]_{\tilde{f}} \\ &\quad = C_{1*}\left(\sum_{\lambda \in W_1} i\left(N^{\lambda}_{(C_1;H)}\right)[\lambda]_{\tilde{f}_1}\right) + \cdots + C_{n*}\left(\sum_{\lambda \in W_n} i\left(N^{\lambda}_{(C_n;H)}\right)[\lambda]_{\tilde{f}_n}\right) \\ &\quad = C_{1*}\left(N_{(C_1;H)}\left(f_1;\, \tilde{f}_1, \tilde{\imath}_1\right)\right) + \cdots + C_{n*}\left(N_{(C_n;H)}\left(f_n;\, \tilde{f}_n, \tilde{\imath}_n\right)\right). \end{aligned}$$

3. The local generalized *H*-Lefschetz number

We define the local generalized H-Lefschetz number $L_H(f;\, \tilde{f}, \tilde{\imath})$ for a local setting of a map f without the restrictions imposed in [5]. Let LS(f) be a setting for f as in Definition 2.5.4, and let $\overline{K} = \tilde{p}_X^{-1}(K) \subseteq \tilde{X}$.

3.1. The definition of $L_H(f;\, \tilde{f}, \tilde{\imath})$

We identify X with a triangulation of X. For L any simplicial complex that is a subdivision of X, a subcomplex K of L is an (f, U)-subcomplex if the underlying space of K is an (f, U)-subset of X. We do not distinguish between a simplicial complex and its underlying space. All simplices are assumed to be oriented.

We study only those subdivisions L of X for which there is a stable (f, U)-subcomplex. Note that the original triangulation of X is independent of f and U, and the choice of L depends on f and U.

Let LS$(f) = \{L, K, f, \tilde{L}, \tilde{K}, \tilde{f}, \tilde{\imath}\}$ be a local setting for which K is a stable (f, U)-subcomplex of L. The covering spaces $\tilde{L}$ and $\tilde{K}$ inherit simplicial structures from L and K.

Recall that a homotopy $h: U \times I \to X$ is admissible if $\bigcup_{t \in I} \operatorname{Fix} h_t$ is a compact subset of U. Similarly, for K an (f, U)-subset of X, a homotopy $h: K \times I \to X$ is admissible if $\bigcup_{t \in I} \operatorname{Fix} h_t$ is compact in the interior of K.

As in [6], there exists a subdivision L' of L with the following property. Let K' be the subdivision of K that is a subcomplex of L', and let $g: K'' \to L'$ be any simplicial approximation to f defined on some subdivision K'' of K'. Then whenever $s' \in \partial K'$ and v is a vertex of the subdivision of s' induced by K'', we have $g(v) \notin s'$. Thus g has no fixed points on ∂K, and the straight-line homotopy from f to g is admissible.

Let $\tilde{K}''$ and $\tilde{L}'$ be subdivisions of $\tilde{K}$ and $\tilde{L}$ for which the covering projections are simplicial maps that preserve orientation.

The homotopy J between f and g may be lifted to a homotopy $\tilde{J}$ with $\tilde{J}_0 = \tilde{f}$. Let $\tilde{g} = \tilde{J}_1$. The map $\tilde{g}: \tilde{K}'' \to \tilde{L}'$ is a simplicial approximation to $\tilde{f}$. Because $\tilde{\imath}: \tilde{K}' \to \tilde{L}'$ covers the inclusion map $i: K' \hookrightarrow L'$, the map $\tilde{\imath}$ is a simplicial map. Recall that $\bar{K} = \tilde{p}_X^{-1}(K)$. Let $C_q(\cdot; \mathbb{Z})$ be the group of oriented simplicial chains, and let W be the group ring $\mathbb{Z}[\tilde{\xi}(\tilde{\pi}_K)]$. We have

$$C_q(\bar{K}'; \mathbb{Z}) \cong \mathbb{Z}[\tilde{\pi}_X] \otimes_W C_q(\tilde{\imath}(\tilde{K}'); \mathbb{Z}).$$

Therefore $\tilde{\imath}(\tilde{K}')$ is a subcomplex of $\tilde{L}'$.

For any simplicial complex R, let R_q be the set of positively oriented q-simplices of R. Let $B'_q \subseteq \tilde{K}'_q$ contain, for each q-simplex $s \in K'_q$, exactly one simplex s' satisfying $\tilde{p}_K(s') = s$. Then B'_q is a $\mathbb{Z}[\tilde{\pi}_K]$-basis for $C_q(\tilde{K}'; \mathbb{Z})$, and $\tilde{\imath}(B'_q)$ is a $\mathbb{Z}[\tilde{\pi}_X]$-basis for $C_q(\bar{K}'; \mathbb{Z})$. Note that, because K is compact, the bases are finite and have the same number of elements.

As in [6], for each $s \in \tilde{K}'_q$ let $s_{\tilde{K}''}$ be the subdivision of s that is a subcomplex of $\tilde{K}''$. Let $\mathscr{O}(s)$ be the set of positively oriented q-simplices in $s_{\tilde{K}''}$. Let

$$\tau: C_*(\tilde{K}'; \mathbb{Z}) \to C_*(\tilde{K}''; \mathbb{Z})$$

be the subdivision chain map that is the identity on vertices of $\tilde{K}'$ and sends each q-simplex k of $\tilde{K}'$ into $C_q(s_{\tilde{K}''}; \mathbb{Z})$ by the formula

$$\tau_q(k) = \sum_{r \in \mathscr{O}(k)} r.$$

For $b \in B'_q$ and $r \in \mathscr{O}(b)$ we have

$$\tilde{g}(r) = \sum_{s \in \tilde{L}'_q} \lambda_{r,s} s \in C_q(\tilde{L}'; \mathbb{Z})$$

with $\lambda_{r,s} \in \{0, 1, -1\}$ and $\lambda_{r,s} \neq 0$ for at most one simplex s.

Let Φ_q be the composition

$$\Phi_q: C_q(\tilde{K}'; \mathbb{Z}) \xrightarrow{\tau_q} C_q(\tilde{K}''; \mathbb{Z}) \xrightarrow{C_q(\tilde{g})} C_q(\tilde{L}'; \mathbb{Z}) \xrightarrow{\mathrm{pr}_q} C_q(\bar{K}'; \mathbb{Z}).$$

Note that the projection pr_q is not a chain map. Thus Φ_q is not a chain map. Let $M_q = [m_{i,j}]$ be the square matrix for Φ_q over $\mathbb{Z}[\tilde{\pi}_X]$.

Let $B'_q = \{b_1, \ldots, b_n\}$. The $\mathbb{Z}[\tilde{\pi}_X]$-trace of Φ_q is

$$\mathrm{tr}(M_q) = \sum_{i=1}^{n} m_{i,i} = \sum_{b \in B'_q} \sum_{\sigma \in \tilde{\pi}_X} \sum_{r \in \mathscr{O}(b)} \lambda_{r, \sigma \tilde{\imath}(b)} \sigma.$$

Let $\mathbb{Z}(R(\tilde{\phi}, \tilde{\xi}))$ be the free Abelian $\mathbb{Z}$-module generated by the orbits of $R(\tilde{\phi}, \tilde{\xi})$, and let $\rho : \mathbb{Z}[\tilde{\pi}_X] \to \mathbb{Z}(R(\tilde{\phi}, \tilde{\xi}))$ be the linear function defined as follows. For each $\theta \in \tilde{\pi}_X$, $\rho : \theta \mapsto [\theta^{-1}]$. We define $T^R(\Phi_q)$ as a trace-like function into $\mathbb{Z}(R(\tilde{\phi}, \tilde{\xi}))$ given by

$$T^R(\Phi_q) = \rho \circ \mathrm{tr}(M_q) \in \mathbb{Z}\big(R(\tilde{\phi}, \tilde{\xi})\big).$$

We define $L_H(f; \tilde{f}, \tilde{\imath})$ to be

$$L_H(f; \tilde{f}, \tilde{\imath}) = \sum_q (-1)^q T^R(\Phi_q).$$

Note that ρ involves a twisting in the sense that we write $[\theta^{-1}]$ where one might expect to see $[\theta]$. This notation corresponds to the notation in [3].

3.2. Properties of $L_H(f; \tilde{f}, \tilde{\imath})$

Proposition 3.2.1. *$L_H(f; \tilde{f}, \tilde{\imath})$ is independent of the choice of basis B_q'.*

Proof. It suffices to prove that $T^R(\Phi_q)$ is independent of the choice of basis B_q'. Let A_q be a $\mathbb{Z}[\tilde{\pi}_K]$-basis for $C_q(\tilde{K}'; \mathbb{Z})$. Then $\tilde{\imath}(A_q)$ is a $\mathbb{Z}[\tilde{\pi}_X]$-basis for $C_q(\overline{K}'; \mathbb{Z})$. If we define $T^R(\Phi_q)$ in terms of A_q, we have

$$T^R(\Phi_q) = \sum_{a \in A_q} \sum_{s \in \mathscr{O}(a)} \sum_{\theta \in \tilde{\pi}_X} \lambda_{s, \theta^{-1}\tilde{\imath}(a)}[\theta].$$

For each $a \in A_q$, there exists a unique $\sigma \in \tilde{\pi}_K$ and a unique $b \in B_q'$ such that $a = \sigma b$. The contribution of a to $T^R(\Phi_q)$ is

$$\sum_{r \in \mathscr{O}(b)} \sum_{\theta \in \tilde{\pi}_X} \lambda_{\sigma r, \theta^{-1}\tilde{\imath}(\sigma b)}[\theta].$$

Thus

$$\begin{aligned}\sum_{s \in \mathscr{O}(a)} \sum_{\theta \in \tilde{\pi}_X} \lambda_{s, \theta^{-1}\tilde{\imath}(a)}[\theta] &= \sum_{r \in \mathscr{O}(b)} \sum_{\theta \in \tilde{\pi}_X} \lambda_{r, \tilde{\phi}(\sigma^{-1})\theta^{-1}\tilde{\xi}(\sigma)\tilde{\imath}(b)}[\theta] \\ &= \sum_{r \in \mathscr{O}(b)} \sum_{\theta \in \tilde{\pi}_X} \lambda_{r, \theta^{-1}\tilde{\imath}(b)}[\theta].\end{aligned}$$

We have

$$\sum_{a \in A_q} \sum_{s \in \mathscr{O}(a)} \sum_{\theta \in \tilde{\pi}_X} \lambda_{s, \theta^{-1}\tilde{\imath}(a)}[\theta] = \sum_{b \in B_q'} \sum_{r \in \mathscr{O}(b)} \sum_{\theta \in \tilde{\pi}_X} \lambda_{r, \theta^{-1}\tilde{\imath}(b)}[\theta]. \qquad \square$$

The following simple lemma is needed in the proof of the next proposition.

Lemma 3.2.2. *Let $p : \tilde{A} \to A$ and $q : \tilde{B} \to B$ be covering maps. Let $f : A \to B$ and $\tilde{f} : \tilde{A} \to \tilde{B}$ be maps satisfying $fp = q\tilde{f}$, and let $g : A' \to B'$ be a simplicial approximation to f. For $h : A \times I \to B$ the straight-line homotopy between f and g, we define $\tilde{h}$ to be the lift of h that begins at $\tilde{f}$. Let $\tilde{g}$ be defined to be $\tilde{h}_1$. Then $\tilde{g}$ is a simplicial approximation to $\tilde{f}$.*

Further, let $r:\tilde{C}\to C$ be a covering map, and let $k:B\to C$ and $\tilde{k}:\tilde{B}\to\tilde{C}$ be maps satisfying $kq=r\tilde{k}$. If $l:B'\to C'$ is a simplicial approximation to k, then $\tilde{l}\tilde{g}=\widetilde{lg}$ is a simplicial approximation to $\tilde{k}\tilde{f}$.

Proposition 3.2.3. *$L_H(f;\ \tilde{f},\ \tilde{\imath})$ is independent of the choice of subdivisions K'' and L' and of the simplicial approximation $g:K''\to L'$.*

Proof. Let $g_1:K_1''\to L_1'$ and $g_2:K_2''\to L_2'$ both be simplicial approximations to f.

Case 1: $K_1''=K_2''$ *and* $L_1'=L_2'$.

Let $K''=K_i''$ and $L'=L_i'$ for $i=1,2$. Let $\tilde{g}_i:\tilde{K}''\to\tilde{L}'$ be the lifts of the g_i as defined in Lemma 3.2.2. We make use of the chain homotopy between $\tilde{g}_1$ and $\tilde{g}_2$,

$$D_q:C_q(\tilde{K}'';\ \mathbb{Z})\to C_{q+1}(\tilde{L}';\ \mathbb{Z})$$

given by

$$D_q\langle v_0,\ldots,v_q\rangle=\sum_{i=0}^{q}(-1)^q\langle\tilde{g}_1(v_0),\ldots,\tilde{g}_1(v_i),\ \tilde{g}_2(v_i),\ldots,\tilde{g}_2(v_q)\rangle.$$

Note that D_q satisfies $\tilde{\phi}(\sigma)D_q=D_q\sigma$ for all $\sigma\in\tilde{\pi}_K$ and is a homomorphism of degree 1 with

$$D_{q-1}\partial_q+\partial_{q+1}D_q=(\tilde{g}_2)_q-(\tilde{g}_1)_q.$$

For any chain map $h:C_*(\tilde{K}'';\ \mathbb{Z})\to C_*(\tilde{L}';\ \mathbb{Z})$, let $\Phi_q(h)=\tau_q h_q \mathrm{pr}_q$. Then $T^R(\Phi_q(\cdot))$ is linear over $\mathbb{Z}$. Our goal is to prove that

$$\sum_q(-1)^q T^R\big(\Phi_q(D_{q-1}\partial_q+\partial_{q+1}D_q)\big)=0.$$

This would force the local generalized H-Lefschetz numbers induced by the two simplicial approximations to be equal.

Let $F_q=\mathrm{pr}_q D_{q-1}\partial_q\tau_q+\partial_{q+1}\mathrm{pr}_{q+1}D_q\tau_q$ and $G_q=\mathrm{pr}_q\partial_{q+1}D_q\tau_q-\partial_{q+1}\mathrm{pr}_{q+1}D_q\tau_q$. To achieve the goal, we prove that $\sum_q(-1)^qT^R(F_q)=0$ and $\sum_q(-1)^qT^R(G_q)=0$.

Note that F_q is a chain map. Let $E_q=\mathrm{pr}_{q+1}D_q\tau_q$. Then E is a chain homotopy between F and 0. For any $\alpha\in\tilde{\pi}_K$ and any $\sigma\in\tilde{\pi}_X$, we have $[\tilde{\xi}(\alpha)\sigma]=[\sigma\tilde{\phi}(\alpha)]$. Thus it can be proven that $T^R(E_{q-1}\partial_q)=T^R(\partial_qE_{q-1})$ for all q, and

$$\sum_q(-1)^qT^R(E_{q-1}\partial_q+\partial_{q+1}E_q)=0=\sum_q(-1)^qT^R\big(\Phi_q(F_q)\big).$$

Next we prove that for any q we have $T^R(G_q)=0$. Let $b\in B_q'$ as before, and consider the contribution of b to the trace of G_q. We have

$$G_q(b)=\sum_{\sigma\in\tilde{\pi}_X}\sum_{s\in B_q'}\theta_{b,\sigma\tilde{\imath}(s)}\sigma\tilde{\imath}(s)$$

with $\theta_{b,\sigma\tilde{\imath}(s)}\in\mathbb{Z}$. We must show that $\theta_{b,\sigma\tilde{\imath}(b)}=0$ for all $b\in B_q'$ and for all $\sigma\in\tilde{\pi}_X$.

Let A_{q+1} be the set of all $(q+1)$-simplices of $\tilde{L}'$ that are not in $\tilde{K}'$ and have a face in $\tilde{K}'$. Recall that pr_{q+1} is not a chain map. In fact, $\mathrm{pr}_q\partial_{q+1}(t)\neq\partial_{q+1}\mathrm{pr}_{q+1}(t)$ if and only if $t\in A_{q+1}$. If $\theta_{b,\sigma\tilde{\imath}(b)}\neq 0$, then for some $r\in\mathscr{O}(b)$ the simplex $D_q(r)$ is in A_{q+1} and must share a face with $\sigma\tilde{\iota}(b)$. This can never happen, because the definitions of g_1 and g_2 force D to have the following property. Let $\tilde{s}$ be a q-simplex of $\tilde{K}''$ with $\tilde{p}_K(\tilde{s}) = s \in \partial K''$, and let $c \in \partial K'$ with $s \in \mathscr{O}(c)$. Then $\tilde{p}_X D_q(\tilde{s})$ and c have no vertices in common. Thus $T^R(G_q) = 0$ for all q, and Case 1 is proven.

Case 2: $L'_1 = L'_2$ *and* $K''_1 \neq K''_2$.

Let $L' = L'_1 = L'_2$, and let K' be the subcomplex of L' that is a subdivision of K. Then K''_i is a subdivision of K' for $i = 1, 2$. Let L''_i be a subdivision of L' that has K''_i as a subcomplex for $i = 1, 2$.

Step 1. We assume in this step that L''_2 is a subdivision of L''_1.

This assumption implies that K''_2 is a subdivision of K''_1. Let $\rho_K : K''_2 \to L''_1$ be a simplicial approximation to the inclusion map. We define $\tilde{\rho}_K$ to be the lift that is a simplicial approximation to $\tilde{\iota}$ as in Lemma 3.2.2.

We compare $\Phi_q(\tilde{g}_1\tilde{\rho}_K)$ with $\Phi_q(\tilde{g}_2)$. Note that

$$C_q(\tilde{K}') \xrightarrow{\tau_q} C_q(\tilde{K}''_2) \xrightarrow{C_q(\tilde{\rho}_K)} C_q(\tilde{K}''_1)$$

is the same homomorphism as

$$C_q(\tilde{K}') \xrightarrow{\tau_q} C_q(\tilde{K}''_1).$$

Thus $\Phi_q(\tilde{g}_1\tilde{\rho}_K)$ equals $\Phi_q(\tilde{g}_2)$, and the two simplicial approximations to f induce the same local generalized H-Lefschetz number.

We have $g_1\rho_K : K''_2 \to L'$ and $g_2 : K''_2 \to L'$ simplicial approximations to f. As in Case 1, $g_1\rho_K$ and g_2 induce the same local generalized H-Lefschetz number. Case 2 is proven when L''_2 is assumed to be a subdivision of L''_1.

Step 2. If L''_2 is not a subdivision of L''_1, let L''_3 be a subdivision of both L''_1 and L''_2. Then there is a subcomplex K''_3 of L''_3 that is a subdivision of both K''_1 and K''_2, and there is a simplicial approximation to f given by $g_3 : K''_3 \to L'$. The work just completed proves that for $i = 1, 2$, g_i and g_3 induce the same local generalized H-Lefschetz number. Case 2 is completed.

Case 3: $L'_1 \neq L'_2$.

For $i = 1, 2$, let K'_i be the subcomplex of L'_i that has K''_i as a subdivision. If s is a simplex, let $\bar{s}$ be the closure of the underlying space of s.

Step 1. We assume that L'_2 is a subdivision of L'_1. This implies that K'_2 is a subdivision of K'_1. Without loss of generality, we may assume that K''_2 is a subdivision of K''_1. (If K''_2 is not a subdivision of K''_1, we use the logic of Step 2 in Case 2.)

Let $\mu : K''_2 \to K''_1$ be a simplicial approximation to the identity map on K, and let $\tilde{\mu}$ be the lift of μ as in Lemma 3.2.2. Note that $g_1 : K''_1 \to L'_1$ and $g_1\mu : K''_2 \to L'_1$ are both simplicial approximations to f. As in Case 2, they induce the same local generalized H-Lefschetz number. Also, $g_2 : K''_2 \to L'_2$ and $\tau g_1\mu : K''_2 \to L'_2$ are both simplicial approximations to f. As in Case 1, they induce the same local generalized H-Lefschetz number. By a tedious calculation, it can be shown that $T^R(\Phi_q)$ is the same for $g_1\mu$ and $\tau g_1\mu$. Thus Step 1 of Case 3 is proven.

Step 2. If L'_2 is not a subdivision of L'_1, we let L'_3 be a common subdivision and use the logic of Step 2 in Case 2. □

Let $\gamma \in \tilde{\pi}_X$. Then $\gamma\tilde{f}$ induces both the homomorphism $\gamma\tilde{\phi}(\cdot)\gamma^{-1} : \tilde{\pi}_K \to \tilde{\pi}_X$ and a Reidemeister action of $\tilde{\pi}_K$ on $\tilde{\pi}_X$. For any $\theta \in \tilde{\pi}_X$, let $[\theta]_{\gamma\tilde{f}}$ and $[\theta]_{\tilde{f}}$ denote the Reidemeister orbits of θ induced by $\gamma\tilde{f}$ and $\tilde{f}$, respectively.

Proposition 3.2.4 (Independence from the lifts $\tilde{f}$ and $\tilde{\imath}$). *For any $\gamma \in \tilde{\pi}_X$, we have $L_H(f;\ \tilde{f}, \tilde{\imath}) = L_H(f;\ \gamma\tilde{f}, \tilde{\imath})$ in the sense that, for any $\theta \in \tilde{\pi}_X$, the coefficient of $[\theta]_{\tilde{f}}$ in $L_H(f;\ \tilde{f}, \tilde{\imath})$ equals the coefficient of $[\theta\gamma^{-1}]_{\gamma\tilde{f}}$ in $L_H(f;\ \gamma\tilde{f}, \tilde{\imath})$. A similar result holds when $\tilde{\imath}$ is replaced by $\gamma\tilde{\imath}$.*

Proof. Let Φ_q^γ be the composition

$$\Phi_q^\gamma : C_q(\tilde{K}';\ \mathbb{Z}) \xrightarrow{\tau_q} C_q(\tilde{K}'';\ \mathbb{Z}) \xrightarrow{C_q(\tilde{g})} C_q(\tilde{L}';\ \mathbb{Z}) \xrightarrow{C_q(\gamma)} C_q(\tilde{L}';\ \mathbb{Z}) \xrightarrow{\mathrm{pr}_q} C_q(\tilde{K}';\ \mathbb{Z}).$$

Note that

$$T^R(\Phi_q^\gamma) = \sum_{b \in B'_q} \sum_{\theta \in \tilde{\pi}_X} \sum_{r \in \mathcal{O}(b)} \lambda_{r,\gamma^{-1}\theta^{-1}\tilde{\imath}(b)} [\theta]_{\gamma\tilde{f}}$$

$$= \sum_{b \in B'_q} \sum_{\theta \in \tilde{\pi}_X} \sum_{r \in \mathcal{O}(b)} \lambda_{r,\theta^{-1}\tilde{\imath}(b)} [\theta\gamma^{-1}]_{\gamma\tilde{f}},$$

$$T^R(\Phi_q) = \sum_{b \in B'_q} \sum_{\theta \in \tilde{\pi}_X} \sum_{r \in \mathcal{O}(b)} \lambda_{r,\theta^{-1}\tilde{\imath}(b)} [\theta]_{\tilde{f}}. \qquad \square$$

Proposition 3.2.5 (Independence from $\tilde{K}$). *$L_H(f;\ \tilde{f}, \tilde{\imath})$ is independent of the choice of (H, f)-admissible covering space $\tilde{K}$ up to the choices of lifts as in Proposition* 3.2.2.

Proof. It suffices to prove that an (H, f)-admissible covering space $\tilde{K}$ and the universal cover $\hat{K}$ for K give rise to the same local generalized H-Lefschetz number.

We choose $\tilde{g}$ as in Lemma 3.2.2, and we choose a lift $\hat{g}$ of $\tilde{g}$ to $\hat{K}$ such that $\tilde{g}j = \hat{g}$. Similarly, we choose $\tilde{\imath}$ and then $\hat{\imath}$ such that $\tilde{\imath}j = \hat{\imath}$. Let $\tilde{B}'_q$ and $\hat{B}'_q$ be bases for $C_q(\tilde{K}';\ \mathbb{Z})$ over $\mathbb{Z}[\tilde{\pi}_K]$ and $\mathbb{Z}[\hat{\pi}_K]$, respectively, with $C_q(j)(\hat{B}'_q) = \tilde{B}'_q$. Note that there is a one-to-one correspondence between $\tilde{B}'_q$ and $\hat{B}'_q$. Let $\hat{\Phi}_q$ and $\tilde{\Phi}_q$ be compositions as in Section 3.1. Then $\hat{\Phi}_q = \tilde{\Phi}_q C_q(j)$. It follows that $\hat{\Phi}_q$ and $\tilde{\Phi}_q$ have the same matrix over $\mathbb{Z}[\tilde{\pi}_X]$. Thus $T^R(\hat{\Phi}_q) = T^R(\tilde{\Phi}_q)$, and the local generalized H-Lefschetz numbers are the same for the two covering spaces. □

Proposition 3.2.6 (Independence from K). *For a fixed triangulation L of X, $L_H(f;\ \tilde{f}, \tilde{\imath})$ is independent of the choice of stable (f, U)-subcomplex K of L (where we always choose K so that $K = \overline{\mathrm{int}\ K}$).*

Proof. Let K_1 and K_2 be stable (f, U)-subcomplexes of L. Then $P=K_1\cup K_2$ is also a stable (f, U)-subcomplex of L. Let $\tilde{P}$ be an (H, f)-admissible covering space for P with $q:\tilde{P}\to P$ the covering map. Choose $\tilde{\imath}$ and $\tilde{f}$ defined on $\tilde{P}$ as usual, let $\tilde{K}_1$ be a component of $q^{-1}(K_1)$.

Let M be the simplicial complex $(P-K_1)\cup\partial K_1$. For s a simplex, let $\bar{s}$ be the closure of the underlying space of s. By Lemma 3.1 of [6], there is a subdivision L' of L and there are corresponding subdivisions M' and P' so that, for all $s\in M'$,

$$f(\bar{s})\cap\left(\bigcup_{v\in s}\operatorname{star}(v)\right)=\emptyset.$$

Let $g:P''\to L'$ be any simplicial approximation to f defined on a subdivision P'' of P'. Let M'' be the obvious subcomplex of P''. For $s\in M'$ and $t\in\mathscr{O}(s)$, we have $g(t)\neq\pm s$.

Let $\tilde{g}:\tilde{P}''\to\tilde{L}'$ be the lift of g as determined in Lemma 3.2.2, and let $\tilde{g}_1=\tilde{g}\mid\tilde{K}_1''$. Let B'_{q1} be a $\mathbb{Z}[\tilde{\pi}_{K_1}]$-basis for $C_q(\tilde{K}_1';\mathbb{Z})$. We extend B'_{q1} to a $\mathbb{Z}[\tilde{\pi}_P]$-basis for $C_q(\tilde{P}';\mathbb{Z})$ to be denoted by B'_{qP}. For $b\in B'_{qP}-B'_{q1}$, the contribution of b to $T^R(\Phi_q(\tilde{g}))$ is zero. Therefore

$$T^R\big(\Phi_q(\tilde{g})\big)=\sum_{b\in B'_{q1}}\ \sum_{r\in\mathscr{O}''(b)}\ \sum_{\sigma\in\tilde{\pi}_X}\lambda_{r,\sigma\tilde{\imath}(b)}[\sigma^{-1}]=T^R\big(\Phi_q(\tilde{g}_1)\big).$$

The same argument holds for K_2. Thus K_1 and K_2 give rise to the same local generalized H-Lefschetz number. □

Proposition 3.2.7 (Independence from L). *$L_H(f;\tilde{f},\tilde{\imath})$ is independent of the choice of L, where we consider only those L that have an (f, U)-subcomplex.*

Proof. The proof follows from Propositions 3.2.6 and 3.2.3. □

Proposition 3.2.8 (Homotopy invariance). *Let $h:U\times I\to X$ be an admissible homotopy. Then $L_H(h_0;\tilde{h}_0,\tilde{\imath})=L_H(h_1;\tilde{h}_1,\tilde{\imath})$ where $\tilde{h}_0$ and $\tilde{h}_1$ are homotopic via a lift of h to an appropriate covering space.*

Proof.

Case 1: *Assume there is a subdivision L of X with a subcomplex K that is a stable (h_t, U)-subcomplex of L for all $t\in I$.*

Let $\tilde{K}$ be any (H, h_0)-admissible covering space for K, and choose lifts $\tilde{h}_0$ and $\tilde{\imath}$. By lifting $h\mid K\times I$ to a map $\tilde{h}:\tilde{K}\times I\to\tilde{X}$ beginning at $\tilde{h}_0$, we see that, for any $t\in I$, $\tilde{K}$ is an (H, h_t)-admissible covering space for K.

Using the method of Proposition 4.4 in [6], we find a suitable subdivision $0=t_0<t_1<\cdots<t_n=1$ of I so that for each $i=0,\ldots,n-1$ there is a vertex map $\rho_i:K''\to L'$ satisfying the following. Fix $\rho_i\cap\partial K=\emptyset$, and for all vertices $v\in K''$, $h(\operatorname{star}(v)\times[t_i,t_{i+1}])\subseteq\operatorname{star}(\rho_i(v))$. Then ρ_i is a simplicial approximation to h_r for all $t_i\leqslant r\leqslant t_{i+1}$, and by Proposition 3.2.3, $L_H(h_{t_i};\tilde{h}_{t_i},\tilde{\imath})=L_H(h_{t_{i+1}};\tilde{h}_{t_{i+1}},\tilde{\imath})$ for $i=0,\ldots,n-1$. Case 1 is proven.

Case 2: *Assume there is no L with a subcomplex that for all $t \in I$ is stable.*

There is a subdivision N of X with a compact subcomplex M such that $\bigcup_{t \in I} \text{Fix } h_t \subseteq \text{int } M$ and $M \subseteq U$. The subcomplex M is an (h_t, U)-subcomplex of N for all t, but it need not be stable for any t. Because M is both an (h_0, U)-subcomplex and an (h_1, U)-subcomplex of N, there is a subdivision L of N with K_i stable (h_i, U)-subcomplexes of L for $i = 0, 1$. Let $K = K_0 \cup K_1 \cup M'$, where M' is the subdivision of M that is a subcomplex of L. Then K is a stable (h_i, U)-subcomplex of L for $i = 0, 1$, and $\bigcup_{t \in I} \text{Fix } h_t \subseteq \text{int } K$.

There is an open subset W of X with $K \subseteq W \subseteq U$ and for which K is a stable $(h_t \mid W, W)$-subcomplex of L for all $t \in I$. As in Case 1, we have $L_H(h_0 \mid W; \tilde{h}_0 \mid W, \tilde{\imath}) = L_H(h_1 \mid W; \tilde{h}_1 \mid W, \tilde{\imath})$. Because K is a stable (h_0, U)-subcomplex, the local generalized H-Lefschetz numbers for h_0 on U and on W are equal. The same is true for h_1. □

Proposition 3.2.9 (The additivity property for $L_H(f; \tilde{f}, \tilde{\imath})$). *Let $f: U \to X$ as usual with $H \lhd \pi_1(X)$. For W_1 and W_2 open, disjoint subsets of U with* Fix $f \subseteq W_1 \cup W_2$, *we define f_i to be $f \mid W_i$ for $i = 1, 2$. There exist local settings for f, f_1 and f_2 for which*

$$L_H(f; \tilde{f}, \tilde{\imath}) = L_H(f_1; \tilde{f}_1, \tilde{\imath}_1) + L_H(f_2; \tilde{f}_2, \tilde{\imath}_2).$$

Proof. Let $\text{LS}(f) = \{L, K, f, \tilde{L}, \tilde{K}, \tilde{f}, \tilde{\imath}\}$. For $i = 1, 2$, we define $f_i := f \mid W_i$.

Case 1: *Assume that for $i = 1, 2$ there exist $K_i \subseteq K$ with the K_i-stable (f_i, W_i)-subcomplexes of L so that* Fix $f \subseteq \text{int } K_1 \cup \text{int } K_2$.

We define $\tilde{K}_i$ to be a component of $\tilde{p}_K^{-1}(K_i)$, and we restrict $\tilde{f}$ and $\tilde{\imath}$ to $\tilde{f}_i : \tilde{K}_i \to \tilde{X}$ and $\tilde{\imath}_i : \tilde{K}_i \to \tilde{X}$. We prove that the local settings $\text{LS}(f_i) = \{L, K_i, f_i, \tilde{L}, \tilde{K}_i, \tilde{f}_i, \tilde{\imath}_i\}$ for $i = 1, 2$ are those for which additivity holds.

For g the usual simplicial approximation to f, let $\tilde{g}_i$ be the restriction of $\tilde{g}$ to $\tilde{K}_i$, and let $\overline{K}_i'$ be $\tilde{p}_X^{-1}(K_i')$ for each i. It suffices to prove that

$$T^R(\Phi_q(\tilde{g})) = T^R(\Phi_q(\tilde{g}_1)) + T^R(\Phi_q(\tilde{g}_2)).$$

Let $B_{iq} = B_q' \cap \tilde{K}_i'$. Then $B_{1q} \cap B_{2q} = \emptyset$. Consider $b \in B_q' - (B_{1q} \cup B_{2q})$. The contribution of b to $T^R(\Phi_q(\tilde{g}))$ is

$$\sum_{r \in \mathscr{O}(b)} \sum_{\theta \in \tilde{\pi}_X} \lambda_{r, \theta^{-1}\tilde{\imath}(b)}[\theta].$$

The simplex $\tilde{p}_K(b)$ contains no fixed point of g, because $\tilde{p}_K(b)$ is a simplex in $K - (K_1 \cup K_2)$. Thus

$$\begin{aligned} T^R(\Phi_q(\tilde{g})) &= \sum_{b \in B_{1q}} \sum_{r \in \mathscr{O}(b)} \sum_{\theta \in \tilde{\pi}_X} \lambda_{r, \theta^{-1}\tilde{\imath}(b)}[\theta] + \sum_{b \in B_{2q}} \sum_{r \in \mathscr{O}(b)} \sum_{\theta \in \tilde{\pi}_X} \lambda_{r, \theta^{-1}\tilde{\imath}(b)}[\theta] \\ &= T^R(\Phi_q(\tilde{g}_1)) + T^R(\Phi_q(\tilde{g}_2)). \end{aligned}$$

Case 2: *Assume there are no subcomplexes K_1 and K_2 as in Case* 1.

We prove that there exists a local setting that does satisfy the requirement for Case 1. Let A be a subdivision of L that is sufficiently fine to have a stable

(f_i, W_i)-subcomplex K_i of A for $i = 1, 2$. Let K_A be the subdivision of K that is a subcomplex of A. If $B = K_A \cup K_1 \cup K_2$, then a local setting for f of the form $\{A, B, f, \tilde{A}, \tilde{B}, \tilde{f}, \tilde{\imath}\}$ satisfies the requirement of Case 1. □

3.3. *Equating the local generalized H-Lefschetz number and the* $(H, \tilde{f}, \tilde{\imath})$*-NR chain*

We prove that for X a connected, triangulable n-manifold with $n \geqslant 3$, for $H \lhd \pi_1(X)$ and for a compactly fixed map $f : U \to X$ as before, the local generalized H-Lefschetz number $L_H(f; \tilde{f}, \tilde{\imath})$ and the NR chain $N_H(f; \tilde{f}, \tilde{\imath})$ are the same. When these hypotheses are met and $H = 1$, $L_H(f; \tilde{f}, \tilde{\imath})$ equals the local obstruction $o(f)$ as defined in [4], and a generalized version of the converse to the Lefschetz fixed point theorem holds.

Theorem 3.3.1. *For a compactly fixed map $f : U \to X$ with X a connected, triangulable n-manifold and $n \geqslant 3$, and for any $H \lhd \pi_1(X)$,*

$$L_H(f; \tilde{f}, \tilde{\imath}) = N_H(f; \tilde{f}, \tilde{\imath}).$$

Remark. In the definition of a local setting the (f, U)-subcomplex K is chosen to be stable with $K = \overline{\text{int } K}$. Thus K is automatically an n-submanifold of X in this theorem.

Proof of Theorem 3.3.1. We begin by using a local version of the Hopf simplicial approximation theorem. See [1] for the global version (with $U = X$). The details for the local version can be found in the Appendix of [5].

By the local version of the Hopf simplicial approximation theorem, we may choose g a simplicial approximation to f and subdivisions K'' and L' that satisfy the following conditions.

(1) $g : K'' \to L'$ with Fix g in the interior of K''.
(2) $|\text{Fix } g| < \infty$.
(3) Each fixed point of g is contained in the interior of an n-simplex of K''.
(4) Each simplex of K'' contains at most one fixed point of g.

As before, let B'_q be a $\mathbb{Z}[\tilde{\pi}_K]$-basis for $C_q(\tilde{K}'; \mathbb{Z})$ consisting of positively oriented simplexes. Let B''_q be the set of all $r \in \tilde{K}''$ such that $r \in \mathscr{O}(b)$ for some $b \in B'_q$.

Assertion 1. *If $s \in B''_q$ and $\tilde{p}_K(s)$ contains no fixed point of g in its interior, then s does not contribute to the trace $T^R(\Phi_q)$. Also, let $\mathscr{F}(b)$ equal the set of all $r \in \mathscr{O}(b)$ for which $\tilde{p}_K(r)$ contains a fixed point of g in its interior. For $b \in B'_q$, the contribution of b to the trace can be reduced to*

$$\sum_{\alpha \in \tilde{\pi}_X} \sum_{r \in \mathscr{F}(b)} \lambda_{r, \alpha^{-1}\tilde{\imath}(b)}[\alpha].$$

Proof of Assertion 1. Let $s \in \mathscr{O}(b) - \mathscr{F}(b)$. We have $g(\tilde{p}_K(s)) \neq \pm\tilde{p}_K(b)$, which implies that for all $\alpha \in \tilde{\pi}_X$ we have $\alpha\tilde{g}(s) \neq \pm\tilde{\imath}(b)$. Thus $\lambda_{s,\alpha^{-1}\tilde{\imath}(b)} = 0$ for all $\alpha \in \tilde{\pi}_X$, and s contributes nothing to the trace. Assertion 1 follows.

By Assertion 1, $T^R(\Phi_q) = 0$ for $q < n$. Thus $L_H(f;\, \tilde{f}, \tilde{\imath}) = (-1)^n T^R(\Phi_n)$. We have

$$T^R(\Phi_n) = \sum_{b \in B'_n} \sum_{\alpha \in \tilde{\pi}_X} \sum_{r \in \mathscr{F}(b)} \lambda_{r,\alpha^{-1}\tilde{\imath}(b)}[\alpha].$$

Assertion 2. *If $r \in \mathscr{F}(b)$ with $b \in B'_n$, then the local index of g on the interior of $\tilde{p}_K(r)$ is equal to $(-1)^n \lambda_{r,\alpha^{-1}\tilde{\imath}(b)}$, where α is the unique element of $\tilde{\pi}_X$ for which $\alpha\tilde{g}(r) = \pm\tilde{\imath}(b)$.*

Proof of Assertion 2. Let $x \in \operatorname{Fix} g$. Let r_x and b_x be the unique simplices for which $b_x \in B'_n$ and $r_x \in \mathscr{O}(b_x)$ with x in the interior of $\tilde{p}_K(r_x)$. Then there is a unique $\alpha_x \in \tilde{\pi}_X$ for which $\alpha_x\tilde{g}(r_x) = \pm\tilde{\imath}(b_x)$. We have $x \in \tilde{p}_K(\operatorname{Coin}(\alpha\tilde{g}, \tilde{\imath}))$. In fact, $\tilde{g}(r_x) = \lambda_{r_x,\alpha_x^{-1}\tilde{\imath}(b_x)}\alpha_x^{-1}\tilde{\imath}(b_x)$, and $\lambda_{r_x,\alpha_x^{-1}\tilde{\imath}(b_x)} = \pm 1$. Because $\tilde{p}_K$ is a simplicial local homeomorphism, $g(\tilde{p}_K(r_x)) = \lambda_{r_x,\alpha_x^{-1}\tilde{\imath}(b_x)}\tilde{p}_K(b_x)$. Using the index as defined in [10] (the degree of $1 - g$), the local index of the isolated fixed point x equals $(-1)^n\lambda_{r_x,\alpha_x^{-1}\tilde{\imath}(b_x)}$, and Assertion 2 is proven.

Let W be a set of Reidemeister representatives for $R(\tilde{\phi}, \tilde{\xi})$. We have

$$T^R(\Phi_n) = \sum_{x \in \operatorname{Fix} g} \lambda_{r_x,\alpha_x^{-1}\tilde{\imath}(b_x)}[\alpha_x] = \sum_{\alpha \in W} \sum_{x \in \mathscr{C}(\alpha)} \lambda_{r_x,\alpha^{-1}\tilde{\imath}(b_x)}[\alpha],$$

where $\mathscr{C}(\alpha) = \tilde{p}_K(\operatorname{Coin}(\alpha\tilde{g}, \tilde{\imath}))$. Thus $T^R(\Phi_n) = (-1)^n \Sigma_{\alpha \in W} i(N_H^\alpha)[\alpha]$, and $L_H(f;\, \tilde{f}, \tilde{\imath}) = N_H(f;\, \tilde{f}, \tilde{\imath})$. □

We would like to detect whether f is homotopic to a map that is fixed point free via an admissible homotopy. That is, we would like to know whether $\min f = 0$.

For $H = 1$, a local obstruction to f having this property is defined in [4]. Briefly, a bundle $\mathscr{B}(f)$ of groups on U is defined in which the fibers are isomorphic to $\mathbb{Z}[\pi_1(X)]$. Then the obstruction $o(f)$ is a class in

$$H_c^n(U;\, \mathscr{B}(f)) = \varinjlim H^n(U, U - C;\, \mathscr{B}(f)),$$

where the limit is taken over all compact $C \subseteq U$. With $\mathscr{T}(U)$ the orientation sheaf on U let $\underline{\mu}(U)$ denote the twisted $\tilde{\pi}_X$-fundamental homology class of U in $H_n^c(U;\, \mathscr{T}(U))$. A coefficient pairing of $\mathscr{T}(U)$ and $\mathscr{B}(f)$ can then be used to give a cap product:

$$H_c^n(U;\, \mathscr{B}(f)) \otimes H_n^c(U;\, \mathscr{T}(U)) \to \mathbb{Z}\big(R(\tilde{\phi}, \tilde{\xi})\big).$$

Theorem 5.12 of [4] states that

$$\langle o(f), \underline{\mu}(U)\rangle = \sum_{\alpha \in W} i(N_H^\alpha)[\alpha].$$

This fact combined with Theorem 3.3.1 provides the following corollary.

Corollary 3.3.2. *For a map $f: U \to X$ as in Theorem* 3.3.1 *and for $H = 1$,*

$$L_1(f; \tilde{f}, \tilde{\iota}) = \langle o(f), \underline{\mu}(U) \rangle.$$

If X is not simply connected, a generalization of the converse of the Lefschetz fixed point theorem holds. We have

$$L_1(f; \tilde{f}, \tilde{\iota}) = 0 \quad \Leftrightarrow \quad \min f = 0.$$

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