A class of semigroups regularized in space and time

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Abstract

We consider \( \alpha \)-times integrated \( C \)-regularized semigroups, which are a hybrid between semigroups regularized in space (\( C \)-semigroups) and integrated semigroups regularized in time. We study the basic properties of these objects, also in absence of exponential boundedness. We discuss their generators and establish an equivalence theorem between existence of integrated regularized semigroups and well-posedness of certain Cauchy problems. We investigate the effect of smoothing regularized semigroups by fractional integration.

Keywords: \( \alpha \)-Times integrated \( C \)-regularized semigroups; Generator; Fractional integral; Abstract Cauchy problem
1. Introduction

Let $A : D(A) \subset X \to X$ be an (unbounded) linear operator in a Banach space $X$. The theory of the abstract Cauchy problem

$$\frac{d}{dt} u(t) = Au(t), \quad u(0) = x$$

leads to the notion of a $C_0$-semigroup. Of course, even if such semigroup exists, (1) admits solutions only for $x \in D(A)$. However, the integrated version

$$u(t) = A \int_0^t u(s) \, ds + x$$

admits unique solutions for all $x \in X$ if and only if $A$ generates a $C_0$-semigroup. Given a $C_0$-semigroup, one obtains also well-posedness for the inhomogeneous problem

$$u(t) = A \int_0^t u(s) \, ds + \int_0^t f(s) \, ds,$$

which is the integrated version of

$$\frac{d}{dt} u(t) = Au(t) + f(t), \quad u(0) = 0,$$

with suitable functions $f$. Needless to say that the theory of $C_0$-semigroups is well established, see, e.g., [6,7,9,18].

In less favorable situations, (2) may admit solutions only for a subset of initial values. One may describe the space of these admissible initial values as the range of some operator $C$ and arrives at the notion of a $C$-regularized semigroup. Another way to picture a $C$-regularized semigroup is to assume that the solutions of (2) which start in $X$ continue to live in a larger space than $X$, and it requires multiplication by a regularizing operator to pull them back into $X$. $C$-regularized semigroups were studied by deLaubenfels [4,5] for the case that the domain $D(A)$ of $A$ may not be dense. Further studies were made by Zheng and Liu [20] and Li [10].

On the other hand, it is possible that (3) admits solutions only for sufficiently smooth inhomogeneities $f$. In the case of a $C_0$-semigroup, solutions exist at least for $f \in L^1([0, T], X)$. If, for instance, solutions exist for $f$ which admit a (possibly fractional) derivative $\frac{d^\alpha}{dt^\alpha} f$ with $\alpha > 0$, then we arrive at the notion of an $\alpha$-times integrated semigroup. Another way to picture this object is to assume that the solutions to (3) live in a space larger than $X$, but smoothing in time by (possibly fractional) integration pulls them back into $X$. For $\alpha \in \mathbb{N}$, the integrated semigroup was introduced by Arendt [1–3] and Neubrander [16,17]. Subsequently, Hieber [8] and Mijatović et al. [14] introduced $\alpha$-times integrated semigroup for real $\alpha > 0$. Furthermore, Mijatović and Pilipović [13] presented $\alpha$-times integrated semigroup for $\alpha \in \mathbb{R}^-$.

Combining regularization with respect to space and time we arrive finally at $\alpha$-times integrated, $C$-regularized semigroups. For integer $\alpha$, these objects have been introduced by Li and Shaw [11] and Liu and Shaw [12]. The extension to fractional order of integration
has started with Xiao and Liang [19] for the case of exponentially bounded regularized semigroups.

This paper presents some basic facts about $\alpha$-times integrated, $C$-regularized semigroups and their generators. Except in one special section, we do not assume exponential boundedness, so that we have to find our way without the powerful help of the Laplace transform. The pivot of the paper is the characterization of integrated regulated semigroups by the class of abstract Cauchy problems which admit solutions (Theorem 10). The homogeneous problem (2) is now replaced by

$$u(t) = A \int_0^t u(s) \, ds + \frac{t^\alpha}{\Gamma(\alpha + 1)} C x. \quad (4)$$

The inhomogeneity in (3) has to be such that $\frac{d^\alpha}{dt^\alpha} f$ exists as a measure with values in the range of $C$. We show that smoothing integrated regularized semigroups by fractional integration yields again integrated regulated semigroups, and so does fractional differentiation, if it still leads to strongly continuous trajectories. We also add a section on exponentially bounded integrated regularized semigroups.

Although these results are straightforward and not surprising, they require some care and the proofs are sometimes a little tedious. We would like to draw the reader’s attention to some fine points, which are not obvious at a first glance:

Regularized semigroups are not necessarily exponentially bounded.

The domain of the generator stays exactly the same, if the semigroups are smoothed by fractional integration or differentiated.

Existence and uniqueness of solutions for (4) leads to existence of an $\alpha$-times integrated, $C$-regularized semigroup. However, $A$ is only a subset of the generator, unless additional assumptions on the commutativity of $A$ and $C$ are guaranteed. In particular, we require the following “cancellation law”:

$$Cx \in D(A) \quad \text{and} \quad ACx = Cy \quad \Rightarrow \quad x \in D(A) \quad \text{and} \quad Ax = y.$$  

The paper is organized as follows: Section 2 presents the definition and gives the basic properties of the generator of an integrated regularized semigroup. Section 3, which contains the main technical work, gives the equivalence of existence of an integrated regularized semigroup and well-posedness of abstract Cauchy problems. Section 4 deals with fractional integration and differentiation of integrated regularized semigroups. In Section 5 we consider aspects of the exponentially bounded case.

2. $\alpha$-Times integrated $C$-regularized semigroups and their generators

We start with the definition of an $\alpha$-times integrated, strongly continuous $C$-regularized semigroup:

**Definition 1.** Let $\alpha \geq 0$ and $C \in B(X)$. A strongly continuous family of bounded linear operators $\{S(t)\}_{t \geq 0} \subset B(X)$ is called an $\alpha$-times integrated $C$-regularized semigroup on $X$ if it satisfies:
(a) $S(0) = 0$ if $\alpha > 0$ and $S(0) = C$ if $\alpha = 0$.
(b) $S(t)C = CS(t)$ for all $t \geq 0$.
(c) For all $x \in X$ and $t, s \geq 0$,
\[ S(t)S(s)x = \begin{cases} 
S(t + s)Cx, & \text{if } \alpha = 0, \\
\frac{1}{\Gamma(\alpha)}\left[\int_t^{s+t} - \int_0^s\right](s + t - r)^{\alpha-1}S(r)Cx \, dr, & \text{else.}
\end{cases} \]

Moreover, $\{S(t)\}_{t \geq 0}$ is said to be nondegenerate if $S(t)x = 0$ for all $t > 0$ implies $x = 0$. We say that $\{S(t)\}_{t \geq 0}$ is exponentially bounded if there are constants $M, \omega > 0$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

**Remark 2.**

(1) If $\alpha = 0$, then an $\alpha$-times integrated $C$-regularized semigroup is just a $C$-regularized semigroup, see [4,5].
(2) If $C = I$, then an $\alpha$-times integrated $C$-regularized semigroup reduces to an $\alpha$-times integrated semigroup, see [1–3,8,16,17].
(3) If $\alpha = 0$ and $C = I$, then an $\alpha$-times integrated $C$-regularized semigroup is just a $C_0$ semigroup, see [6,7,9,18].
(4) For $\alpha = n \in \mathbb{N}$, $\alpha$-times integrated $C$-regularized semigroups have been investigated by Li and Shaw [11] and Liu and Shaw [12].

Before we proceed, we define some shorthand notation:

**Definition 3.** Let $\alpha > 0$. For $t > 0$ we define
\[ g_\alpha(t) = t^{\alpha-1} \frac{1}{\Gamma(\alpha)}. \]

**Remark 4.** The following properties are well known and can be easily proved by straightforward calculation (here and in the rest of the paper asterisk $\ast$ will denote convolution and hat $\hat{}$ denotes the Laplace transform):

\[ g_1(t) = 1, \quad g_\alpha \ast g_\beta = g_{\alpha + \beta}, \quad \int_0^t g_\beta(\tau) \, d\tau = (g_1 \ast g_\beta)(t) = g_{\beta+1}(t), \quad \hat{g}_\alpha(s) = s^{-\alpha}. \]

**Theorem 5.** If $\{S(t)\}_{t \geq 0}$ is a nondegenerate $\alpha$-times integrated $C$-regularized semigroup, then $C$ is injective.

**Proof.** We show that $\ker(C) = \{0\}$. Pick $x \in \ker(C)$. Then for any $t, s \geq 0$, we have
\[ S(t)S(s)x = \left[\int_t^{s+t} - \int_0^s\right]g_\alpha(t + s - r)S(r)Cx \, dr = 0. \]
Fix $s > 0$. Since $\{S(t)\}_{t \geq 0}$ nondegenerate, we infer $S(s)x = 0$. Using nondegeneracy again, we obtain $x = 0$. \qed

The rest of this section is devoted to the discussion of the generator of an $\alpha$-times integrated $C$-regularized semigroup.
Definition 6. Let $\alpha \geq 0$ and $\{S(t)\}_{t \geq 0}$ be a nondegenerate $\alpha$-times integrated $C$-regularized semigroup. The generator $A$ of $S(t)$ is defined by the following property: $x \in D(A)$ and $Ax = y$ iff

$$S(t)x = \frac{t^\alpha}{\Gamma(\alpha + 1)} Cx + \int_0^t S(s)y \, ds$$

for all $t \geq 0$.

The assumption that $\{S(t)\}_{t \geq 0}$ is nondegenerate implies that the operator $A$ is well defined. The following gives some properties of generator of a nondegenerate $\alpha$-times integrated $C$-regularized semigroup $\{S(t)\}_{t \geq 0}$.

Theorem 7. Let $A$ be the generator of a nondegenerate $\alpha$-times integrated $C$-regularized semigroup $\{S(t)\}_{t \geq 0}$. Then

(a) $A$ is a closed linear operator.
(b) If $x \in D(A)$, then $Cx \in D(A)$ and $ACx = CAx$.
(c) If $Cx \in D(A)$ and $ACx = Cy$, then $x \in D(A)$ and $Ax = y$.

Proof. Since $S(t)$ and $C$ are linear operators, then it is clear that $A$ is a linear operator. To prove $A$ closed, let $\{x_n\} \subset D(A)$ with $x_n \to x$ and $Ax_n = y_n \to y$. We show that $x \in D(A)$ and $Ax = y$. By definition of $A$, we have

$$S(t)x_n = \int_0^t S(s)y_n \, ds + g_{\alpha+1}(t)Cx_n.$$

Taking limits on both sides and using the uniform boundedness of $S(t)$, we infer that

$$S(t)x = \int_0^t S(s)y \, ds + g_{\alpha+1}(t)Cx.$$

Hence $Ax = y$.

To prove (b), let $Ax = y$ and multiply (5) from the left by $C$. Since $S(t)$ and $C$ commute, we obtain

$$S(t)Cx = \int_0^t S(s)Cy \, ds + g_{\alpha+1}(t)C(Cx).$$

This shows $ACx = Cy$.

To prove (c), let $ACx = Cy$, this says

$$S(t)Cx = \int_0^t S(s)Cy \, ds + g_{\alpha+1}(t)CCx.$$
Since $C$ commutes with $S(t)$ and $C$ is injective, we may cancel one $C$ from the equation above and obtain (5), thus $Ax = y$. □

**Theorem 8.** Let $\alpha \geq 0$. The generator $A$ of a nondegenerate $\alpha$-times integrated $C$-regularized semigroup $\{S(t)\}_{t \geq 0}$ satisfies

(a) for any $x \in D(A)$ and $t \geq 0$, $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$;
(b) for any $x \in X$ and $t \geq 0$, $\int_0^t S(r)x \, dr \in D(A)$ and

$$A \int_0^t S(r)x \, dr = S(t)x - g_{\alpha+1}(t)Cx;$$

(c) $C^{-1}AC = A$.

**Proof.** (a) This is proved as in Theorem 7(b) with $S(t)$ instead of $C$.

To prove (b), we must show that for any $s, t \geq 0$ and $x \in X$, we have

$$S(s) \int_0^t S(r)x \, dr = g_{\alpha+1}(s)C \int_0^t S(r)x \, dr + \int_0^s S(u)\{S(t)x - g_{\alpha+1}(t)Cx\} \, du,$$

i.e.,

$$\int_0^t S(s)S(r)x \, dr - \int_0^s S(r)S(t)x \, dr$$

$$= g_{\alpha+1}(s) \int_0^t S(u)Cx \, du - g_{\alpha+1}(t)C \int_0^s S(u)Cx \, du.$$

By symmetry, we may assume $0 < s < t$. Then

$$\int_0^t S(s)S(r)x \, dr - \int_0^s S(r)S(t)x \, dr$$

$$= \int_0^t \left(\int_0^{s+r} - \int_0^r\right) g_\alpha(s + r - u)S(u)Cx \, du \, dr$$

$$- \int_0^s \left(\int_0^{t+r} - \int_0^r\right) g_\alpha(t + r - u)S(u)Cx \, du \, dr$$

$$= \int_0^t \left(\int_0^{s+r} - \int_0^r\right) g_\alpha(r - u)S(u)Cx \, du \, dr$$
\[-\int_{t}^{s+t} \int_{r}^{s} \int_{0}^{t} g_{\alpha}(r-u)S(u)x \, dr \, du \]
\[= \left( \int_{0}^{s+t} \int_{u}^{s} \int_{0}^{t} \right) - \left( \int_{0}^{s+t} \int_{u}^{s} \int_{0}^{t} \right) g_{\alpha}(r-u)S(u)x \, dr \, du \]
\[-\left( \int_{0}^{s+t} \int_{u}^{s} \int_{0}^{t} \right) g_{\alpha}(r-u)S(u)x \, dr \, du \]
\[= \int_{0}^{s} \left( \int_{0}^{s+t} \int_{0}^{t} + \int_{0}^{t} \int_{0}^{s+t} \right) \int_{s}^{u} g_{\alpha}(r-u)S(u)x \, dr \, du \]
\[+ \int_{s}^{t} \left( \int_{0}^{s+t} \int_{0}^{t} \right) + \int_{0}^{t} \int_{u}^{s+t} \int_{u}^{t} g_{\alpha}(r-u)S(u)x \, dr \, du \]
\[-\int_{0}^{s} \int_{0}^{t} \int_{0}^{s+t} g_{\alpha}(r-u)S(u)x \, dr \, du \]
\[+ \int_{s}^{t} \int_{0}^{u+s} g_{\alpha}(r-u)S(u)x \, dr \, du \]
\[= \int_{0}^{s} \left( g_{\alpha+1}(s) - g_{\alpha+1}(t) \right) S(u)x \, du + \int_{s}^{t} g_{\alpha+1}(s) S(u)x \, du \]
\[= g_{\alpha+1}(s) \int_{0}^{t} S(u)x \, du - g_{\alpha+1}(t) \int_{0}^{s} S(u)x \, du. \]

Thus, it is proved that \( \int_{0}^{t} S(r)x \, dr \in D(A) \) and \( A \int_{0}^{t} S(r)x \, dr = S(t)x - \frac{t^{\alpha}}{\Gamma(\alpha+1)}Cx. \)

(c) This is an equivalent formulation for Theorem 7(b) and (c). \( \square \)

3. Abstract Cauchy problems

Generators of \( \alpha \)-times integrated \( C \)-regularized semigroups, like the generators of \( C_{0} \)-semigroups, are characterized by the solvability of an abstract Cauchy problem in the state space. The basic problem is

\[ u'(t) = Au(t) + f(t), \quad t > 0, \quad u(0) = 0, \quad (6) \]
where \( f \in C([0, \infty); X) \). A function \( u \) is a solution of the abstract Cauchy problem (6) if
\[
u \in C^1((0, \infty); X) \cap C([0, \infty); [D(A)])
\]
and satisfies (6).

However, solutions to this problem are only to be expected if \( f \) is sufficiently smooth. We will therefore investigate the integrated form of (6):

\[
u(t) = A \int_0^t u(s) ds + \int_0^t f(s) ds.
\]

(7)

A function \( u \) is a solution to (7) if
\[
u \in C([0, \infty); X), \int_0^t u(s) ds \in D(A), \text{ and (7) is satisfied for all } t \geq 0.
\]

Notice that Theorem 8(b) states that an \( \alpha \)-times integrated \( C \)-regularized semigroup solves (7) with \( f = g_\alpha Cx \).

**Lemma 9.** Let \( \{S(t)\}_{t \geq 0} \) be an \( \alpha \)-times integrated \( C \)-regularized semigroup on \( X \) with generator \( A \). Let \( u \in C([0, T], D(A)) \cap C^1([0, T], X) \) such that \( u(0) = 0 \) and

\[
d\frac{d}{dt} u(t) = Au(t) + f(t),
\]

(8)

where \( f \in C([0, T], X) \). Then

\[
\int_0^t g_\alpha(t-s)Cu(s) ds = \int_0^t S(t-s)f(s) ds.
\]

(9)

In particular, if \( f(t) = 0 \) for all \( t > 0 \), then \( u = 0 \).

**Proof.** Since \( u(s) \in D(A) \), we may fix \( s \) and differentiate the equation

\[
S(\tau)u(s) = A \int_0^\tau S(\sigma)u(s) d\sigma + g_{\alpha+1}(\tau)Cu(s)
\]

and obtain

\[
\frac{d}{d\tau} S(\tau)u(s) = S(\tau)Au(s) + g_\alpha(\tau)Cu(s).
\]

From this identity we infer

\[
\frac{d}{ds} \left[ S(t-s)u(s) \right] \\
= \left( \frac{d}{ds} S(t-s) \right) u(s) + S(t-s) \left( \frac{d}{ds} u(s) \right) \\
= -AS(t-s)u(s) - g_\alpha(t-s)Cu(s) + S(t-s)Au(s) + S(t-s)f(s) \\
= -g_\alpha(t-s)Cu(s) + S(t-s)f(s).
\]

We integrate this identity and obtain
0 = S(t)u(0) - S(0)u(t)
\quad = \int_0^t \left( \frac{d}{ds} S(t-s)u(s) \right) ds = \int_0^t (-g_\alpha(t-s)Cu(s) + S(t-s)f(s)) ds.

If \( f = 0 \), then we infer that \( g_\alpha * Cu = 0 \). By Titchmarsh’s theorem and injectivity of \( C \), we infer that \( u = 0 \). □

**Theorem 10.** Let \( C \) be a bounded injective linear operator on \( X \), and \( A : D(A) \subset X \to X \) be a linear operator satisfying properties (a)–(c) of Theorem 7. Let \( \alpha \geq 0 \). Then the following properties are equivalent:

(a) \( A \) is the generator of an \( \alpha \)-times integrated, \( C \)-regularized semigroup \( S(t) \).
(b) For each \( x \in X \), there exists a unique solution \( u(t; x) \) to

\[
    u(t; x) = A \int_0^t u(s; x) ds + g_\alpha + 1 Cx. \tag{10}
\]

In this case, the solution is \( u(t; x) = S(t)x \).
(c) If \( f = Ch \), where \( w = g_{1-\alpha} * h \) is a function of bounded variation with values in \( X \), then there exists a unique solution \( u \) to \( (7) \). In this case the solution is \( u(t) = \int_0^t S(t-s) dw(s) \).

**Proof.** We show first that (a) implies (c): To prove uniqueness it is sufficient that \( (7) \) with \( f = 0 \) admits only the trivial solution. Suppose that \( u \) solves \( (7) \) with \( f = 0 \), and let \( v(t) = \int_0^t u(s) ds \). We obtain then \( \frac{d}{dt} v(t) = Av(t) \) and from Lemma 9 we infer that \( v = 0 \). Consequently \( u = 0 \).

To prove existence, let \( f = Ch \) and \( w = g_{1-\alpha} * h \) be of bounded variation. We put

\[
    u(t) = \int_0^t S(t-s) dw(s).
\]

Fix \( t > 0 \). Notice that \( \{ w(s) : s \in [0, t] \} \) is compact, so that the functions \( \tau \to S(\tau)w(s) \) are equicontinuous on \([0, t]\). We may therefore choose partitions \( 0 = s_0^N < s_1^N < \cdots < s_m^N < t \) such that

\[
    \sum_{j: s_j^N \leq \tau} S(\tau - s_j^N)(w(s_{j+1}^N) - w(s_j^N)) \to u(\tau)
\]

uniformly for \( \tau \in [0, t] \), and

\[
    \sum_{j: s_j^N \leq t} g_{1+\alpha}(t-s_j^N)(w(s_{j+1}^N) - w(s_j^N)) \to \int_0^t g_{1+\alpha}(\tau-s) dw(s).
\]

Using Theorem 8(b), we compute
\[
A \int_0^t \sum_{s_j^N \leq \tau} S(\tau - s_j^N)(w(s_{j+1}^N) - w(s_j^N)) d\tau
= A \sum_{j: s_j^N \leq t} \int_{s_j^N}^{s_{j+1}^N} S(\tau - s_j^N)(w(s_{j+1}^N) - w(s_j^N)) d\tau
= \sum_{j: s_j^N \leq t} \left[ S(t - s_j^N)(w(s_{j+1}^N) - w(s_j^N)) - g_{\alpha+1}(t - s_j^N)C(w(s_{j+1}^N) - w(s_j^N)) \right].
\]

We take limits and exploit the closedness of \(A\). Notice also that \(g_{\alpha+1}\) is the antiderivative of \(g_{\alpha}\).

\[
A \int_0^t u(\tau) d\tau = u(t) - C \int_0^t g_{\alpha+1}(t - s)dw(s)
= u(t) - \int_0^t g_{\alpha}(t - s)Cw(s)ds = u(t) - (g_{\alpha} \ast g_{1-\alpha} \ast f)(t)
= u(t) - \int_0^t f(s)ds.
\]

It is obvious that (c) implies (b), putting
\[
w(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ x & \text{if } t > 0. \end{cases}
\]

To show that (b) implies (a), we put \(S(t)x = u(t; x)\), where \(u\) solves (10). The operator
\[
S: X \to C([0, T], X),
\]
\[
x \mapsto u(\cdot; x)
\]
is evidently a linear operator. From the closedness of \(A\) it is easily seen that \(S\) is a closed linear operator. Hence \(S\) is bounded, and the operators \(S(t)\) for \(t \in [0, T]\) are uniformly bounded linear operators. Since \(A\) satisfies property (b) of Theorem 7, we infer that \(Cu(\cdot; x)\) solves (10) with \(Cx\) instead of \(x\), thus \(S(t)Cx = u(t; Cx) = Cu(t; x) = CS(t)x\).

We want now to show that
\[
S(t)S(s) = \left( \int_s^{s+t} - \int_0^t \right) g_{\alpha}(s + \tau - \tau)S(\tau)Cx d\tau.
\]

For this purpose fix \(s > 0\), put \(u(t) = u(t; x)\) and
\[
v(t) = \left( \int_s^{s+t} - \int_0^t \right) g_{\alpha}(s + \tau - \tau)S(\tau)Cx d\tau.
\]
We have to show that \( v(t) = S(t)S(s)x = u(t; S(s)x) \), i.e., that \( v \) solves (10) with \( u(s) \) instead of \( x \). For shorthand we introduce the functions

\[
\tilde{u}(t) = u(t + s) \quad \text{and} \quad \tilde{g}_\alpha(t) = g_\alpha(t + s).
\]

Notice that \( v \) may be rewritten by convolution

\[
v(t) = C(\tilde{g}_\alpha * u)(t).
\]

Integration of \( v \) will be expressed by convolution with the unit function 1. From (10) we obtain

\[
A(1 * \tilde{u})(t) = u(t + s) - g_{\alpha+1}(t + s)Cx - u(s) + g_{\alpha+1}(s)Cx
\]

= \((\tilde{u} - \tilde{g}_{\alpha+1}Cx - u(s)1 + g_{\alpha+1}(s)Cx1)(t)\).

We compute now

\[
A(1 * v)(t)
= CA\left(1 \ast (g_\alpha \ast \tilde{u} - \tilde{g}_\alpha \ast u)\right)(t)
= C\left(g_\alpha \ast A(1 \ast \tilde{u}) - \tilde{g}_\alpha \ast A(1 \ast u)\right)(t)
= C\left(g_\alpha \ast \tilde{u} - \tilde{g}_{\alpha+1}Cx - u(s) + g_{\alpha+1}(s)Cx\right)
\]

= \((\tilde{u} - \tilde{g}_{\alpha+1}Cx - u(s)1 + \tilde{g}_\alpha(u - g_{\alpha+1}Cx))(t)\)

= \((\tilde{u} - \tilde{g}_{\alpha+1}Cx - u(s)1 + \tilde{g}_\alpha(u - g_{\alpha+1}Cx))(t)\).

A straightforward integration by parts shows that

\[
(g_{\alpha \ast \tilde{g}_{\alpha+1}} - \tilde{g}_{\alpha \ast g_{\alpha+1}})(t)
= \int_0^t g_{\alpha+1}(t - \tau)g_{\alpha+1}(\tau + s) - g_{\alpha+1}(t - \tau)g'_{\alpha+1}(\tau + s) d\tau
= -g_{\alpha+1}(t - \tau)g_{\alpha+1}(\tau + s)|_0^t
= g_{\alpha+1}(t)g_{\alpha+1}(s)
= (g_{\alpha \ast 1})(t)g_{\alpha+1}(s).
\]

Thus

\[
v(t) = A(1 \ast v)(t) + g_{\alpha+1}Cu(s).
\]

Finally we have to show that \( A \) is the generator of \( \{S(t)\} \). For this purpose we denote the generator by \( B \).

First let \( x \in D(B) \) with \( Bx = y \). We have to show that \( x \in D(A) \) with \( Ax = y \). By definition of the generator and (10), we have

\[
\int_0^t S(s)Bx \, ds = \int_0^t S(s)y \, ds = S(t)x - g_{\alpha+1}Cx = A \int_0^t S(s)x \, ds.
\]
The left side of this equation has a derivative. Using the closedness of $A$, we can differentiate and have $S(s)x \in D(A)$ with $AS(s)x = S(s)y$. Next

\[ g_{\alpha+1}(t)Cx = S(t)x - A \int_0^t S(s)x \, ds = S(t)x - \int_0^t S(s)y \, ds. \]

Therefore $Cx \in D(A)$ with

\[ g_{\alpha+1}(t)ACx = S(t)y - A \int_0^t S(s)y \, ds = g_{\alpha+1}Cy. \]

Hence $ACx = Cy$. By assumption, $A$ satisfies property (c) of Theorem 7. Hence $x \in D(A)$ with $Ax = y$.

Now let $x \in D(A)$ with $Ax = y$. We have to show that $x \in D(B)$ with $Bx = y$, i.e., putting

\[ v(t) = \int_0^t S(s)y \, ds + g_{\alpha+1}Cx, \]

we have to show that $v(t) = S(t)x$. We show that $v$ is the unique solution of (10). We will use the closedness of $A$ and property (b) from Theorem 7:

\[ A \int_0^t v(\tau) \, d\tau = A \int_0^t \left( \int_0^\tau S(s)y \, ds + g_{\alpha+1}(\tau)Cx \right) \, d\tau \]
\[ = \int_0^\tau \left( A \int_0^\tau S(s)y \, ds + g_{\alpha+1}(\tau)ACx \right) \, d\tau \]
\[ = \int_0^\tau \left( S(\tau)y - g_{\alpha+1}(\tau)Cy + g_{\alpha+1}(\tau)ACx \right) \, d\tau \]
\[ = \int_0^\tau S(\tau)y \, d\tau \]
\[ = v(t) - g_{\alpha+1}(t)Cx. \]

Notice that part (c) of Theorem 7 is needed in the theorem above:

**Example 11.** There exists a bounded injective linear operator $C$ and a closed linear operator $A$ with $AC \subset CA$, such that with $\alpha = 1$ problem (10) admits a unique solution for each $x \in X$. However, the generator of the corresponding 1-times integrated, $C$-regularized semigroup is a proper extension of $A$. 

Proof. Let \( \{T(t)\}_{t \geq 0} \) be any \( C_0 \)-semigroup on a Banach space \( X \) generated by \( B \) and let \( C \) be an injective bounded linear operator which satisfies \( CT(t) = T(t)C \) for all \( t \geq 0 \) and \( \text{Rg}(C) = Y \subsetneq X \). Define

\[
S(t)x := \int_0^t T(s)Cx \, ds.
\]

Then \( \{S(t)\}_{t \geq 0} \) is a 1-times integrated \( C \)-regularized semigroup, where \( B \) is also the generator of \( \{S(t)\}_{t \geq 0} \). Further, let \( A = B|_{D(B) \cap Y} \). Then for all \( x \in X \) the function \( u(t; x) = S(t)x \) is the unique solution of

\[
u(t; x) = A \int_0^t u(s; x) \, ds + tCx.
\]

But \( A \) is not the generator of \( \{S(t)\}_{t \geq 0} \). \( \square \)

**Theorem 12.** Let \( \alpha \geq 0 \), \( C \) be an injective linear operator on Banach space \( X \) and \( A \) be closed linear operator. If \( A \) is a generator of an \( \alpha \)-times integrated \( C \)-regularized semigroup \( \{S(t)\}_{t \geq 0} \), then this semigroup is unique.

**Proof.** This follows from the fact that the semigroup provides the unique solution to (7). \( \square \)

4. Fractional calculus of \( \alpha \)-times integrated \( C \)-regularized semigroups

**Definition 13** [15]. Let \( \alpha > 0 \) and let \( f \) be integrable on any finite subinterval of \( [0, \infty) \). Then for \( t > 0 \) we call

\[
D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f(\zeta) \, d\zeta = g_\alpha \ast f(t)
\]

the fractional integral of \( f \) of order \( \alpha \).

If \( 0 < \alpha < n \) with an integer \( n \), and \( D_t^{\alpha-n} f \in W^{n,1}([0, T], X) \) for all \( T > 0 \), then

\[
D_t^{\alpha} f(t) = \frac{d^n}{dt^n} D_t^{\alpha-n} f(t)
\]

is called the fractional derivative of order \( \alpha \) of \( f \). For sake of completeness we put \( D_t^0 f = f \).

**Remark 14.** It is obvious from Remark 4 that \( D_t^{\alpha} D_t^{\beta} f = D_t^{\alpha+\beta} f \) for positive and negative \( \alpha \).
Theorem 15. Let \( \{S(t)\}_{t \geq 0} \) be a \( C \)-regularized \( \beta \)-times integrated semigroup on a Banach space \( X \). For any \( \alpha > 0 \), we define
\[
T(t)x := D_1^{-\alpha}S(t)x = \int_0^t g_\alpha(t - \tau)S(\tau)x \, d\tau, \quad \forall x \in X.
\]
Then \( \{T(t)\}_{t \geq 0} \) is an \((\alpha + \beta)\)-times integrated \( C \)-regularized semigroup on Banach space \( X \) with generator \( A \).

Proof. From Theorem 10 we know that \( u(t; x) = S(t)x \) is the unique solution of
\[
u(t; x) = A t \int_0^t u(s; x) \, ds + g_{\beta + 1}(t)Cx.
\]
Since \( A \) is the generator of \( \{S(t)\} \), it satisfies the properties (a)–(c) of Theorem 7. Again by Theorem 10 all we have to show is that \( v(t; x) = T(t)x \) is the unique solution of
\[
v(t; x) = A t \int_0^t v(s; x) \, ds + g_{\alpha + \beta + 1}(t)Cx.
\]
Uniqueness follows easily, since any solution of (13) with \( x = 0 \) solves also (12) and therefore must be the zero solution. For shorthand we write \( u \) for \( u(\cdot; x) \) and \( v \) for \( v(\cdot; x) \). Now take convolutions in (12) and use the closedness of \( A \):
\[
v(t) = (g_\beta * u(\cdot))(t) = g_\beta * \left( A(1 * u) + g_{\alpha + 1}Cx \right)(t)
= A(1 * g_\beta * u)(t) + g_{\alpha + \beta + 1}(t)Cx = A t \int_0^t v(s) \, ds + g_{\alpha + \beta + 1}(t)Cx.
\]

Theorem 16. Let \( 0 \leq \alpha \leq \beta < 1 \) and \( \{S(t)\}_{t \geq 0} \) be a differentiable \( \beta \)-times integrated \( C \)-regularized semigroup on \( X \) with generator \( A \). Assume that \( T(t)x = D_1^{\alpha}S(t)x \) exists and is strongly continuous, i.e., for all \( x \in X \), \( \int_0^t g_{1-\alpha}(t - s)S(s)x \, ds \) is continuously differentiable. Then \( T(t) \) is an \((\beta - \alpha)\)-times integrated, \( C \)-regularized semigroup with generator \( A \).

Proof. Put \( W(t)x = D_1^{\alpha - 1}S(t)x \). From Theorem 15 we know that \( \{W(t)\} \) is a \((\beta + 1 - \alpha)\)-times integrated, \( C \)-regularized semigroup with generator \( A \). By Theorem 10, we know that for fixed \( x \) the function \( u(t) = W(t)x \) is the unique solution of
\[
u(t) = A t \int_0^t u(s) \, ds + g_{2 + \beta - \alpha}Cx.
\]
By the same theorem, all we have to show is that \( v(t) = \frac{d}{dt} u(t) \) is the unique solution to

\[
v(t) = A \int_0^t v(s) \, ds + g_{1+\beta-\alpha} Cx.
\]

The left side of the equation

\[
u(t) - g_{2+\beta-\alpha}(t)Cx = A \int_0^t u(s) \, ds
\]

is continuously differentiable. Using the closedness of \( A \), we may differentiate and obtain

\[
u'(t) - g_{1+\beta-\alpha}(t)Cx = Au(t),
\]

which can easily be rewritten as

\[
v(t) - g_{1+\beta-\alpha}(t)Cx = A \int_0^t v(s) \, ds.
\]

Uniqueness of the solution follows as in the proof of Theorem 15.

Finally we turn to the case of a Lipschitz continuous, 1-times integrated \( C \)-regularized semigroup. We will show that it corresponds to a strongly continuous \( C \)-regularized semi-group on a closed subspace of \( X \).

**Definition 17** [2]. An \( \alpha \)-times integrated \( C \)-regularized semigroup is called locally Lipschitz continuous if, for all \( \tau > 0 \), there exists a constant \( k(\tau) > 0 \) such that

\[
\| S(t)x - S(s)x \| \leq k(\tau)|t - s|\|x\| \quad \text{for all } t, s \in [0, \tau].
\]

**Theorem 18.** Let \( \{S(t)\}_{t \geq 0} \) be a Lipschitz continuous, 1-times integrated \( C \)-regularized semigroup on Banach space \( X \) with generator \( A \). Then the following are satisfied:

(a) \( \frac{d}{dt} S(t)x \) exists for all \( x \in D(A) \).
(b) \( \frac{d}{dt} S(t)x =: T(t)x \) admits a continuous extension to \( \overline{D(A)} \).
(c) \( \{T(t)\}_{t \geq 0} \) is a \( C \)-regularized, strongly continuous semigroup on \( Y = \overline{D(A)} \).
(d) The generator of \( \{T(t)\}_{t \geq 0} \) on \( Y \) is \( A_0 \), where \( D(A_0) = \{x \in D(A): Ax \in \overline{D(A)}\} \) and \( A_0 x = Ax \).

**Proof.** For \( x \in D(A) \), we have \( S(t)x = tCx + \int_0^t S(s)Ax \, ds \). Hence \( \frac{d}{dt} S(t)x \) exists for all \( x \in D(A) \) and

\[
T(t)x = \frac{d}{dt} S(t)x = Cx + S(t)Ax = A \int_0^t T(s)x \, ds + Cx.
\]
For \( x \in D(A) \) we have that \( S(t)x \) is in \( D(A) \). Consequently \( T(t)x = \frac{d}{dt} S(t)x \) in \( D(A) \). Using the Lipschitz continuity of \( S(t) \), we have for \( 0 \leq t < \tau \) and \( x \in D(A) \) that

\[
\| T(t)x \| = \lim_{h \to 0} h^{-1} \| S(t+h)x - S(t)x \| \leq k(\tau) \| x \| .
\]

Therefore \( T(t) : D(A) \to Y = \overline{D(A)} \) can be extended to a bounded linear operator \( T(t) : Y \to Y \).

By closedness of \( A \), we can extend the equation

\[
T(t)x = A \int_0^t T(s)x \, ds + Cx
\]

to the case \( x \in Y \). Notice that for \( x \in Y \) we also have \( \int_0^t T(s)x \, ds \in Y \) and \( T(t)x - Cx \in Y \). Therefore the equation above uses only the part of \( A \) in \( Y \). With \( u(t) = T(t)x \) we have

\[
u(t) = A_0 \int_0^t u(s) \, ds + Cx .
\]

Thus (15) admits a solution for each \( x \in Y \). Since \( S(t) \) is an integrated \( C \)-regularized semigroup, the solutions to

\[
u(t) = A \int_0^t u(s) \, ds + tCx
\]

are unique. This implies that the only solution to (15) with \( x = 0 \) is the zero solution. Thus the solutions of (15) are unique. From Theorem 10 we infer that \( T(t) \) is a \( C \)-regularized semigroup on \( Y \) with generator \( A_0 \).

\( \square \)

5. Exponentially bounded \( \alpha \)-times integrated \( C \)-regularized semigroups

It is well known that every \( C_0 \)-semigroup is exponentially bounded, but this property is not necessarily satisfied for \( \alpha \)-times integrated \( C \)-regularized semigroups, as the following examples show:

Example 19. Let \( X = l_1 = \{ (x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n| < \infty \} \) and \( C : l_1 \to l_1 \) be defined by

\[
C(x) = \left( e^{-n^2} x_n \right) \quad \text{for} \quad x = (x_n) \in l_1 .
\]

Define \( S(t) \) by

\[
S(t)x = \left( e^{nt-n^2} x_n \right) \quad \text{for} \quad x = (x_n) \in l_1 \quad \text{and} \quad t \geq 0 .
\]

Then \( \{ S(t) \}_{t \geq 0} \) is a nondegenerate 0-times integrated \( C \)-regularized semigroup, but it is not exponentially bounded.
In fact, 
\[ \| S(2n) \| = \sup \{ \| S(2n)x \| : \| x \| = 1 \} \geq \| S(2n)e_n \| = e^{n^2} \]
for \( n = 1, 2, 3, \ldots \). Notice also that \( C \) is a bounded injective linear operator with dense range.

**Example 20.** Let \( X = L^2(0, \infty) \) and \( \{ S(t) \}_{t \geq 0} \) be a family of linear operators on \( X \) defined by
\[
(S(t)f)(x) = \frac{1}{x} (e^{tx} - 1) e^{-x^2} f(x).
\]
It is clear that \( \{ S(t) \}_{t \geq 0} \) is a 1-times integrated \( C \)-regularized semigroup with \( (Cf)(x) = e^{-x^2} f(x) \) for all \( x > 0, f \in X \), but it is not exponentially bounded.

As for \( C_0 \)-semigroups, there is a close connection between exponentially bounded integrated regularized semigroups and the resolvents of their generators. The link is given by the Laplace transform which we denote by \( \hat{f} \).

**Theorem 21.** Let \( A \) be the generator of an exponentially bounded, \( \alpha \)-times integrated \( C \)-regularized semigroup \( \{ S(t) \}_{t \geq 0} \) (i.e., \( \| S(t) \| \leq Me^{\omega t} \) for some \( M, \omega \in \mathbb{R}^+ \)). Then for \( \Re(\lambda) > \omega \) the operator \( \lambda I - A \) is injective and its range contains the range of \( C \). In particular,
\[
(\lambda I - A)^{-1}Cx = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)x \, dt = \lambda^\alpha \hat{S}(\lambda)x.
\]

**Proof.** Fix some \( \lambda \) with \( \Re(\lambda) > \omega \). For \( x \in X \) we define
\[
Bx = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)x \, dt.
\]
\( B \) is a bounded linear operator since \( \{ S(t) \}_{t \geq 0} \) is exponentially bounded. Integration by parts yields
\[
Bx = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)x \, dt = \lambda^{\alpha+1} \int_0^\infty e^{-\lambda t} \int_0^t S(s)x \, ds \, dt.
\]
Using the closedness of \( A \) and Remark 4, we infer
\[
ABx = \lambda^{\alpha+1} \int_0^\infty e^{-\lambda t} A \int_0^t S(s)x \, ds \, dt
\]
\[
= \lambda^{\alpha+1} \int_0^\infty e^{-\lambda t} \left( S(t)x - g_{\alpha+1}(t)Cx \right) \, dt
\]
\[ = \lambda B x - \lambda^{\alpha + 1} \hat{g}_{\alpha + 1}(\lambda) C x \]
\[ = \lambda B x - C x. \]

Thus \((\lambda I - A) B x = C x\) for all \(x \in X\). Furthermore by generator properties, for every \(x \in D(A)\) we have \(A S(t) x = S(t) A x\). Hence

\[
B A x = \lambda^{\alpha + 1} \int_0^\infty e^{-\lambda t} \int_0^t S(s) A x \, ds \, dt \\
= \lambda^{\alpha + 1} \int_0^\infty e^{-\lambda t} \left( A \int_0^t S(s) x \, ds \right) dt \\
= \lambda B x - C x.
\]

Thus \(B(\lambda I - A) x = C x\) for all \(x \in D(A)\). We conclude that for all \(x \in X\) we have

\[
(\lambda I - A)^{-1} C x = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t) x \, dt. \quad \square
\]

**Lemma 22.** Let \(A\) be a closed linear operator in a Banach space \(X\) and let \(f\) and \(g\) be Banach space valued functions such that the Laplace transforms \(\hat{f}(\lambda)\) and \(\hat{g}(\lambda)\) exist for \(\Re(\lambda) > \omega\). Assume that in this half-plane \(\hat{f}(\lambda) \in D(A)\) with \(A \hat{f}(\lambda) = \hat{g}(\lambda)\). Then \(f(t) \in D(A)\) with \(A f(t) = g(t)\) for almost all \(t\).

**Proof.** For \(\rho > \omega\) and all \(t > 0\)

\[
(1 * 1 * f)(t) = \frac{1}{2\pi i} \int_{\rho - i \infty}^{\rho + i \infty} e^{\lambda t} \lambda^{-2} \hat{f}(\lambda) \, d\lambda,
\]
\[
(1 * 1 * g)(t) = \frac{1}{2\pi i} \int_{\rho - i \infty}^{\rho + i \infty} e^{\lambda t} \lambda^{-2} A \hat{f}(\lambda) \, d\lambda.
\]

Using the closedness of \(A\), we obtain that \(A(1 * 1 * f)(t) = (1 * 1 * g)(t)\). Differentiating twice and using the closedness of \(A\), again we obtain \(A f(t) = g(t)\) almost everywhere. \(\square\)

**Theorem 23.** Let \(C\) be a bounded injective linear operator in a Banach space \(X\). Let \(A : D(A) \subset X \to X\) be a linear operator satisfying properties (a)–(c) of Theorem 7 and such that \(\lambda - A\) is injective for \(\Re(\lambda) > \omega\). Let \(\alpha \in (0, 1]\) and \(\{S(t)\}_{t \geq 0}\) be a strongly continuous, exponentially bounded family of linear operators in \(X\) such that for all \(x \in X\) and \(\Re(\lambda) > \omega\),

\[
(\lambda - A)^\alpha \hat{S}(\lambda) x = C x.
\]

Then \(\{S(t)\}\) is an \(\alpha\)-times integrated, \(C\)-regularized semigroup on \(X\) with generator \(A\).
**Proof.** Use Lemma 22 and Remark 4 on 

\[ A\lambda^{-1} \hat{S}(\lambda) = \hat{S}(\lambda) - \lambda^{-\alpha - 1} C x \]

to obtain

\[ A \int_0^t S(s) x \, ds = S(t) x - g_{\alpha + 1} C x. \]

Therefore \( u(t; x) = S(t) x \) is a solution to (10). Suppose that the solution is not unique. Then there exists some \( v \in C([0, \infty), X) \) such that

\[ v(t) = A \int_0^t v(s) \, ds. \]  

(16)

We put \( w(t) = \int_0^t v(s) \, ds \). Then \( w \) and \( Cw \) are in \( C([0, \infty), D(A)) \) and solve (16). We will now construct an exponentially bounded solution of (16). First define

\[ y(t) = g_{1 - \alpha} * S(\cdot) Aw(1) + 1.Cw(1). \]

Notice that

\[ (\lambda - A) \hat{S}(\lambda) w(1) = \lambda^{-\alpha} C w(1), \]

so that

\[ \hat{y}(\lambda) = \lambda^{-\alpha - 1} A \hat{S}(\lambda) w(1) + \lambda^{-1} C w(1) \]
\[ = \lambda^{-\alpha - 1} (\lambda \hat{S}(\lambda) w(1) - \lambda^{-\alpha} C w(1)) + \lambda^{-1} C w(1) \]
\[ = \lambda^\alpha \hat{S}(\lambda) w(1), \]

and

\[ (\lambda - A) \hat{y}(\lambda) = C w(1). \]

Using Lemma 22, we see that \( y \) solves

\[ y(t) - A \int_0^t y(s) \, ds = C w(1). \]

Now we patch the desired solution:

\[ z(t) = \begin{cases} 
C w(t) & \text{if } t < 0, \\
y(t - 1) & \text{if } t \geq 0.
\end{cases} \]

Then \( z \) is exponentially bounded and satisfies (16). This is clear for \( t \leq 1 \). For \( t > 1 \) we compute

\[ A \int_0^t z(s) \, ds = A \left( \int_0^1 C w(s) \, ds + \int_1^t y(s - 1) \, ds \right) \]
\[ = C w(1) + y(t - 1) - C w(1) = z(t). \]
Hence,
\[(\lambda - A)\hat{z}(\lambda) = 0\]
for all \(\lambda\) with sufficiently large real part, in contradiction to the injectivity of \((\lambda - A)\).

**Remark 24.** Again, property (c) of Theorem 7 is needed. However, if \((\lambda - A)\) is bijective from \(D(A)\) onto \(X\) for some \(\lambda \in \mathbb{C}\), then properties (a) and (b) imply property (c) of Theorem 7.

**Proof.** Suppose (a) and (b) hold, and \(Cx \in D(A)\) with \(ACx = Cy\). Let \(z \in D(A)\) be such that
\[(\lambda - A)z = \lambda x - y.\]
By property (b), we have
\[(\lambda - A)Cz = (\lambda - A)Cx.\]
By injectivity of \(\lambda - A\) and \(C\), we infer that \(z = x\). Hence \(x \in D(A)\) with \(Ax = y\).

**References**
