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Axiomatizing the subsumption and subword preorders on finite and infinite partial words

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Dedicated to Professor Masami Ito on the occasion of his 60th birthday

Abstract

We consider two-sorted algebras of finite and infinite partial words equipped with the subsumption preorder and the operations of series and parallel product and omega power. It is shown that the valid equations and inequations of these algebras can be described by an infinite collection of simple axioms, and that no finite axiomatization exists. We also prove similar results for two related preorders, namely for the induced partial subword preorder and the partial subword preorder. Along the way of proving these results, we provide a concrete description of the free algebras in the corresponding varieties in terms of generalized series–parallel partial words. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

The notion of word, i.e., isomorphism class of a (finite) total order equipped with a labeling function, may be generalized in two directions to obtain partial words. On the one hand, a partial word is an isomorphism class of labeled total orders equipped with a partially defined labeling function, see [3]. In this paper, following [10], we define a partial word as an isomorphism class of a labeled partial order.

Partially ordered structures, and in particular partial words or pomsets for partially ordered multisets [15], have been used extensively to give semantics to concurrent languages [15, 8, 14, 7, 1, 2], and to Petri nets [10, 19, 13, 20, 18], to mention a few references. The event structures of Winskel [21, 22] are partial words enriched with a conflict relation subject to certain conditions. A wide variety of operations have been studied on partial words. The operations of series product $P \cdot Q$ and parallel product

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$P \otimes Q$ have a fundamental role in many applications. In [6], the authors also considered the omega power operation P^{ω} which provides solutions to fixed-point equations of the sort

$$X = P \cdot X$$

for *finite nonempty* P. The models considered in [6] were two-sorted, since the series product and omega power operations $P \cdot Q$ and P^{ω} were restricted to finite partial words P. The restriction was due to the fact that only those partial words P represent the behavior of concurrent processes which satisfy the condition that each vertex of P generates a finite principal ideal. It was shown in [6] that the equational theory of partial words equipped with the operations of series and parallel product and omega power can be axiomatized by an infinite collection of simple equations, and that no finite axiomatization exists. In this paper, we consider the same algebras of partial words, but also equipped with the *subsumption preorder* [15, 9] defined by $P \leqslant Q$ if and only if there is a monotonic and label-preserving bijective function $Q \to P$. We prove that the valid inequations of these algebras have a finite axiomatization over the set of valid equations, and exhibit a finite relative inequational axiomatization. We also show that no finite inequational axiomatization exists and establish similar results for two related preorders: the *induced partial subword preorder* and the *partial subword preorder*. Along the way of proving these results, we provide a concrete description of the free algebras in the corresponding varieties in terms of generalized series–parallel partial words. Our arguments make use of the theorems proved in [6], and our results extend the axiomatization of the subsumption order in [9].

2. Partial words

We consider finite nonempty and countably infinite posets $P = (P, \leqslant_P, l_P)$ whose elements, called vertices, are labeled in a set A. Thus, l_P is a function $P \to A$. A morphism of A-labeled posets is a monotonic function which preserves the labeling. An isomorphism is a morphism which is an order isomorphism. An *A-labeled partial word*, or just partial word [10], for short, is an isomorphism class of A-labeled posets. We will identify isomorphic labeled posets with the partial word they represent.

Some notation: For each nonnegative integer n, we denote the set $\{1, \ldots, n\}$ by $[n]$. Moreover, we denote $[\omega] = \{1, 2, \ldots\}$.

Suppose that $P = (P, \leqslant_P, l_P)$ and $Q = (Q, \leqslant_Q, l_Q)$ are partial words. We define several operations, some of which will require that P is finite.

Series product. If P is finite, then the series product of P and Q is constructed by taking the disjoint union of P and Q and making each vertex of Q larger than any vertex of P. Thus, assuming without loss of generality that P and Q are disjoint,

$$P \cdot Q = (P \cup Q, \leqslant_{P \cdot Q}, l_{P \cdot Q}),$$

where, for any $u, v \in P \cup Q$,

$$u \leqslant_{P \cdot Q} v \Leftrightarrow (u \in P \text{ and } v \in Q) \text{ or } u \leqslant_P v \text{ or } u \leqslant_Q v,$$
$$l_{P \cdot Q}(u) = \begin{cases} l_P(u) & \text{if } u \in P, \\ l_Q(u) & \text{if } u \in Q. \end{cases}$$

Parallel product. The parallel product of P and Q is constructed as the disjoint union of P and Q. Thus,

$$P \otimes Q = (P \cup Q, \leqslant_{P \otimes Q}, l_{P \otimes Q}),$$

where we again assume that P and Q are disjoint. Moreover, for any $u, v \in P \cup Q$,

$$u \leqslant_{P \otimes Q} v \Leftrightarrow u \leqslant_P v \text{ or } u \leqslant_Q v,$$
$$l_{P \otimes Q}(u) = \begin{cases} l_P(u) & \text{if } u \in P, \\ l_Q(u) & \text{if } u \in Q. \end{cases}$$

Omega power. Assume that P is finite. The omega power of P, denoted P^ω, is the series product of P with itself ω-times. Thus,

$$P^\omega = (P \times [\omega], \leqslant_{P^\omega}, l_{P^\omega}),$$

where

$$(u, i) \leqslant_{P^\omega} (v, j) \Leftrightarrow i < j \text{ or } (i = j \text{ and } u \leqslant_P v),$$
$$l_{P^\omega}((u, i)) = l_P(u),$$

for all $(u, i), (v, j) \in P \times [\omega]$.

More generally, given disjoint finite partial words $P_i, i \geqslant 0$, we define $P = P_0 \cdot P_1 \cdot \ldots$ as the partial word on the set $\bigcup_{i \geqslant 0} P_i$ equipped with the partial order and labeling

$$u \leqslant_P v \Leftrightarrow i < j \text{ or } (i = j \text{ and } u \leqslant_{P_i} v),$$
$$l_P(u) = l_{P_i}(u),$$

for all vertices $u \in P_i$ and $v \in P_j$.

Let $\mathbf{Pw}_f(A)$ denote the collection of all finite (nonempty) A-labeled partial words. Moreover, let $\mathbf{Pw}_\omega(A)$ stand for the set of all (countably) infinite A-labeled partial words. Then we have the two-sorted algebra

$$\mathbf{Pw}(A) = (\mathbf{Pw}_f(A), \mathbf{Pw}_\omega(A), \cdot, \otimes, {}^\omega),$$

where for all $P, Q \in \mathbf{Pw}_f(A) \cup \mathbf{Pw}_\omega(A)$, $P \cdot Q$ and P^ω are defined if and only if $P \in \mathbf{Pw}_f(A)$. Moreover, $P \cdot Q$ and $P \otimes Q$ are in $\mathbf{Pw}_f(A)$ if and only if $P, Q \in \mathbf{Pw}_f(A)$. Thus, if $Q \in \mathbf{Pw}_\omega(A)$, then $P \cdot Q \in \mathbf{Pw}_\omega(A)$, and if P or Q is in $\mathbf{Pw}_\omega(A)$, then $P \otimes Q \in \mathbf{Pw}_\omega(A)$. We call the sets $\mathbf{Pw}_f(A)$ and $\mathbf{Pw}_\omega(A)$ the *carriers of finite and infinite sort*, respectively.

Example 2.1. In the following examples, we assume that a, b, c are letters in A. If u, v are vertices of a poset, ordered by $\leqslant$, we write $u < v$ to mean that $u \leqslant v$ and $u \neq v$.

(1) Consider the partial word $P \in \mathbf{Pw}_f(A)$ determined by the poset with vertices $v_1, \ldots, v_6$ such that v_1 is below v_2, v_3 and v_4, v_2 and v_3 are below v_4, and v_5 is below v_6, and where v_1, v_4, v_5 are labeled a, and the other vertices are labeled b. Identifying letter a and b with the singleton partial word labeled a and b, respectively, P can be given by the expression $(a \cdot (b \otimes b) \cdot a) \otimes (a \cdot b)$, where we take advantage of the associativity of $\cdot$.

(2) Let Q_1 be the partial word in $\mathbf{Pw}_f(A)$ determined by the poset with vertices $v_1, \ldots, v_4$ where $v_1 < v_3$ and $v_2 < v_4$, and there are no other nontrivial order relations. Each vertex is labeled a. Let Q_2 denote the partial word obtained from Q_1 by additionally requiring $v_2 < v_3$, and let Q_3 be the partial word in which, in addition to the order relations of Q_2, also $v_1 < v_4$ holds. Then $Q_1 = (a \cdot a) \otimes (a \cdot a) = a^2 \otimes a^2$ and $Q_3 = (a \otimes a) \cdot (a \otimes a) = (a \otimes a)^2$. On the other hand, Q_2 has no decomposition into a series or parallel product of two partial words.

(3) The partial word $R = a \cdot (b \cdot (c \otimes c))^\omega \in \mathbf{Pw}_\omega(A)$ is determined by the poset on the vertices (i, j), where $i \geqslant 0$, and $j = 1$ if i is 0 or odd, and $j = 1, 2$ if $i > 0$ is even. We have $(i, j) < (i', j')$ iff $i < i'$. Thus, vertices $(i, 1)$ and $(i, 2)$, where $i > 0$ is even, are parallel. Moreover, vertex 0 is labeled a, and a vertex (i, j) with $i > 0$ is labeled b if i is odd, and c if i is even.

Theorem 2.2 (Bloom and Ésik [6]). *The variety $\mathscr{V}$ generated by the algebras $\mathbf{Pw}(A)$ is axiomatized by the equations*

$$x \cdot (y \cdot u) = (x \cdot y) \cdot u, \tag{1}$$

$$u \otimes (v \otimes w) = (u \otimes v) \otimes w, \tag{2}$$

$$u \otimes v = v \otimes u, \tag{3}$$

$$(x \cdot y)^\omega = x \cdot (y \cdot x)^\omega, \tag{4}$$

$$(x^n)^\omega = x^\omega, \quad n \geqslant 2 \textit{ is prime}. \tag{5}$$

Here the variables x, y are of finite sort, and u, v, w can have either finite or infinite sort. (More precisely, in [6] the parallel product of two partial words is defined only if both partial words are finite or both are infinite, causing only a little change in the above result and its proof.) Note that (5) holds in $\mathscr{V}$ for all integers $n \geqslant 2$, and that the equation

$$x \cdot x^\omega = x^\omega \tag{6}$$

also holds in $\mathscr{V}$.

Remark 2.3. Eqs. (1), (4) and (5) define Wilke algebras [17] that have been used to construct syntactic algebras for ω-languages.

2.1. Varieties of preordered algebras

We will consider two-sorted *preordered algebras* $M=(M_f, M_\omega, \cdot, \otimes, ^\omega, \leqslant)$ equipped with binary operations $\cdot, \otimes$, a unary operation $^\omega$, and a preorder, i.e., a reflexive and transitive relation $\leqslant$ *preserved by the operations*. The preorder $\leqslant$ is defined on both M_f and M_ω, and in the algebras considered in Section 5, it may as well relate elements of M_f and M_ω. When the preorder is antisymmetric, the preordered algebra is sometimes referred to as an *ordered algebra* [4]. For preordered algebras $M=(M_f, M_\omega, \cdot, \otimes, ^\omega, \leqslant)$ and $M'=(M'_f, M'_\omega, \cdot, \otimes, ^\omega, \leqslant)$, a *morphism* $h: M \to M'$ is a pair of functions $h_f: M_f \to M'_f$, $h_\omega: M_\omega \to M'_\omega$ which jointly preserve the operations and the preorder. An *isomorphism* $M \to M'$ is a morphism $h: M \to M'$ such that the inverses h_f^{-1} and h_ω^{-1} exist and determine a morphism $M' \to M$. Thus, e.g., for all $x, y \in M_f$, $x \leqslant y$ in M iff $xh_f \leqslant yh_f$ in M'. We call M a *subalgebra* of M' if $M_f \subseteq M'_f$, $M_\omega \subseteq M'_\omega$, and if the inclusions determine a morphism $M \to M'$ such that for all $x, y \in M_f \cup M_\omega$, if $x \leqslant y$ in M' then $x \leqslant y$ in M. Moreover, we call M' a *morphic image* or *quotient* of M if there is a morphism $M \to M'$ whose components are surjective functions. *Direct products* are defined as usual, the preorder on the direct product is the pointwise preorder. A *variety* of preordered algebras is any class of preordered algebras closed under taking subalgebras, quotients and direct products. Birkhoff's Variety Theorem can be extended to preordered algebras in a natural way (see [4] for the case of ordered algebras). Given a set E of equations $t=t'$ and a set E' of inequations $t \leqslant t'$ between terms, let $\mathbf{Mod}(E, E')$ denote the class of all models of E and E', i.e., the class of all preordered algebras satisfying all equations in E as well as all inequations in E'. Then $\mathbf{Mod}(E, E')$ is a variety, and any variety is of this form. The proof relies on the existence of free preordered algebras in any class closed under subalgebras and direct products. We say that an algebra M with carriers M_f and M_ω is *freely generated by a pair of sets* (A, B) *in a class* $\mathscr{K}$ of preordered algebras, or that M is a *free algebra on* (A, B) *in* $\mathscr{K}$, if $M \in \mathscr{K}$ and there is a pair of functions $\eta_f: A \to M_f$, $\eta_\omega: B \to M_\omega$ with the following property: For any $M' \in \mathscr{K}$ with carriers M'_f and M'_ω and for any functions $h_f: A \to M'_f$ and $h_\omega: B \to M'_\omega$ there is a unique morphism $h^\sharp = (h_f^\sharp, h_\omega^\sharp): M \to M'$ with $\eta_f \circ h_f^\sharp = h_f$ and $\eta_\omega \circ h_\omega^\sharp = h_\omega$. It is clear that if both M and M' are freely generated by (A, B) in $\mathscr{K}$, then M and M' are isomorphic.

The free preordered algebra on a pair of sets (A, B) in the variety $\mathbf{Mod}(E, E')$ can be constructed from the algebra F freely generated by (A, B) in the Birkhoff variety $\mathbf{Mod}(E)$ by equipping F with the least preorder such that the resulting algebra satisfies the inequations in E'. In particular, for all E', the free preordered algebras in $\mathbf{Mod}(E, \emptyset)$ and $\mathbf{Mod}(E, E')$ have the same underlying $\leqslant$-free reduct. The ordered algebras in the variety of preordered algebras $\mathbf{Mod}(E, E')$ form a variety $\mathscr{V}$ of ordered algebras as defined in [4]. We call $\mathscr{V}$ the *variety of ordered algebras contained in* $\mathbf{Mod}(E, E')$. It consists of those ordered algebras satisfying all equations in E as well as all inequations in E'. The free algebra on (A, B) in $\mathscr{V}$ may be constructed as a quotient of the preordered algebra F freely generated by the sets (A, B) in $\mathbf{Mod}(E, E')$: We identify two elements x, y of F if both $x \leqslant y$ and $y \leqslant x$ hold. The partial order is the induced partial order.

3. The subsumption preorder

In this section, we study some properties of the subsumption preorder on partial words.

Definition 3.1. Suppose that $P = (P, \leqslant_P, l_P)$ and $Q = (Q, \leqslant_Q, l_Q)$ are partial words. We define $P \leqslant Q$ in the *subsumption preorder* if there is a bijective morphism from Q to P, i.e., a one-to-one and onto function $\varphi : Q \to P$ that preserves the partial order and the labeling.

Example 3.2. (1) Consider the partial words Q_1, Q_2 and Q_3 given in Example 2.1. Since the identity map on the set $\{v_1, v_2, v_3, v_4\}$ is a bijective morphism $Q_1 \to Q_2$, we have that $Q_2 \leqslant Q_1$ in the subsumption order. In the same way, $Q_3 \leqslant Q_2$ and $Q_3 \leqslant Q_1$. On the other hand, the identity function is not a morphism $Q_2 \to Q_1$. In fact, none of the relations $Q_1 \leqslant Q_2$, $Q_1 \leqslant Q_3$ and $Q_2 \leqslant Q_3$ holds.

(2) Consider the partial words a^ω and $a^\omega \otimes a^\omega$, determined by the labeled partial orders $(P, \leqslant_P, l_P)$ and $(Q, \leqslant_Q, l_Q)$, where $P = \{u_i : i \geqslant 0\}$, $Q = \{v_{i,j} : i \geqslant 0, j = 1, 2\}$, with $u_i \leqslant_P u_{i'}$ iff $i \leqslant i'$ and $v_{i,j} \leqslant_Q v_{i',j'}$ iff $i \leqslant i'$ and $j = j'$, and where $l_P(u_i) = l_Q(v_{i,j}) = a$, for all $i \geqslant 0$ and $j = 1, 2$. The function $v_{i,1} \mapsto u_{2i}$, $v_{i,2} \mapsto u_{2i+1}$ is a monotonic bijection that preserves the labels. Thus, $a^\omega \leqslant a^\omega \otimes a^\omega$. On the other hand, there is no monotonic bijection $P \to Q$, since any such function should map some vertices u_i and u_k to $v_{0,1}$ and $v_{0,2}$, respectively. But then the function does not preserve the order, for u_i and u_k are related in P, but $v_{0,1}$ and $v_{0,2}$ are parallel.

(3) The subsumption order is *not* a partial order. To see this, consider the partial words

$$a^\omega \otimes R,$$

$$a^\omega \otimes a^\omega \otimes R,$$

where R is a countably infinite discrete partial word whose vertices are labeled a. We may represent R as the labeled poset $(\{w_i : i \geqslant 0\}, \leqslant_R, l_R)$ where $w_i \leqslant_R w_j$ iff $i = j$, and where $l_R(w_i) = a$, for all $i \geqslant 0$. Thus, when P and Q denote the labeled posets of the previous example, $a^\omega \otimes R$ is represented by the disjoint union of P with R, and $a^\omega \otimes a^\omega \otimes R$ by the disjoint union of Q with R. Now the function defined by $u_i \mapsto v_{i,1}$, $w_{2i} \mapsto v_{i,2}$ and $w_{2i+1} \mapsto w_i$, $i \geqslant 0$, is a monotonic label-preserving bijection from the disjoint union of P and R onto the disjoint union of Q and R. Moreover, the function defined by $v_{i,1} \mapsto u_{2i}$, $v_{i,2} \mapsto u_{2i+1}$ and $w_i \mapsto w_i$, $i \geqslant 0$ is a monotonic bijection from the disjoint union of Q and R to the disjoint union of P and R. This proves that both

$$a^\omega \otimes R \leqslant a^\omega \otimes a^\omega \otimes R$$

and

$$a^\omega \otimes a^\omega \otimes R \leqslant a^\omega \otimes R$$

hold.

Nevertheless, the subsumption order is a partial order on an important subclass of the partial words.

Definition 3.3. We call an A-labeled partial word $P = (P, \leqslant_P, l_P)$ ω-linearizable if P is either finite or it has linearization to an ω-chain, i.e,. its elements can be enumerated in a sequence $u_0, u_1, \ldots$ such that $u_i \leqslant u_j$ implies $i \leqslant j$, for all $i, j \geqslant 0$.

The following fact is clear.

Proposition 3.4. *A partial word P is ω-linearizable if and only if each principal ideal of P is finite.*

Of course, an ideal of P is a nonempty downward closed subset of P, and a filter is a nonempty upward closed subset. A principal ideal is an ideal which is generated by a single vertex, i.e., consists of the vertices $\leqslant$ than a given vertex of P.

The *width* of a partial word P, denoted $w(P)$, is the maximal number of pairwise parallel vertices of P, if this number is finite. Otherwise the width of P is ω.

Proposition 3.5. *Suppose that P and Q are ω-linearizable partial words of finite width. If $P \leqslant Q$ and f is a bijective morphism $P \to Q$, then f is an isomorphism.*

Proof. Suppose first that P and Q are finite. Since $P \leqslant Q$, there is a bijective morphism $g: Q \to P$. Let $h = f \circ g$, so that h is a bijective morphism $P \to P$. Since P is finite, there is an integer $n > 0$ such that h^n is a the identity function $P \to P$. Thus, for all $x, y \in P$, if $f(x) \leqslant_Q f(y)$, then

$$x = h^n(x) = h^{n-1}(g(f(x))) \leqslant_P h^{n-1}(g(f(y))) = h^n(y) = y.$$

This proves that f is an isomorphism.

Suppose now that P and Q are infinite. Let g denote a bijective morphism $Q \to P$. For each $n \geqslant 0$, let P_n denote the subposet of P determined by the vertices of height at most n, and define Q_n in the same way. Since P and Q have finite width, it follows that P_n and Q_n are finite. Moreover, $P = \bigcup_n P_n$ and $Q = \bigcup_n Q_n$, and since f and g are injective morphisms, $f^{-1}(Q_n) \subseteq P_n$ and $g^{-1}(P_n) \subseteq Q_n$, for each $n \geqslant 0$. Since the sets P_n and Q_n are finite, it follows now that $f^{-1}(Q_n) = P_n$ and $g^{-1}(P_n) = Q_n$, for each n, so that by the first part of the proof, the restriction of f to P_n is an isomorphism $P_n \to Q_n$. Since this holds for every n, and since $P = \bigcup_n P_n$ and $Q = \bigcup_n Q_n$, it follows now that f itself is an isomorphism. $\square$

Corollary 3.6. *The subsumption preorder is a partial order on the set of ω-linearizable partial words with finite width.*

Equipped with the subsumption order, the structure

$$\mathbf{Pw}^{\leqslant}(A) = (\mathbf{Pw}_f(A), \mathbf{Pw}_\omega(A), \cdot, \otimes, {}^\omega, \leqslant)$$

is a two-sorted preordered algebra, so that the operations preserve the preorder $\leqslant$. The (*weak*) *interchange laws* are the following inequations:

$$(x \otimes y)\cdot(u \otimes v) \leqslant (x\cdot u) \otimes (y\cdot v), \tag{7}$$

$$x\cdot(u \otimes v) \leqslant (x\cdot u) \otimes v, \tag{8}$$

$$(x \otimes y)\cdot u \leqslant (x\cdot u) \otimes y, \tag{9}$$

$$x\cdot u \leqslant x \otimes u, \tag{10}$$

$$(x \otimes y)^\omega \leqslant x^\omega \otimes y^\omega, \tag{11}$$

where x, y are finite sort variables and u and v can independently have finite or infinite sort.

Proposition 3.7. *The weak interchange laws hold in all algebras* $\mathbf{Pw}^{\leqslant}(A)$.

Proof. We only prove that (7) holds, i.e., $(P \otimes Q)\cdot(R \otimes S) \leqslant (P\cdot R) \otimes (Q\cdot S)$, for all $P, Q \in \mathbf{Pw}_f(A)$ and $R, S \in \mathbf{Pw}_f(A) \cup \mathbf{Pw}_\omega(A)$. But this is clear, since $(P \otimes Q)\cdot(R \otimes S)$ is constructed by taking the disjoint union of P, Q, R, S and making each vertex in $R \cup S$ larger than any vertex in $P \cup Q$. The partial word $(P\cdot R) \otimes (Q\cdot S)$ is constructed as the same disjoint union, by making each vertex of R only larger than the vertices in P, and each vertex of S only larger than those in Q. □

4. Axiomatizing the subsumption preorder

Let $\mathscr{V}^{\leqslant}$ denote the collection of all preordered two-sorted algebras equipped with the above operations satisfying the equations that hold in $\mathscr{V}$ (equivalently, the equations given in Theorem 2.2), and the weak interchange laws. Thus, $\mathscr{V}^{\leqslant}$ is a variety of two-sorted preordered algebras. In this section we prove:

Theorem 4.1. *An equation or inequation holds in all algebras* $\mathbf{Pw}^{\leqslant}(A)$ *if and only if it holds in* $\mathscr{V}^{\leqslant}$.

Thus, the algebras $\mathbf{Pw}^{\leqslant}(A)$ generate the variety $\mathscr{V}^{\leqslant}$. Theorem 4.1 will follow from Theorem 4.6 which gives a concrete description of the free algebras in $\mathscr{V}^{\leqslant}$. To formulate this result, we need to consider partial words over a pair of disjoint sets. So suppose that A and B are disjoint. Let $\mathbf{Pw}_\omega(A, B)$ denote those $(A \cup B)$-labeled partial words which are either infinite or contain a vertex labeled in B, and such that any vertex labeled in B is maximal. Note that $\mathbf{Pw}_f(A)$ is disjoint from $\mathbf{Pw}_\omega(A,B)$. Moreover, if $P \in \mathbf{Pw}_f(A)$ and $Q \in \mathbf{Pw}_\omega(A,B)$, then $P \cdot Q \in \mathbf{Pw}_\omega(A,B)$, and if $P, Q \in \mathbf{Pw}_f(A) \cup \mathbf{Pw}_\omega(A,B)$ such that P or Q is in $\mathbf{Pw}_\omega(A,B)$, then $P \otimes Q \in \mathbf{Pw}_\omega(A,B)$. Thus, we have a two-sorted algebra $\mathbf{Pw}(A,B) = (\mathbf{Pw}_f(A), \mathbf{Pw}_\omega(A,B), \cdot, \otimes, {}^\omega)$, defined in the same way as the algebra $\mathbf{Pw}(A)$, so that $\mathbf{Pw}(A,\emptyset)$ is just $\mathbf{Pw}(A)$. Equipped with the subsumption preorder, this algebra is denoted $\mathbf{Pw}^\leqslant(A,B)$. Note that $\mathbf{Pw}(A,B) \in \mathscr{V}$ and $\mathbf{Pw}^\leqslant(A,B) \in \mathscr{V}^\leqslant$.

Definition 4.2. A partial word $P \in \mathbf{Pw}_f(A)$ is *series–parallel* if P belongs to the least subalgebra of $\mathbf{Pw}(A,B)$ (or $\mathbf{Pw}(A)$) containing the singletons. Similarly, a partial word $P \in \mathbf{Pw}_\omega(A,B)$ is *generalized series–parallel* if P is contained in the subalgebra of $\mathbf{Pw}(A,B)$ generated by the singletons.

Example 4.3. Consider the partial words given in Example 2.1. P, Q_1, Q_3 are series–parallel, and R is generalized series–parallel. On the other hand, Q_2 is not series–parallel. The partial words $R, a^\omega \otimes R$ and $a^\omega \otimes a^\omega \otimes R$ of Example 3.2 are not generalized series–parallel. In fact, every generalized series–parallel partial word has finite width, as implied by Theorem 4.8. Let S denote the partial word $(\{u_i\colon i \geqslant 0\}, \leqslant_S, l_S)$ ordered by the relation $u_i \leqslant u_j$ iff $j \leqslant i$ and equipped with the labeling function $l_S(u_i) = a$, for all $i \geqslant 0$. Then S is not generalized series–parallel.

Let $\mathbf{SP}_A$ denote the collection of all series–parallel partial words in $\mathbf{Pw}_f(A)$, and $\mathbf{SP}^\omega_{A,B}$ the collection of all generalized series–parallel partial words in $\mathbf{Pw}_\omega(A,B)$. These two sets determine a subalgebra of $\mathbf{Pw}(A,B)$ denoted $\omega\mathbf{SP}_{A,B}$. Equipped with the subsumption preorder, $\omega\mathbf{SP}_{A,B}$ is a preordered algebra denoted $\omega\mathbf{SP}^\leqslant_{A,B}$. We have that $\omega\mathbf{SP}_{A,B} \in \mathscr{V}$ and $\omega\mathbf{SP}^\leqslant_{A,B} \in \mathscr{V}^\leqslant$.

By Proposition 3.5, if $P, Q \in \mathbf{Pw}_\omega(A,B)$ such that both $P \leqslant Q$ and $Q \leqslant P$ hold in the subsumption preorder, and if P and Q are ω-linearizable and have finite width, then P and Q are isomorphic. Thus, since any generalized series–parallel partial word is ω-linearizable and has finite width, see below, we have

Corollary 4.4. *The subsumption preorder is a partial order on* (*generalized*) *series–parallel partial words.*

Thus, $\mathbf{SP}^\leqslant_{A,B}$ is in fact an ordered algebra.

The following result is a variation of Theorem 5.1 in [6]. (We identify each label with the corresponding singleton partial word.)

Theorem 4.5 (Bloom and Ésik [6]). *For any pair* (A,B) *of disjoint sets,* $\mathbf{SP}_{A,B}$ *is freely generated in* $\mathscr{V}$ *by* (A,B).

The meaning of this result is the following. Given any algebra $M = (M_f, M_\omega, \cdot, \otimes, {}^\omega)$ in $\mathscr{V}$ and functions $h_f : A \to M_f$, $h_\omega : B \to M_\omega$, there exist unique functions $h_f^\sharp : \mathbf{SP}_A \to$

M_f and $h_\omega^\sharp : \mathbf{SP}_{A,B}^\omega \to M_\omega$ extending h_f and h_ω, respectively, which together preserve all operations.

We use this result to prove:

Theorem 4.6. *For any pair (A,B) of disjoint sets, $\mathbf{SP}_{A,B}^\leqslant$ is freely generated in $\mathscr{V}^\leqslant$ by (A,B).*

Before proving this result, we recall the graph theoretic characterization of the series–parallel partial words from [10, 16], and the characterization of the generalized series–parallel partial words from [6]. We say that a partial word P is *N-free* if P does not have four distinct vertices u_1, u_2, v_1, v_2 whose induced partial order determines an N, i.e., $u_1 < v_1$, $u_2 < v_2$, $u_2 < v_1$ and any other two vertices are incomparable.

Theorem 4.7 (Grabowski [10], Valdes et al. [16]). *A partial word $P \in \mathbf{Pw}_f(A)$ is series–parallel if and only if P is N-free.*

Theorem 4.8 (Bloom and Ésik [6]). *A partial word $P \in \mathbf{Pw}_\omega(A, B)$ is a generalized series–parallel partial word if and only if the following hold.*
(1) *P is N-free.*
(2) *P is ω-linearizable.*
(3) *P has a finite number of filters (up to isomorphism).*

It follows that each generalized series–parallel partial word P has finite width. Indeed, otherwise P would contain an n-generated filter for each $n \geqslant 1$, contradicting the last condition. Note also that the last two conditions hold for all finite $P \in \mathbf{Pw}_\omega(A, B)$.

Corollary 4.9. *Suppose that $P \in \mathbf{SP}_A \cup \mathbf{SP}_{A,B}^\omega$. If Q is a finite $(A \cup B)$-labeled partial word determined by a nonempty subset of P, then $Q \in \mathbf{SP}_A$ or $Q \in \mathbf{SP}_{A,B}^\omega$ depending on whether or not P has a vertex labeled in B.*

Corollary 4.10. *Suppose that $P \in \mathbf{SP}_{A,B}^\omega$ and Q is an infinite filter of P. Then $Q \in \mathbf{SP}_{A,B}^\omega$.*

Thus, if $P = R \cdot S$ or $P = R \otimes S$ with $P \in \mathbf{SP}_A$, then $R, S \in \mathbf{SP}_A$. Moreover, if $P = R \cdot S \in \mathbf{SP}_{A,B}^\omega$, then $R \in \mathbf{SP}_A$ and $S \in \mathbf{SP}_{A,B}^\omega$, and if $P = R \otimes S \in \mathbf{SP}_{A,B}^\omega$, then $R, S \in \mathbf{SP}_A \cup \mathbf{SP}_{A,B}^\omega$ and either R or S is in $\mathbf{SP}_{A,B}^\omega$. Also, if $P = S^\omega \in \mathbf{SP}_{A,B}^\omega$, then $S \in \mathbf{SP}_A$. Below we will use these facts without mention.

For later use, we note the following.

Lemma 4.11. *Suppose that $P \in \mathbf{SP}_{A,B}^\omega$, R is a partial word in $\mathbf{SP}_A \cup \mathbf{SP}_{A,B}^\omega$ determined by a subset of P and Q is a partial word determined by a finite nonempty subset of P. Then the partial word determined by $R \cup Q$ is in $\mathbf{SP}_A \cup \mathbf{SP}_{A,B}^\omega$.*

Proof. Any filter of $R \cup Q$ is the union of a filter of R with a subset of Q. Moreover, the partial word determined by $R \cup Q$ is N-free and ω-linearizable. $\square$

By Theorem 4.8, or by a straightforward induction on the number of operations needed to generate an infinite partial word $P \in \mathbf{SP}^{\omega}_{A,B}$ from the singletons, it is easy to see that P is either disconnected, i.e., $P = R \otimes S$ for some (nonempty) R, S; or eventually disconnected, i.e., $P = R \cdot S$, where S is disconnected; or P is directed,[1] in which case P is of the form $R \cdot S^{\omega}$, where R and S are finite. Moreover, if $P \in \mathbf{SP}^{\omega}_{A,B}$ is finite and not a singleton, then P is either disconnected or $P = R \cdot S$ for some R and S.

Lemma 4.12. *Suppose that $P \leqslant R = R_1 \cdot R_2$, where $R_1 \in \mathbf{Pw}_f(A)$, and $P, R_2 \in \mathbf{Pw}_f(A)$ or $P, R_2 \in \mathbf{Pw}_{\omega}(A, B)$. Then there exist $P_1 \in \mathbf{Pw}_f(A)$ and $P_2 \in \mathbf{Pw}_f(A)$ or $P_2 \in \mathbf{Pw}_{\omega}(A, B)$ with $P = P_1 \cdot P_2$ and $P_i \leqslant R_i$, $i = 1, 2$.*

Proof. Without loss of generality, we may assume that P and R have the same vertex set and labeling and that the partial order $\leqslant_P$ is an extension of the order $\leqslant_R$. Thus, we have $v_1 \leqslant_P v_2$ for any $v_1 \in R_1$ and $v_2 \in R_2$. It follows that $P = P_1 \cdot P_2$, where for $i = 1, 2$, P_i is R_i with the partial order inherited from P. It is clear that $P_i \leqslant R_i$, $i = 1, 2$. $\square$

Lemma 4.13. *Let $P = P_1 \otimes P_2 \leqslant R$ for some partial words P, P_1, P_2 and R in $\mathbf{Pw}_f(A) \cup \mathbf{Pw}_{\omega}(A, B)$. Then there exist partial words R_1 and R_2 in $\mathbf{Pw}_f(A) \cup \mathbf{Pw}_{\omega}(A, B)$ with $R = R_1 \otimes R_2$ and $P_i \leqslant R_i$, $i = 1, 2$.*

Proof. Assume that P and R have the same vertex set and labeling, and that the partial order $\leqslant_P$ is an extension of $\leqslant_R$. For $i = 1, 2$, let R_i be P_i with the partial order inherited from R. $\square$

Proof of Theorem 4.6. Suppose that $M = (M_f, M_{\omega}, \cdot, \otimes, {}^{\omega}, \leqslant)$ is a preordered algebra in $\mathscr{V}^{\leqslant}$. Given maps $h_f : A \to M_f$ and $h_{\omega} : B \to M_{\omega}$, by Theorem 4.5 there is a unique morphism

$$h^{\sharp} = (h^{\sharp}_f, h^{\sharp}_{\omega}) : \omega\mathbf{SP}_{A,B} \to M$$

extending (h_f, h_{ω}) which preserves the operations. We need to show that the extension preserves the preorder. Below we will write $h^{\sharp}$ for both $h^{\sharp}_f$ and $h^{\sharp}_{\omega}$.

The fact that $h^{\sharp}$ preserves the preorder on $\mathbf{SP}_A$ can be derived from the main completeness result (Theorem 5.9) in [9]. In order to make the paper self-contained, we give our simple semantic argument here. This argument is also shorter than the one given in [9]. Suppose that $P \leqslant Q$, where $P, Q \in \mathbf{SP}_A$. We argue by induction on the *size*

[1] A partial word is directed if any two of its vertices have an upper bound.

(i.e., number of vertices) of Q to show that $Ph^\sharp \leqslant Qh^\sharp$. Without loss of generality, we assume that P and Q have the same vertex set and labeling, and that the partial order on P is an extension of the order on Q. When Q is a singleton we have $P=Q$, so that $Ph^\sharp = Qh^\sharp$. Suppose that $Q=Q_1 \cdot Q_2$, where Q_1 and Q_2 are in $\mathbf{SP}_A$. Then, by Lemma 4.1, we can write $P=P_1 \cdot P_2$, where $P_1, P_2 \in \mathbf{SP}_A$ with $P_i \leqslant Q_i$, $i=1,2$. By induction, $P_i h^\sharp \leqslant Q_i h^\sharp$, $i=1,2$, so that

$$\begin{aligned} Ph^\sharp &= P_1 h^\sharp \cdot P_2 h^\sharp \\ &\leqslant Q_1 h^\sharp \cdot Q_2 h^\sharp \\ &= Qh^\sharp, \end{aligned}$$

using the fact that $h^\sharp$ preserves series product and that series product preserves the preorder.

The nontrivial case is that Q is disconnected. If P is also disconnected, then by Lemma 4.13 we can write $P=P_1 \otimes P_2$ and $Q=Q_1 \otimes Q_2$, where P_1, P_2, Q_1, Q_2 are in $\mathbf{SP}_A$ with

$$P_i \leqslant Q_i, \quad i=1,2. \tag{12}$$

Since the sizes of Q_1 and of Q_2 are strictly less than the size of Q, we have

$$P_i h^\sharp \leqslant Q_i h^\sharp, \quad i=1,2 \tag{13}$$

by the induction hypothesis. Thus, using the fact that $h^\sharp$ preserves parallel product and that parallel product preserves the preorder,

$$\begin{aligned} Ph^\sharp &= P_1 h^\sharp \otimes P_2 h^\sharp \\ &\leqslant Q_1 h^\sharp \otimes Q_2 h^\sharp \\ &= (Q_1 \otimes Q_2) h^\sharp \\ &= Qh^\sharp. \end{aligned}$$

If P is connected, then write $P=R \cdot S$, where $R, S \in \mathbf{SP}_A$. Since Q is disconnected, there exist Q_1 and Q_2 in $\mathbf{SP}_A$ with $Q=Q_1 \otimes Q_2$. Define

$$R_i = Q_i \cap R, \tag{14}$$

$$S_i = Q_i \cap S, \quad i=1,2, \tag{15}$$

and equip each set with the partial order (and labeling) inherited from Q. Below, we will assume that none of the sets R_1, R_2, S_1, S_2 is empty, since the argument can be modified easily in the other cases. Note that $R_i, S_i \in \mathbf{SP}_A$, $i = 1, 2$, and

$$R \leqslant R_1 \otimes R_2, \tag{16}$$

$$S \leqslant S_1 \otimes S_2. \tag{17}$$

Thus,

$$Rh^\sharp \leqslant R_1 h^\sharp \otimes R_2 h^\sharp, \tag{18}$$

$$Sh^\sharp \leqslant S_1 h^\sharp \otimes S_2 h^\sharp, \tag{19}$$

by the induction assumption. Moreover, for $i = 1, 2$, $R_i \cdot S_i \leqslant Q_i$, so that

$$(R_i \cdot S_i)h^\sharp \leqslant Q_i h^\sharp, \tag{20}$$

again by the induction assumption. Thus, using (18)–(20),

$$\begin{aligned} Ph^\sharp &= Rh^\sharp \cdot Sh^\sharp \\ &\leqslant (R_1 h^\sharp \otimes R_2 h^\sharp) \cdot (S_1 h^\sharp \otimes S_2 h^\sharp) \\ &\leqslant (R_1 h^\sharp \cdot S_1 h^\sharp) \otimes (R_2 h^\sharp \cdot S_2 h^\sharp) \\ &= (R_1 \cdot S_1)h^\sharp \otimes (R_2 \cdot S_2)h^\sharp \\ &\leqslant Q_1 h^\sharp \otimes Q_2 h^\sharp \\ &= Qh^\sharp \end{aligned}$$

by the interchange law (7) and the fact that the operations preserve the preorder and $h^\sharp$ preserves the operations. The argument is similar when one of the R_i and/or one of the S_i is empty. One uses (8), (9), or (10).

Suppose now that $P \leqslant Q$ in $\mathbf{SP}^\omega_{A,B}$. Without loss of generality, we may again assume that P and Q have the same vertex set and labeling, and that the partial order $\leqslant_P$ is an extension of the partial order $\leqslant_Q$. We need to show that $Ph^\sharp \leqslant Qh^\sharp$ holds in M_ω. When Q is finite, the argument is the same as above. So we assume that Q and thus P are infinite. If $w(Q) = 1$ then $P = Q$, so that $Ph^\sharp = Qh^\sharp$. We proceed by induction on $w(Q)$. Assuming that $w(Q) > 1$, there are several cases.

Case 1: Q is disconnected. If P is also disconnected, then by Lemma 4.13 we can write $P = P_1 \otimes P_2$, $Q = Q_1 \otimes Q_2$, where the P_i and Q_i are partial words in $\mathbf{SP}_A \cup \mathbf{SP}^\omega_{A,B}$ satisfying (12) such that $w(Q_i) < w(Q)$, $i = 1, 2$. Thus, by the induction hypothesis, or by the previous argument, also (13) holds. The proof can be completed as above.

If P is connected but not directed, then write $P = R \cdot S$, where $R \in \mathbf{SP}_A$, $S \in \mathbf{SP}^\omega_{A,B}$, and S is disconnected. Since Q is disconnected, there exist Q_1 and Q_2 in $\mathbf{SP}_A \cup \mathbf{SP}^\omega_{A,B}$ with $Q = Q_1 \otimes Q_2$. Clearly, $w(Q_1), w(Q_2) < w(Q)$. Define R_i and S_i, $i = 1, 2$ as in (14) and (15), and equip each set with the partial order and labeling inherited from Q. Below, we will again assume that none of the sets R_1, R_2, S_1, S_2 is empty, since the argument can be modified easily in the other cases. Note that $R_i \in \mathbf{SP}_A$, $S_i \in \mathbf{SP}_A \cup \mathbf{SP}^\omega_{A,B}$,

$i=1,2$ such that S_1 or S_2 is in $\mathbf{SP}^{\omega}_{A,B}$, and (16) and (17) hold. Thus, since $R_1 \otimes R_2$ is in $\mathbf{SP}_A$ and $S_1 \otimes S_2$ is a disconnected partial word in $\mathbf{SP}^{\omega}_{A,B}$ with $w(S_1 \otimes S_2) \leqslant w(S)$, also (18) and (19) hold. Using (7), it follows as above that $Ph^{\sharp} \leqslant Qh^{\sharp}$.

Suppose now that P is directed. Write $Q = Q_1 \otimes Q_2$ as before, so that $w(Q_1), w(Q_2) < w(Q)$. Since P is directed, there exist $P_0, P_1 \in \mathbf{SP}_A$ with $P = P_0 \cdot P_1^{\omega}$, i.e.,

$$P = P_0 \cdot P_1 \cdot P_2 \cdot \ldots,$$

where the P_j, $j \geqslant 2$ are disjoint copies of P_1. For each $i=1,2$ and $j \geqslant 0$, let

$$P_{i,j} = Q_i \cap P_j,$$
$$Q_{i,j} = Q_i \Big\backslash \bigcup_{k=0}^{j-1} P_{i,k} = \bigcup_{k \geqslant j} P_{i,k},$$

and equip each set $P_{i,j}$ with the partial order and labeling inherited from P, and each $Q_{i,j}$ with the partial order and labeling inherited from Q_i. It is clear that each $Q_{i,j}$ is either empty or a filter of Q_i (and of Q). Now let $0 < j_0 < j_1 < \cdots$ be any sequence such that $P_0 \cdot \ldots \cdot P_{j_0 - 1}$ contains all minimal vertices of Q and for all $t \geqslant 1$, $P_0 \cdot \ldots \cdot P_{j_t - 1}$ contains all minimal vertices in the set obtained from Q by removing all vertices belonging to the product $P_0 \cdot \ldots \cdot P_{j_{t-1} - 1}$, i.e., all minimal vertices of $Q_{1, j_{t-1}}$ and $Q_{2, j_{t-1}}$.

Below we will make use of the following fact.

Fact: Suppose that Q is an ω-linearizable partial word, F is a filter of Q isomorphic to Q and $G = Q - F$ is finite and nonempty. If $P \leqslant G$ and G contains all minimal vertices of Q, then $P^{\omega} \leqslant Q$.

Returning now to the main proof, suppose now that both Q_1 and Q_2 are infinite. Since up to isomorphism Q_1 and Q_2 have a finite number of filters, there exist $t_1 < t_2$ with

$$Q_{i, j_{t_1}} = Q_{i, j_{t_2}}, \quad i = 1, 2. \tag{21}$$

Now let

$$R_i = P_{i,0} \cdot \ldots \cdot P_{i, j_{t_1} - 1},$$
$$S_i = P_{i, j_{t_1}} \cdot \ldots \cdot P_{i, j_{t_2} - 1}, \quad i = 1, 2.$$

By (21), and since S_i contains all minimal vertices of $Q_{i, j_{t_1}}$, $i = 1, 2$, it follows from the above fact that

$$S_i^{\omega} \leqslant Q_{i, j_{t_1}},$$
$$R_i \cdot S_i^{\omega} \leqslant Q_i, \quad i = 1, 2,$$

so that

$$(R_i h^{\sharp}) \cdot (S_i h^{\sharp})^{\omega} \leqslant Q_i h^{\sharp}, \quad i = 1, 2,$$

by the induction hypothesis and the fact that $h^{\sharp}$ preserves the operations. Since

$$P_0 \cdot P_1^{j_{t_1} - 1} = P_0 \cdot P_1 \cdot \ldots \cdot P_{j_{t_1} - 1} \leqslant R_1 \otimes R_2$$

and

$$P_1^{j_{t_2}-j_{t_1}} = P_{j_{t_1}} \cdot \ldots \cdot P_{j_{t_2}-1} \leqslant S_1 \otimes S_2,$$

and since $R_1 \otimes R_2$ and $S_1 \otimes S_2$ are finite, we have

$$P_0 h^\sharp \cdot (P_1 h^\sharp)^{j_{t_1}-1} = P_0 h^\sharp \cdot P_1 h^\sharp \cdot \ldots \cdot P_{j_{t_1}-1} h^\sharp \leqslant R_1 h^\sharp \otimes R_2 h^\sharp$$

and

$$(P_1 h^\sharp)^{j_{t_2}-j_{t_1}} \leqslant S_1 h^\sharp \otimes S_2 h^\sharp.$$

Thus, using Eqs. (6) and (5) and the interchange laws (7) and (11), and the fact that the operations preserve $\leqslant$ and $h^\sharp$ preserves the operations, it follows that

$$\begin{aligned}
Ph^\sharp &= P_0 h^\sharp \cdot (P_1 h^\sharp)^\omega \\
&= P_0 h^\sharp \cdot (P_1 h^\sharp)^{j_{t_1}-1} \cdot ((P_1 h^\sharp)^{j_{t_2}-j_{t_1}})^\omega \\
&\leqslant (R_1 h^\sharp \otimes R_2 h^\sharp) \cdot (S_1 h^\sharp \otimes S_2 h^\sharp)^\omega \\
&\leqslant (R_1 h^\sharp \otimes R_2 h^\sharp) \cdot ((S_1 h^\sharp)^\omega \otimes (S_2 h^\sharp)^\omega) \\
&\leqslant (R_1 h^\sharp \cdot (S_1 h^\sharp)^\omega) \otimes (R_2 h^\sharp \cdot (S_2 h^\sharp)^\omega) \\
&\leqslant Q_1 h^\sharp \otimes Q_2 h^\sharp \\
&= Qh^\sharp.
\end{aligned}$$

If Q_1 is infinite and Q_2 is finite, say, then choose the integers $t_1 < t_2$ such that $Q_{1,j_{t_1}} = Q_{1,j_{t_2}}$ and $Q_{2,j_{t_1}}$ is empty. Moreover, let k denote the least integer such that $Q_{2,k}$ is empty. Then let R_1 and S_1 be defined as before and

$$R_2 = P_{2,0} \cdot \ldots \cdot P_{2,k-1}.$$

We have

$$\begin{aligned}
R_1 \cdot S_1^\omega &\leqslant Q_1, \\
R_2 &\leqslant Q_2.
\end{aligned}$$

Thus, using the induction assumption and the fact that $h^\sharp$ is monotonic on finite partial words,

$$\begin{aligned}
(R_1 h^\sharp) \cdot (S_1 h^\sharp)^\omega &\leqslant Q_1 h^\sharp, \\
R_2 h^\sharp &\leqslant Q_2 h^\sharp.
\end{aligned}$$

The proof can be completed using (9) as above:

$$\begin{aligned}
Ph^\sharp &= P_0 h^\sharp \cdot (P_1 h^\sharp)^\omega \\
&= P_0 h^\sharp \cdot (P_1 h^\sharp)^{j_{t_1}-1} \cdot ((P_1 h^\sharp)^{j_{t_2}-j_{t_1}})^\omega \\
&\leqslant (R_1 h^\sharp \otimes R_2 h^\sharp) \cdot (S_1 h^\sharp)^\omega
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (R_1h^\sharp\cdot(S_1h^\sharp)^\omega)\otimes R_2h^\sharp\\
&\leqslant Q_1h^\sharp\otimes Q_2h^\sharp\\
&= Qh^\sharp.
\end{aligned}
$$

Case 2: Q is eventually disconnected. Then write $Q=Q_1\cdot Q_2$, where $Q_1\in\mathbf{SP}_A$ and $Q_2\in\mathbf{SP}^\omega_{A,B}$ and Q_2 is disconnected. Since $P\leqslant Q$, there exist $P_1\in\mathbf{SP}_A$ and $P_2\in\mathbf{SP}^\omega_{A,B}$ with $P=P_1\cdot P_2$ and $P_i\leqslant Q_i$, $i=1,2$. Since $Q_1\in\mathbf{SP}_A$, we have $P_1h^\sharp\leqslant Q_1h^\sharp$. Also, since Q_2 is disconnected and $w(Q_2)\leqslant w(Q)$, we have $P_2h^\sharp\leqslant Q_2h^\sharp$, by the previous case. Thus,

$$
\begin{aligned}
Ph^\sharp &= P_1h^\sharp\cdot P_2h^\sharp\\
&\leqslant Q_1h^\sharp\cdot Q_2h^\sharp\\
&= Qh^\sharp.
\end{aligned}
$$

Case 3: Q is directed. Then we can write $Q=Q_0\cdot Q_1^\omega$ for some $Q_0,Q_1\in\mathbf{SP}_A$. Since $P\leqslant Q$, there exist partial words $P_0,P_1,P_2,\ldots$ in $\mathbf{SP}_A$ with

$$
\begin{aligned}
P_0 &\leqslant Q_0,\\
P_i &\leqslant Q_1,\quad i\geqslant 1,\\
P &= P_0\cdot P_1\cdot P_2\cdot\ \ldots\ .
\end{aligned}
$$

Since P has a finite number of filters, there exist $1\leqslant i<j$ with

$$P_i\cdot P_{i+1}\cdot\ldots = P_j\cdot P_{j+1}\cdot\ldots$$

Thus, letting

$$
\begin{aligned}
R_0 &= P_0\cdot P_1\cdot\ \ldots\ \cdot P_{i-1},\\
R_1 &= P_i\cdot\ \ldots\ \cdot P_{j-1},\\
S_0 &= Q_0\cdot Q_1^{i-1},\\
S_1 &= Q_1^{j-i},
\end{aligned}
$$

we have $R_i\leqslant S_i$, $i=0,1$ and $P=R_0\cdot R_1^\omega$. Since the R_i and S_i are in $\mathbf{SP}_A$, also $R_ih^\sharp\leqslant S_i h^\sharp, i=0,1$. Thus,

$$
\begin{aligned}
Ph^\sharp &= R_0h^\sharp\cdot(R_1h^\sharp)^\omega\\
&\leqslant S_0h^\sharp\cdot(S_1h^\sharp)^\omega\\
&= Q_0h^\sharp\cdot(Q_1h^\sharp)^{i-1}\cdot((Q_1h^\sharp)^{j-i})^\omega\\
&= Q_0h^\sharp\cdot(Q_1h^\sharp)^\omega\\
&= Qh^\sharp. \qquad\square
\end{aligned}
$$

Proof of Theorem 4.1. By Theorem 2.2 and Proposition 3.7, all of the valid equations and inequations of $\mathscr{V}^\leqslant$ hold in the algebras $\mathbf{Pw}^\leqslant(A)$ (and $\mathbf{Pw}^\leqslant(A,B)$). Suppose that $M=(M_f,M_\omega,\cdot,\otimes,{}^\omega,\leqslant)$ is in $\mathscr{V}^\leqslant$. Then let $A=M_f$, $B=M_\omega$, and consider the identity functions $h_f:A\to M_f$ and $h_\omega:B\to M_\omega$. By Theorem 4.6, the pair of functions $h=(h_f,h_\omega)$ extends to a morphism $h^\sharp=(h_f^\sharp,h_\omega^\sharp):\omega\mathbf{SP}^\leqslant_{A,B}\to M$. Since the functions $h_f^\sharp$ and $h_\omega^\sharp$ are surjective and preserve $\leqslant$, it follows that any equation or inequation that holds in the algebras $\omega\mathbf{SP}^\leqslant_{A,B}$ holds in M. Since (in)equations are preserved by taking preordered subalgebras, we conclude that any (in)equation that holds in the algebras $\mathbf{Pw}^\leqslant(A,B)$ also holds in M. The result follows from Lemma 4.14. □

Lemma 4.14. *Suppose that A and B are disjoint sets, and let $C=A\cup B$. Then $\mathbf{Pw}^\leqslant(A,B)$ is isomorphic to a preordered subalgebra of $\mathbf{Pw}^\leqslant(C)$ and $\omega\mathbf{SP}^\leqslant_{A,B}$ is isomorphic to a preordered subalgebra of $\omega\mathbf{SP}^\leqslant_C$.*

Proof. For each $P\in\mathbf{Pw}_f(A)$ and $Q\in\mathbf{Pw}_\omega(A,B)$, let $Ph_f=P$ and Qh_ω the partial word obtained from Q by replacing each maximal vertex labeled by a letter $b\in B$ by the partial word b^ω. Then $h=(h_f,h_\omega)$ preserves the operations, and for any two partial words $P,Q\in\mathbf{Pw}_f(A)$ or $P,Q\in\mathbf{Pw}_\omega(A,B)$, $P\leqslant Q$ if and only if $Ph\leqslant Qh$. For the second claim, note that if P is (generalized) series–parallel, then so is Ph. □

Corollary 4.15. *The variety $\mathscr{V}^\leqslant$ is generated by the single algebra $\omega\mathbf{SP}^\leqslant_{A_0}$ or $\mathbf{Pw}^\leqslant(A_0)$, where $A_0=\{a\}$ is a singleton set.*

Proof. By Lemma 4.14, it is sufficient to show that each algebra $\omega\mathbf{SP}^\leqslant_A$ can be embedded into $\omega\mathbf{SP}_{A_0}$, for any finite set $A=\{a_1,\ldots,a_n\}$. But it is clear that the extension of the assignment

$$a_i\mapsto a\cdot(a\otimes\cdots\otimes a)\cdot a,$$

where the parallel product of a with itself is taken i times, is an injective morphism $h:\omega\mathbf{SP}^\leqslant_A\to\omega\mathbf{SP}^\leqslant_{A_0}$. Moreover, $P\leqslant Q$ if and only if $Ph\leqslant Qh$, for all P,Q in $\mathbf{SP}_A\cup\mathbf{SP}^\omega_A$. □

4.1. No finite axiomatization

It has been shown in [6] that the variety $\mathscr{V}$ has no finite axiomatization. Since for every two-sorted algebra M, we have $M\in\mathscr{V}$ if and only if $M^\leqslant\in\mathscr{V}^\leqslant$, where $M^\leqslant$ is M equipped with the total relations on its carriers, it follows that $\mathscr{V}^\leqslant$ has no finite axiomatization either. In this subsection we point out that not even the variety of ordered algebras contained in $\mathscr{V}^\leqslant$ has a finite axiomatization. We modify an argument from [6]. Suppose that p is a prime number. Let F_p denote the set of positive integers, and let $I_p=\{1,p,\top\}$. The preorder $\leqslant$ on F_p is the discrete order. Moreover, for all $u,v\in I_p$, $u\leqslant v$ if and only if $u=v$ or $v=\top$. For all $a,b\in F_p$ and $u,v\in I_p$, define

$$a\cdot b=a+b,$$
$$a\otimes b=a+b,$$

$$a \cdot u = u,$$
$$a \otimes u = \top,$$
$$u \otimes a = \top,$$
$$u \otimes v = \top,$$
$$a^\omega = \begin{cases} 1 & \text{if } p \text{ divides } a, \\ p & \text{otherwise.} \end{cases}$$

The resulting two-sorted ordered algebra $M_p = (F_p, I_p, \cdot, \otimes, {}^\omega, \leqslant)$ satisfies Eqs. (1)–(4), inequations (7)–(11), as well as Eq. (5) for all primes $n \neq p$. On the other hand, $(1^p)^\omega = p^\omega = p$ and $1^\omega = 1$, showing that (5) does not hold in M_p. Thus, by Theorem 4.1 and the compactness theorem we have:

Theorem 4.16. *Neither $\mathscr{V}^\leqslant$ nor the variety of ordered algebras contained in $\mathscr{V}^\leqslant$ has a finite axiomatization.*

5. The subword preorder

Definition 5.1. Suppose that P and Q are partial words in $\mathbf{Pw}_f(A) \cup \mathbf{Pw}_\omega(A,B)$. We say that P is an *induced partial subword* of Q, denoted $P \sqsubseteq Q$, if there is an order reflecting morphism $P \to Q$, i.e., a morphism $f : P \to Q$ such that $x \leqslant_P y$ if and only if $f(x) \leqslant_Q f(y)$ holds for all $x, y \in P$. Moreover, we say that P is a *partial subword of Q*, denoted $P \preceq Q$, if there is an induced partial subword $R \sqsubseteq Q$ such that $P \leqslant R$ in the subsumption preorder.

It is clear that $P \preceq Q$ holds whenever $P \sqsubseteq Q$ or $P \leqslant Q$.

Example 5.2. The partial words a, $a \cdot a$, $a \otimes a$, $a \cdot (a \otimes a)$, $(a \otimes a) \cdot a$ and $(a \otimes a) \cdot (a \otimes a)$ are all induced partial subwords, and hence partial subwords of $(a \otimes a) \cdot (a \otimes a)$. Since $a^3 = a \cdot a \cdot a \leqslant a \cdot (a \otimes a)$, also a^3 is a partial subword of $(a \otimes a) \cdot (a \otimes a)$. In the same way, a^4, $a^2 \cdot (a \otimes a)$, $(a \otimes a) \cdot a^2$ are all partial subwords of $(a \otimes a) \cdot (a \otimes a)$. On the other hand, Q_1 and Q_2 of Example 2.1 are not partial subwords of $(a \otimes a) \cdot (a \otimes a)$. The partial word a^ω is an induced partial subword of $a^\omega \otimes a^\omega$.

Remark 5.3. If P and Q are finite with $P \preceq Q$ and $Q \preceq P$, then P is in fact isomorphic to Q, see below. The same fact holds for the relation $\sqsubseteq$. Thus, both $\sqsubseteq$ and $\preceq$ are partial orders on $\mathbf{Pw}_f(A)$. On the other hand, for $P = (a \cdot b)^\omega$ and $Q = (b \cdot a)^\omega$ we have that $P \sqsubseteq Q$ and $Q \sqsubseteq P$, but P and Q are not isomorphic.

Note that both $\sqsubseteq$ and $\preceq$ are preorders preserved by the operations $\cdot$, $\otimes$ and ${}^\omega$. Thus,

$$\mathbf{Pw}(A,B)^\sqsubseteq = (\mathbf{Pw}_f(A), \mathbf{Pw}_\omega(A,B), \cdot, \otimes, {}^\omega, \sqsubseteq),$$
$$\mathbf{Pw}(A,B)^\preceq = (\mathbf{Pw}_f(A), \mathbf{Pw}(A,B), \cdot, \otimes, {}^\omega, \preceq)$$

are both preordered algebras. They include

$$\omega\mathbf{SP}^{\sqsubseteq}_{A,B} = (\mathbf{SP}_A, \mathbf{SP}^{\omega}_{A,B}, \cdot, \otimes, {}^{\omega}, \sqsubseteq),$$
$$\omega\mathbf{SP}^{\preceq}_{A,B} = (\mathbf{SP}_A, \mathbf{SP}^{\omega}_{A,B}, \cdot, \otimes, {}^{\omega}, \preceq)$$

as subalgebras.

Proposition 5.4. *The following inequations hold in the preordered algebras* $\mathbf{Pw}(A,B)^{\sqsubseteq}$:

$$x \leqslant x \cdot u, \tag{22}$$

$$u \leqslant x \cdot u, \tag{23}$$

$$u \leqslant u \otimes v, \tag{24}$$

where x *ranges over partial words in* $\mathbf{Pw}_f(A)$ *and* u *and* v *range over partial words in* $\mathbf{Pw}_f(A) \cup \mathbf{Pw}_\omega(A,B)$.

Corollary 5.5. *The weak interchange laws and the inequations* (22)–(24) *hold in all algebras* $\mathbf{Pw}(A,B)^{\preceq}$.

Let $\mathscr{V}^{\sqsubseteq}$ denote the varieties of preordered algebras axiomatized by the equations defining $\mathscr{V}$ together with the inequations (22)–(24). Moreover, let $\mathscr{V}^{\preceq}$ denote the subvariety of $\mathscr{V}^{\sqsubseteq}$ consisting of all algebras in $\mathscr{V}^{\sqsubseteq}$ satisfying the weak interchange laws. We thus have $\mathbf{Pw}(A,B)^{\sqsubseteq}, \omega\mathbf{SP}^{\sqsubseteq}_{A,B} \in \mathscr{V}^{\sqsubseteq}$ and $\mathbf{Pw}(A,B)^{\preceq}, \omega\mathbf{SP}^{\preceq}_{A,B} \in \mathscr{V}^{\preceq}$.

Theorem 5.6. *The algebra* $\omega\mathbf{SP}^{\sqsubseteq}_{A,B}$ *is freely generated by* (A,B) *in* $\mathscr{V}^{\sqsubseteq}$.

Proof. Suppose that $M = (M_f, M_\omega, \cdot, \otimes, {}^{\omega}, \leqslant)$ is in $\mathscr{V}^{\sqsubseteq}$ and $h = (h_f, h_\omega) : (A,B) \to (M_f, M_\omega)$, so that h_f and h_ω are functions $A \to M_f$ and $B \to M_\omega$, respectively. We know that h extends to a unique morphism $\omega\mathbf{SP}_{A,B} \to M$. Let h denote this morphism also. We need to show that for all $P, Q \in \mathbf{SP}_A \cup \mathbf{SP}^{\omega}_{A,B}$, if $P \sqsubseteq Q$ then $Ph \leqslant Qh$. We show this by induction on the rank of Q, i.e., by induction on the least number of operations needed to generate Q from the singletons. When Q is a singleton, we have $P = Q$ so that our claim holds obviously. Suppose now that $Q = Q_1 \cdot Q_2$ for some $Q_1 \in \mathbf{SP}_A$ and $Q_2 \in \mathbf{SP}_A \cup \mathbf{SP}^{\omega}_{A,B}$ such that $\mathrm{rank}(Q_i) < \mathrm{rank}(Q)$ $i = 1,2$. Then three cases arise. If $P \sqsubseteq Q_1$, then, using the induction assumption and (22),

$$\begin{aligned} Ph &\leqslant Q_1 h \\ &\leqslant Q_1 h \cdot Q_2 h \\ &= Qh. \end{aligned}$$

The case that $P \sqsubseteq Q_2$ is handled in the same way using (23). Finally, if $P = P_1 \cdot P_2$ with $P_i \sqsubseteq Q_i$, $i = 1, 2$, then we have

$$\begin{aligned} Ph &= P_1 h \cdot P_2 h \\ &\leqslant Q_1 h \cdot Q_2 h \\ &= Qh, \end{aligned}$$

by the induction assumption. If $Q = Q_1 \otimes Q_2$ for some Q_1 and Q_2 in $\mathbf{SP}_A \cup \mathbf{SP}^{\omega}_{A,B}$ with $\mathrm{rank}(Q_i) < \mathrm{rank}(Q)$, $i = 1, 2$, then $Ph \leqslant Qh$ is proved in the same way using (24). Assume now that $Q = R^{\omega}$, where $R \in \mathbf{SP}_A$ with $\mathrm{rank}(R) < \mathrm{rank}(Q)$. If P is finite we have $P \sqsubseteq R^n$ for some $n > 0$. As before, we can prove that $Ph \leqslant R^n h$. Thus,

$$\begin{aligned} Ph &\leqslant R^n h \\ &\leqslant R^n h \cdot (Rh)^{\omega} \\ &= (Rh)^n \cdot (Rh)^{\omega} \\ &= (Rh)^{\omega} \\ &= Qh, \end{aligned}$$

by (23), (5) and (6). Assume now that P is infinite. Then, since $P \in \mathbf{SP}^{\omega}_{A,B}$, we can write

$$P = P_0 \cdot P_1^{\omega}$$

for some $P_0, P_1 \in \mathbf{SP}_A$ such that $P_0 \sqsubseteq R^m$ and $P_1 \sqsubseteq R^n$ for some $m, n > 0$. As before, we can prove that $P_0 h \leqslant R^m h$ and $P_1 h \leqslant R^n h$. Thus,

$$\begin{aligned} Ph &= P_0 h \cdot (P_1 h)^{\omega} \\ &\leqslant R^m h \cdot (R^n h)^{\omega} \\ &= (Rh)^m \cdot ((Rh)^n)^{\omega} \\ &= (Rh)^{\omega} \\ &= Qh. \quad \square \end{aligned}$$

Corollary 5.7. *The algebra* $\omega\mathbf{SP}^{\preceq}_{A,B}$ *is freely generated by* (A, B) *in* $\mathscr{V}^{\preceq}$.

Proof. Suppose that $M = (M_f, M_{\omega}, \cdot, \otimes, ^{\omega}, \leqslant)$ is in $\mathscr{V}^{\preceq}$ and $h_f : A \to M_f$, $h_{\omega} : B \to M_{\omega}$. We know that $h = (h_f, h_{\omega})$ extends to a unique morphism $\omega\mathbf{SP}_{A,B} \to M$ that we also denote by h. To complete the proof, we must show that for all $P, Q \in \mathbf{SP}_A \cup \mathbf{SP}^{\omega}_{A,B}$, if $P \preceq Q$ then $Ph \leqslant Qh$. But by Lemma 5.8, if $P \preceq Q$, then there is some R in $\mathbf{SP}_A \cup \mathbf{SP}^{\omega}_{A,B}$ with $P \leqslant R$ and $R \sqsubseteq Q$. Thus, by Theorems 4.6 and 5.6, $Ph \leqslant Rh \leqslant Qh$. $\square$

Lemma 5.8. *Suppose that* $P, Q \in \mathbf{SP}_A \cup \mathbf{SP}^{\omega}_{A,B}$ *with* $P \preceq Q$. *If* S *is a partial word determined by a subset of* Q *with* $P \leqslant S$, *then* S *has a nonempty subset* R *such that the partial word determined by* R *is in* $\mathbf{SP}_A \cup \mathbf{SP}^{\omega}_{A,B}$ *and* $P \leqslant R$.

Proof. We prove this lemma by induction on the rank of P. However, let us remark that our claim holds if P is finite. This is immediate from the fact that any finite induced partial subword of a (generalized) series–parallel partial word is series–parallel. In the rest of the proof we may thus assume that P is infinite. The basis case that the rank of P is 0 is trivial. Assuming that the rank of P is positive, several cases arise.

Case 1: $P = P_1 \cdot P_2$ for some $P_1 \in \mathbf{SP}_A$ and $P_2 \in \mathbf{SP}^{\omega}_{A,B}$ such that $\mathrm{rank}(P_i) < \mathrm{rank}(P)$, $i = 1, 2$. Without loss of generality, we may assume that P and S have the same vertices and labeling and the partial order on P extends the partial order on S. Thus, P_1 and P_2 are subsets of S and hence of Q. Let $S_i, i = 1, 2$ denote the partial word determined by P_i equipped with the partial order and labeling inherited from Q. Since S_1 is finite, we have that $S_1 \in \mathbf{SP}_A$. Let $R_1 = S_1$. Now let F denote the filter generated by S_2 in Q. Then F is in $\mathbf{SP}^{\omega}_{A,B}$, $F \cap S_1 = \emptyset$, and $P_2 \leqslant S_2$. Thus, by the induction assumption, there exists some $R_2 \subseteq S_2$ such that the partial word determined by R_2 is in $\mathbf{SP}^{\omega}_{A,B}$ and $P_2 \leqslant R_2$. Let R denote the partial subword of Q induced by the set $R_1 \cup R_2$. By Lemma 4.11, we have $R \in \mathbf{SP}^{\omega}_{A,B}$. Moreover, $P \leqslant R$.

Case 2: $P = P_1 \otimes P_2$ for some $P_1, P_2 \in \mathbf{SP}_A \cup \mathbf{SP}^{\omega}_{A,B}$ such that $\mathrm{rank}(P_i) < \mathrm{rank}(P)$. Again, we may assume that P has the same vertices and labeling as S and that the partial order on P extends the order on S. For $i = 1, 2$, let S_i denote the partial word induced by P_i as a subset of Q (and of S). Note that $S = S_1 \otimes S_2$. By the induction hypothesis, there exists $R_i \subseteq S_i$, $i = 1, 2$ such that the partial word determined by R_i is in $\mathbf{SP}_A \cup \mathbf{SP}^{\omega}_{A,B}$, moreover, $P_i \leqslant R_i$. Since $S = S_1 \otimes S_2$, the partial word determined by $R_1 \cup R_2$ is isomorphic to $R_1 \otimes R_2$. Since also $P = P_1 \otimes P_2 \leqslant R_1 \otimes R_2$, the proof of this case is complete.

Case 3: $P = P_1^{\omega}$, for some $P_1 \in \mathbf{SP}_A$ with $\mathrm{rank}(P_1) < \mathrm{rank}(P)$. In this case, we argue by induction on the rank of Q. The basis case that $\mathrm{rank}(Q) = 0$ is trivial. When $\mathrm{rank}(Q) > 0$, three subcases arise.

Subcase 1: $Q = Q_1 \cdot Q_2$, where $\mathrm{rank}(Q_i) < \mathrm{rank}(Q)$, $i = 1, 2$. Then Q_1 is finite, so that there exist some $S' \subseteq S_2 \cap Q_2$ such that $P \leqslant S'$ holds for the partial word induced by S'. Our claim is immediate from the induction assumption.

Subcase 2: $Q = Q_1 \otimes Q_2$, where $\mathrm{rank}(Q_i) < \mathrm{rank}(Q)$, $i = 1, 2$. Write $P = U_0 \cdot U_1 \cdot \ldots$, where each U_i is isomorphic to P_1. Without loss of generality, we may assume that P and S have the same vertices and labeling, and that the partial order on P extends the order on S. Define

$$P_{ij} = U_i \cap S \cap Q_j, \quad i \geqslant 0, \ j = 1, 2.$$

Equipped with the partial order and labeling inherited from P, each P_{ij} determines a partial word. For convenience, here we also allow the case that some P_{ij} are empty. Since the size of each P_{ij} is bounded by the size of P_1, there is an infinite sequence $i_0 < i_1 < \cdots$ such that $P_{i_0 1}, P_{i_1 1}, \ldots$ are all isomorphic to a partial word P_1', and $P_{i_0 2}, P_{i_1 2}, \ldots$ are all isomorphic to a partial word P_2'. Suppose that neither P_1' nor P_2' is empty. Clearly, there exist $S_1' \subseteq S \cap Q_1$ and $S_2' \subseteq S \cap Q_2$ such that for the partial word induced by these sets in Q we have $(P_1')^{\omega} \leqslant S_1'$ and $(P_2')^{\omega} \leqslant S_2'$. Thus, by induction, there exist some $R_1 \subseteq S_1'$, $R_2 \subseteq S_2'$ with $(P_1')^{\omega} \leqslant R_1$, $(P_2')^{\omega} \leqslant R_2$ and such that

$R_1, R_2 \in \mathbf{SP}^{\omega}_{A,B}$. Now let $R = R_1 \cup R_2$. Then in Q, R determines the partial word $R_1 \otimes R_2$, moreover, $P = (P_1)^{\omega} \leqslant (P_1')^{\omega} \otimes (P_2')^{\omega} \leqslant R_1 \otimes R_2 = R$. If P_2' is empty, say, then P_1' is just P, and thus we have $P \leqslant S_1$. The result is immediate from the induction assumption.

Subcase 3: $Q = (Q_1)^{\omega}$, for some $Q_1 \in \mathbf{SP}_A$. Write $Q = Q_0' \cdot Q_1' \cdot \ldots$, where each Q_i' is isomorphic to Q_1. There exists a sequence $0 = i_0 < i_1 < \ldots$ and sets $S_0 \subseteq Q_{i_0}' \cdot \ldots \cdot Q_{i_1 - 1}' \cap S$, $S_1 \subseteq Q_{i_1}' \cdot \ldots \cdot Q_{i_2 - 1}' \cap S$, etc, such that the following hold for all j:

- The partial words induced by the S_j are all isomorphic to a partial word R_1.
- $P_1 \leqslant R_1$.

Thus, letting R denote the partial word determined by the union of the S_j, we have that R is isomorphic to $(R_1)^{\omega}$ and $P \leqslant R$. □

Corollary 5.9. *An equation or inequation holds in all preordered algebras* $\mathbf{Pw}(A)^{\sqsubseteq}$ *if and only if it holds in* $\mathscr{V}^{\sqsubseteq}$. *An equation or inequation holds in all preordered algebras* $\mathbf{Pw}(A)^{\preceq}$ *if and only if it holds in* $\mathscr{V}^{\preceq}$.

It is easy to modify the construction used in Section 4 to show that $\mathscr{V}^{\sqsubseteq}$ and $\mathscr{V}^{\preceq}$ as well as the varieties of ordered algebras contained in $\mathscr{V}^{\sqsubseteq}$ and $\mathscr{V}^{\preceq}$ have no finite axiomatization. One needs to consider the same algebras M_p with the partial order extended such that $x \leqslant y$ for all $x \in F_p$ and $y \in I_p$.

The proof of the next fact is the same as that of Corollary 4.15.

Corollary 5.10. *The variety* $\mathscr{V}^{\sqsubseteq}$ *is generated by the single algebra* $\mathbf{SP}^{\sqsubseteq}_{A_0}$ *or* $\mathbf{Pw}(A_0)^{\sqsubseteq}$, *where* $A_0 = \{a\}$ *is a singleton set. Similarly,* $\mathscr{V}^{\preceq}$ *is generated by each of* $\mathbf{SP}^{\preceq}_{A_0}$ *and* $\mathbf{Pw}(A_0)^{\preceq}$.

6. Adding 1

Naturally, one might wish to include the empty partial word 1 in the carrier $\mathbf{Pw}_f(A)$ of finite partial words. This can be done in at least two different meaningful ways. First, it makes sense to define 1^{ω} to be also empty. But then, we need to include the empty partial word, and hence the whole of $\mathbf{Pw}_f(A)$ in the carrier $\mathbf{Pw}_{\omega}(A, B)$ of partial words of infinite sort. The equations of the resulting structures can be axiomatized over the equational theory of $\mathscr{V}$ by

$$1 \cdot u = u, \tag{25}$$

$$x \cdot 1 = x, \tag{26}$$

$$u \otimes 1 = u, \tag{27}$$

$$y \otimes 1^{\omega} = y, \tag{28}$$

$$x \cdot 1^{\omega} = x \otimes 1^{\omega}, \tag{29}$$

where x is of finite sort, y is of infinite sort, and the sort of u can be both finite and infinite. Moreover, the inequations satisfied by the subsumption preorder may be captured by (7) and (11). For the induced partial subword preorder one needs the inequations

$$1 \leqslant u, \tag{30}$$

$$1^\omega \leqslant y, \tag{31}$$

and for the partial subword preorder (7), (11), (30), (31). (Here, we assume that the empty partial word 1^ω is not a partial subword of any partial word in $\mathbf{Pw}_f(A)$.)

The other way is to define $1^\omega = \perp$, where $\perp$ is a designated element of the set B. In this case, the valid equations can be captured by the equations that hold in $\mathscr{V}$ together with (25), (26) and (27). For the subsumption preorder, one also needs (7), (9), (10) and (11). To capture the induced partial subword preorder one needs the axiom (30), and for the partial subword preorder, the axioms (7), (9), (10), (11) and (30).

7. Future work

By an analysis of the proof of Theorem 4.6, it is possible to show that the (in)equational theory of $\mathscr{V}^{\leqslant}$ decidable. Similarly, it can be proved that the (in)equational theories of $\mathscr{V}^{\sqsubseteq}$ and $\mathscr{V}^{\preceq}$ are decidable. It was shown in [6] that the equational theory of $\mathscr{V}$ is decidable in polynomial time. It is an open question whether this holds for $\mathscr{V}^{\leqslant}$, $\mathscr{V}^{\sqsubseteq}$ and $\mathscr{V}^{\preceq}$.

One might also wish to consider a single-sorted (preordered) algebra of countable partial words with no restriction on the applicability of the operations. We will address these models in a forthcoming paper. The word case (without parallel product) is studied in [5].

Recently, Lodaya and Weil have studied recognizable, rational and regular subsets of $\mathbf{SP}_A$, see [11, 12]). They have shown that for sets of bounded width, these three concepts are equivalent. It is quite clear how to define recognizable and rational subsets of $\mathbf{SP}_A^\omega$. We plan to extend the notion of regularity to subsets of $\mathbf{SP}_A^\omega$ so that the equivalence result of Lodaya and Weil carries over.

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