JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 21, 506-509 (1968)

On a Nonlinear Eigenvalue Problem*

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Submitted by P. D. Lax

In a recent paper in this Journal [1], R. E. L. Turner has treated the following eigenvalue problem, in which the eigenvalue parameter appears quadratically:

$$Ax - \lambda^2 Bx - \lambda x = 0, \tag{1}$$

where x lies in a Hilbert space H, and A and B are positive definite compact self-adjoint operators.

Turner showed that the spectrum consists of two sequences of real eigenvalues, the positive eigenvalues converging to zero and the negative eigenvalues tending to minus infinity. Moreover, he obtained nonlinear analogues of the classical variational principles for these eigenvalues. As a corollary of one of these principles, Turner found that if H is finite-dimensional, the eigenvectors corresponding to only the positive eigenvalues (or only the negative eigenvalues) span H.

The purpose of this note is to show how Turner's results can be obtained and his corollary can be generalized by observing that the equation (1) is equivalent to the self-adjoint system

$$\begin{aligned} Ay &= \lambda Ax \\ Ax - y &= \lambda By \end{aligned} \tag{2}$$

for the vector $\{x, y\}$ in $H \times H$.

We note that λ appears linearly in this problem. It then follows from classical theory that the largest eigenvalue λ_1 can be characterized by

$$\lambda_1 = \max_{x,y \in H} \frac{2\,(Ax, y) - (y, y)}{(Ax, x) + (By, y)}.$$

We observe that

$$2(Ax, y) - (y, y) \leqslant (Ax, Ax) \leqslant \| A \| \, [(Ax, x) + (By, y)],$$

* Results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of a Ford Foundation Grant.

so that the ratio on the right is bounded above. The existence of the maximum follows from the fact that when $(Ax, x) + (By, y) = 1$, the inequality $2(Ax, y) - (y, y) \geqslant 0$ implies the bound $(y, y) \leqslant 4 \| A \|$, and from the compactness of A and the weak semicontinuity of (y, y). Successive eigenvalues can be defined as usual by

$$\lambda_n = \max_{(Ax, x_i)+(By, y_i)=0} \frac{2(Ax, y) - (y, y)}{(Ax, x) + (By, y)}, \qquad i = 1, \ldots, n-1 \tag{3}$$

where $\{x_i , y_i\}$ is the eigenvector corresponding to λ_i. Since all vectors of the form $\{x, 0\}$ render the Rayleigh quotient zero, there will be infinitely many nonnegative eigenvalues if H is infinite dimensional. It is clear from (2) and the fact that A is positive definite that $\lambda = 0$ is not an eigenvalue. Suppose that the positive eigenvalues have a positive point of accumulation $\tilde{\lambda}$. Then there is a sequence $\lambda_n \to \tilde{\lambda}$, and the corresponding eigenvectors $\{x_n , y_n\}$ satisfy

$$(Ax_m , x_n) + (By_m , y_n) = \delta_{mn} , \qquad -(y_n , y_n) + 2(Ax_n , y_n) \to \tilde{\lambda} > 0.$$

Then $(y_n , y_n) \leqslant 4 \| A \|$, and since $x_n = y_n/\lambda_n$, the sequence (x_n , x_n) is also bounded. But

$$(A(x_n - x_m), x_n - x_m) + (B(y_n - y_m), y_n - y_m) = 2,$$

contradicting the compactness of A and B. We conclude that (3) defines an infinite sequence of eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ which converges to zero.

If we define κ to be the maximum of the Rayleigh quotient in (3) among vectors $\{x, y\}$ with $(Ax, x_n) + (By, y_n) = 0$ for all $\{x_n , y_n\}$ corresponding to positive eigenvalues, then κ is an eigenvalue and by definition $\kappa \leqslant 0$. Since $\lambda = 0$ is not an eigenvalue, $\kappa < 0$. Suppose there exists a $z \neq 0$ in H which satisfies

$$(Az, x_n) = 0$$

for all $\{x_n , y_n\}$ corresponding to positive eigenvalues. Then the vector $\{z, 0\}$ satisfies the constraints used in defining κ and gives the value zero to the Rayleigh quotient in (3), which contradicts the inequality $\kappa < 0$. We conclude that there is no $z \neq 0$ such that Az is orthogonal to all the x_n. Thus we have the following generalization of Turner's Corollary:

THEOREM. *The eigenvectors $\{x_n\}$ of* (1) *which correspond only to the positive eigenvalues are complete in H with respect to the norm* $(Ax, x)^{1/2}$.

In order to obtain a variational principle for the negative eigenvalues, it is only necessary to observe that replacing λ by $(-1/\lambda)$ in Eq. (1) gives an equation of the same form, but with A and B interchanged. In this way, we find that

$$-\frac{1}{\lambda_n^-} = \max_{(Ax,\bar{x}_i)+(By,\bar{y}_i)=0} \frac{-(y,y)+2(Bx,y)}{(Bx,x)+(Ay,y)}, \qquad i = 1,\ldots, n-1$$

where $\{\bar{x}_i, \bar{y}_i\}$ is the eigenvector corresponding to λ_i^-. Again we find that $(-1/\lambda_n^-) \searrow 0$, so that $\lambda_1^- \geqslant \lambda_2^- \geqslant \cdots \searrow -\infty$. Moreover, the $\{\bar{x}_n\}$ are complete with respect to the norm $(Bx, x)^{1/2}$.

Remark. A simple illustration of these results is obtained when $B = A$. In this case Eq. (1) becomes

$$Ax = \frac{\lambda}{1-\lambda^2}\, x.$$

If $\mu_1 \geqslant \mu_1 \geqslant \cdots \nearrow 0$ are the eigenvalues and $x_1, x_2, \ldots$ the corresponding eigenvectors of A, we find the eigenvalues

$$\lambda_n^{\pm} = \frac{-1}{2\mu_n} \pm \sqrt{\frac{1}{4\mu_n{}^2} + 1}.$$

The λ_n^+ are positive and the λ_n^- are negative, and the same eigenvector x_n is associated with both λ_n^+ and λ_n^-. Obviously its is sufficient to take the x_n, considered as eigenvectors associated with either the λ_n^+ or λ_n^- alone, to form a complete set.

Note that as $n \to \infty$

$$\lambda_n^+ \sim \mu_n \searrow 0$$

while

$$\lambda_n^- \sim -\frac{1}{\mu_n} \searrow -\infty.$$

In order to obtain Turner's variational principle, we note from the first equation in (2) that when $\{x, y\}$ is an eigenfunction, $y = \lambda x$. Moreover, we see by taking the scalar product of equation (1) with x that a positive eigenvalue λ can be expressed as the positive root $Q(x)$ of the equation

$$Q^2(Bx, x) + Q(x, x) - (Ax, x) = 0.$$

If, for any x (not necessarily an eigenfunction), we denote the positive root of this equation by $Q(x)$, we can restrict our consideration in the variational principle (3) to vectors of the form $\{x, Q(x)\, x\}$. Substituting, we find that the Rayleigh quotient has the value $Q(x)$. Moreover, the orthogonality conditions

$(Ax, x_i) + (By, y_i) = 0$ become $(Ax, x_i) + Q(x)(Bx, y_i) = 0$. But $Ax_i = \lambda_i^2 Bx_i + \lambda_i x_i$ while $y_i = \lambda_i x_i$, so that

$$(Ax, x_i) + Q(x)(Bx, y_i) = \lambda_i[(\lambda_i + Q(x)) B(x, x_i) + (x, x_i)].$$

Also, $\lambda_i = Q(x_i)$. Therefore with the choice $y = Q(x)\, x$ the variational principle (3) assumes the form

$$\lambda_n = \max_{[Q(x)+Q(x_i)](Bx,x_i)+(x,x_i)=0} Q(x), \qquad i = 1,\dots, n-1$$

which was given by Turner. The maximum-minimum and minimum-maximum characterizations of Turner can be derived in a similar manner.

Reference

1. R. E. L. Turner. Some variational principles for a nonlinear eigenvalue problem. *J. Math. Anal. Appl.* **17** (1967), 151-160.