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On common fixed points in G-metric spaces using (E.A) property☆

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ABSTRACT

In this paper, we introduce some new types of pairs of mappings (f, g) on G-metric spaces called G-weakly commuting of type G_f and G–R-weakly commuting of type G_f. We obtain also several common fixed point results by using the (E.A) property.

1. Introduction and preliminaries

In 1976, Jungck [1] proved a common fixed point theorem for commuting maps, but his results required the continuity one of the maps. In ordinary metric space, Sessa [2] introduced a weaker version of the commutativity for a pair of self maps. In this remarkable paper, it is shown that a weakly commuting pair of maps in metric space is commuting, but the converse may not be true.

Later, Jungck [3] improved his results by introducing the notion of compatible mappings in order to generalized the concepts of weak commutativity and showed that weak commuting map are compatible, but the reverse implication may not hold. In 1996, Jungck [4] defined a more general notion, weakly compatible maps. A pair of self mappings is weakly compatible if they commute at their coincidence points.

Thus we have a one way implication, namely Commuting maps ⇒ Weakly Commuting maps ⇒ Compatible maps ⇒ Weakly Compatible maps. Recently, various authors have introduced coincidence point results for a various classes of mappings on metric spaces, for more detail on coincidence point theory and related results see [5–8].

However, the study of common fixed point of non-compatible mappings has been initiated by Pant [9,10].

In 2002, Aamri and El Moutawakil [11] defined a new property called the (E.A) property which generalizes the concept of non-compatible mappings and proved some common fixed point theorems.

Definition 1 (*[11]*). Let S and T be two self mappings of a metric space (X, d). We say that T and S satisfy the (E.A) property if there exists a sequence (x_n) such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in X$.

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In 2005, Zead Mustafa and Brailey Sims [12] introduced the notion of G-metric spaces as a generalization of the concept of ordinary metric spaces and they have obtained some fixed point results for mappings satisfying different contractive conditions. Then, based on the notion of generalized metric spaces, many authors obtained some fixed point results under some contractive conditions, see [13–21].

In this paper, we introduce some new types of maps f and g on G-metric spaces called G-weakly commuting of type G_f and G–R-weakly commuting of type G_f. Also, we obtain several common fixed point results using the (E.A) property.

The following definitions and results will be needed in the sequel.

Definition 2. A G-metric space is a pair (X, G), where X is a nonempty set, and G is a nonnegative real-valued function defined on $X \times X \times X$ such that for all $x, y, z, a \in X$ we have:

(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$;
(G3) $G(x, x, y) \le G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables); and
(G5) $G(x, y, z) \le G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

The function G is called a *G-metric* on X.

Every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \quad \text{for all } x, y \in X. \tag{1.1}$$

Example 1. Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x),$$

for all $x, y, z \in X$, is a G-metric on X.

Definition 3 (*[12]*)**.** A sequence (x_n) in a G-metric space X is said to converge if there exists $x \in X$ such that $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one say that the sequence (x_n) is *G-convergent* to x (through this paper we mean by $\mathbb{N}$ the set of all natural numbers). We call x the limit of the sequence (x_n) and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1 (*[12]*)**.** *Let X be a G-metric space. Then the following statements are equivalent.*

(1) *(x_n) is G-convergent to x.*
(2) *$G(x_n, x_n, x) \to 0$, as $n \to \infty$.*
(3) *$G(x_n, x, x) \to 0$, as $n \to \infty$.*
(4) *$G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.*

Definition 4 (*[12]*)**.** In a G-metric space X, a sequence (x_n) is said to be G-Cauchy if given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \ge N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2 (*[12]*)**.** *In a G-metric space X, the following statements are equivalent.*

(1) *The sequence (x_n) is G-Cauchy.*
(2) *For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \ge N$.*

Proposition 3 (*[12]*)**.** *Let X be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 5 (*[12]*)**.** A G-metric space X is said to be complete if every G-Cauchy sequence in X is G-convergent in X.

Definition 6 (*[22]*)**.** Let f and g be self maps of a set X. If $w = fx = gx$ for some $x \in X$. Then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Recall that, a pair of self mappings is called weakly compatible if they commute at their coincidence points.

Proposition 4 (*[22]*)**.** *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

In [23], the authors proved the following theorem.

Theorem 1.1. *Let (X, d) be a metric space, g be a continuous self mapping of X and $f : X \to X$ satisfying the following conditions:*

(1) *$f(g(x)) = g(f(x))$ for every $x \in X$,*
(2) *$f(X) \subset g(X)$.*

If there exists a constant $0 \le \alpha < 1$ such that for every $x, y \in X$

$$d(fx, fy) \le \alpha \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\} \tag{1.2}$$

then f and g have a unique common fixed point.

2. Main results

2.1. New concepts and some properties

First, we introduce the following concepts as follows.

Definition 7. A pair of self mappings (f, g) of a G-metric space (X, G) is said to be G-weakly commuting of type G_f if

$$G(fgx, gfx, ffx) \le G(fx, gx, fx), \quad \forall x \in X. \tag{2.1}$$

Definition 8. A pair of self mappings (f, g) of a G-metric space (X, G) is said to be G–R-weakly commuting of type G_f if there exists some positive real number R such that

$$G(fgx, gfx, ffx) \le RG(fx, gx, fx), \quad \forall x \in X. \tag{2.2}$$

Remark 1. The G-weakly commuting maps of type G_f are G–R-weakly commuting of type G_f. Reciprocally, if $R \le 1$, then G–R-weakly commuting maps of type G_f are G-weakly commuting of type G_f.

If we interchange f and g in (2.1) and (2.2), then the pair of mappings (f, g) is called G-weakly commuting of type G_g and G–R-weakly commuting of type G_g, respectively.

Example 2. Let $X = [0, 2]$ be endowed with the G-metric $G(x, y, z) = |x - y| + |y - z| + |x - z|$, for all $x, y, z \in X$. Define $f(x) = 2 - x$, $g(x) = x$, then by an easy calculation, one can show that $G(fgx, gfx, ffx) = 4|x - 1|$ and $G(fx, gx, fx) = 4|x - 1|$. Then, the pair (f, g) is G-weakly commuting of type G_f and G–R-weakly commuting of type G_f.

Example 3. Let $X = [1, 3]$ be endowed with the G-metric $G(x, y, z) = |x - y| + |y - z| + |x - z|$, for all $x, y, z \in X$. Define $f(x) = \frac{1}{2}x + 1$, $g(x) = \frac{2}{3}x + 1$, then for $x = 1$ we see that $G(fgx, gfx, ffx) = \frac{1}{2}$ and $G(fx, gx, fx) = \frac{1}{3}$. Therefore, the pair (f, g) is not G-weakly commuting of type G_f, but it is G–R-weakly commuting of type G_f for $R \ge \frac{3}{2}$.

The following example shows that a pair of mappings (f, g) that is G-weakly commuting of type G_f doesn't need to be G-weakly commuting of type G_g.

Example 4. Let $X = [0, 1]$ be endowed with the G-metric $G(x, y, z) = \max\{|x - y, |y - z|, |x - z|\}$, for all $x, y, z \in X$. Define $f(x) = \frac{1}{4}x^2$, $g(x) = x^2$, then we see that $G(fgx, gfx, ffx) = \frac{15}{64}x^4$ and $G(fx, gx, fx) = \frac{3}{4}x^2$, while, by an easy calculation, one can show that for $x = 1$ we have

$$G(g(f(x)), f(g(x)), g(g(x))) = \frac{15}{16} \not\le G(g(x), f(x), g(x)) = \frac{3}{4}.$$

Therefore, the pair (f, g) isn't G-weakly commuting of type G_g, but it is G-weakly commuting of type G_f.

Lemma 1. *If f and g are G-weakly commuting of type G_f or G–R-weakly commuting of type G_f, then f and g are weakly compatible.*

Proof. Let x be a coincidence point of f and g, i.e. $f(x) = g(x)$, then if the pair (f, g) is G-weakly commuting of type G_f, we have

$$G(f(g(x)), g(f(x)), f(g(x))) = G(f(g(x)), g(f(x)), f(f(x))) \le G(f(x), g(x), f(x)) = 0.$$

It follows $f(g(x)) = g(f(x))$, then they commute at their coincidence point.

Similarly, if the pair (f, g) is G–R-weakly commuting of type G_f, we have

$$G(f(g(x)), g(f(x)), f(g(x))) = G(f(g(x)), g(f(x)), f(f(x))) \le RG(f(x), g(x), f(x)) = 0,$$

thus $f(g(x)) = g(f(x))$, then the pair (f, g) is weakly compatible. □

The converse of Lemma 1 fails (for the case of G-weakly commutativity). The following example confirms this statement.

Example 5. Let $X = [1, +\infty)$ and $G(x, y, z) = |x - y| + |y - z| + |x - z|$. Define $f, g : X \to X$ by $f(x) = 2x - 1$ and $g(x) = x^2$, $x \in X$. We see that $x = 1$ is the only coincidence point and $f(g(1)) = f(1) = 1$ and $g(f(1)) = g(1) = 1$, so f and g are weakly compatible.

But, by an easy calculation, one can see that for $x = 2$ we have,

$$G(f(g(x)), g(f(x)), f(f(x))) = 8 \not\le 2 = G(f(x), g(x), f(x)).$$

Therefore, f and g are not G-weakly commuting of type G_f.

Now, we rewrite Definition 1 on G-metric spaces.

Definition 9. Let S and T be two self mappings of a G-metric space (X, G). We say that T and S satisfy the (E.A) property if there exists a sequence (x_n) such that (Tx_n) and (Sx_n) G-converge to t for some $t \in X$, that is, thanks to Proposition 1,

$$\lim_{n\to\infty} G(Tx_n, Tx_n, t) = \lim_{n\to\infty} G(Sx_n, Sx_n, t) = 0.$$

Remark 2. In view of (1.1) and Example 1, Definition 1 is equivalent to Definition 9.

In the following example, we show that if f and g satisfy the (E.A) property, we have not necessarily that (f, g) is G-weakly commuting of type G_f.

Example 6. We return to Example 5. Let $x_n = 1 + \frac{1}{n}$. We have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(1 + \frac{2}{n}) = 1$, and $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty}(1 + \frac{1}{n})^2 = 1$, therefore, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = 1 \in [1, \infty)$. Then f and g satisfy the (E.A) property, but we know that (f, g) is not G-weakly commuting of type G_f.

Again, we state the following:

Definition 10. Let (X, G) be a G-metric space and let $T : X \longrightarrow X$ be a mapping. For $A \subset X$, let $\delta(A) = \sup\{G(a, b, c), a, b, c \in A\}$ and $\forall x, y, z \in X$, define,

$$O(x, T, n) = \{x, T(x), T^2(x), \ldots, T^n(x)\},$$
$$O(x, T, \infty) = \{x, T(x), T^2(x), T^3(x), \ldots\}.$$

Definition 11. Let $(x_n)_{n=0}^{\infty}$ be a sequence of elements of X, then for i, j, let

$$O(x_i, j) = \{x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+j}\},$$
$$O(x_i, \infty) = \{x_i, x_{i+1}, x_{i+2}, \ldots\}.$$

2.2. Some common fixed point results

Following to Matkowski [24], let Φ be the set of all functions ϕ such that $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\phi \in \Phi$, then ϕ is called a Φ-map. If ϕ is a Φ-map, then it is easy to show that:

(1) $\phi(t) < t$ for all $t \in (0, +\infty)$.
(2) $\phi(0) = 0$.

Our first result is given by the following:

Theorem 2.1. *Let (X, G) be a G-metric space and suppose mappings $f, g : X \to X$ satisfy the following conditions:*

(1) *f and g are G- weakly commuting of type G_f,*
(2) *$f(X) \subseteq g(X)$,*
(3) *$g(X)$ is a G-complete subspace of X,*
(4) *$G(f(x), f(y), f(z)) \le \phi(M(x, y, z))$, for all $x, y, z \in X$, where,*

$$M(x,y,z) = \max \left\{ \begin{array}{c} G(g(x), g(y), g(z)), G(g(x), f(y), g(x)), \\ G(g(y), f(x), g(y)), G(g(z), f(x), g(z)), \\ G(g(z), f(y), g(z)), G(g(y), f(z), g(y)), \\ G(g(x), f(z), g(x)) \end{array} \right\}. \tag{2.3}$$

If there exists $x_0 \in X$ such that $\delta(O(x_0, f, \infty)) < \infty$ then f and g have a unique common fixed point.

Proof. Let $x_1 \in X$ such that $f(x_0) = g(x_1)$ and $x_2 \in X$ where $f(x_1) = g(x_2)$ then by induction we can define a sequence $(y_n) \in X$ as follows

$$y_n = f(x_n) = g(x_{n+1}), \quad n \in N.$$

If there exist $k \ge 0$ and $n \ge 1$ such that $\delta(O(y_k, n)) = 0$, then immediately we will have $y_k = y_{k+1}$, hence $g(x_{k+1}) = f(x_k) = y_k = y_{k+1} = f(x_{k+1})$, then x_{k+1} is a common fixed point.

Throughout this proof, we assume $\delta(O(y_k, n)) > 0$, for every $k \ge 0$ and $n \ge 1$.

Claim 1. *For all* $m, n \geq 0$ *we have*

$$\delta(O(y_m, n)) \leq \phi^m(\delta(O(y_0, n+m))). \tag{2.4}$$

Proof of Claim 1. We will prove the claim by induction on m.

Let $1 \leq i < j < l \leq n+1$, then (2.3) implies that

$$G(y_i, y_j, y_l) = G(f(x_i), f(x_j), f(x_l)) \leq \phi(M(x_i, x_j, x_l)),$$

where,

$$M(x_i, x_j, x_l) = \max \left\{ \begin{array}{c} G(y_{i-1}, y_{j-1}, y_{l-1}), G(y_{i-1}, y_j, y_{i-1}), \\ G(y_{j-1}, y_i, y_{j-1}), G(y_{l-1}, y_i, y_{l-1}), \\ G(y_{l-1}, y_j, y_{l-1}), G(y_{j-1}, y_l, y_{j-1}), \\ G(y_{i-1}, y_l, y_{i-1}) \end{array} \right\}. \tag{2.5}$$

Therefore,

$$\delta(O(y_1, n)) \leq \phi(\delta(O(y_{i-1}, l-i+1))).$$

In the previous equation, we must have $i = 1$, since otherwise if $i > 1$, with $i - 1 \geq 1$ and $l \leq n+1$ we have using the fact that ϕ is nondecreasing and $\delta(O(y_1, n)) > 0$,

$$\delta(O(y_1, n)) \leq \phi(\delta(O(y_{i-1}, l-i+1))) \leq \phi(\delta(O(y_1, n))) < \delta(O(y_1, n)),$$

which is a contradiction. Hence,

$$\delta(O(y_1, n)) \leq \phi(\delta(O(y_0, n+1))). \tag{2.6}$$

Then, (2.4) holds for $m = 1$. Suppose it is true for $m = k$, that is

$$\delta(O(y_k, n)) \leq \phi^k(\delta(O(y_0, n+k))). \tag{2.7}$$

We will prove it holds for $m = k+1$. Let $k+1 \leq i < j < l \leq n+k+1$, again by (2.3)

$$G(y_i, y_j, y_l) = G(f(x_i), f(x_j), f(x_l)) \leq \phi(M(x_i, x_j, x_l)).$$

A similar argument as (2.6) yields that

$$\delta(O(y_{k+1}, n)) \leq \phi(\delta(O(y_k, n+1))).$$

Thus, by this and having in mind (2.7) and the fact that ϕ is nondecreasing, we get that

$$\delta(O(y_{k+1}, n)) \leq \phi(\delta(O(y_k, n+1))) \leq \phi[\phi^k(\delta(O(y_0, n+1+k)))] = \phi^{k+1}(\delta(O(y_0, n+k+1))). \tag{2.8}$$

Hence,

$$\delta(O(y_{k+1}, n)) \leq \phi^{k+1}(\delta(O(y_0, n+k+1))). \tag{2.9}$$

So, by induction on m, we get

$$\delta(O(y_m, n)) \leq \phi^m(\delta(O(y_0, n+m))), \tag{2.10}$$

that is, Claim 1 is proved. □

But, $\phi^m(\delta(O(y_0, n+m))) \leq \phi^m(\delta(O(y_0, \infty)))$ and since x_0 is taken in order that $\delta(O(x_0, f, \infty)) < \infty$, so $\delta(O(y_0, \infty)) < \infty$, then whenever $n \to \infty$ we will have

$$\delta(O(y_0, n+m)) \to \delta(O(y_0, \infty)).$$

Similarly, as $n \to \infty$, we have

$$\delta(O(y_m, n)) \to \delta(O(y_m, \infty)).$$

Therefore, Eq. (2.10) implies that

$$\delta(O(y_m, \infty)) \leq \phi^m(\delta(O(y_0, \infty))). \tag{2.11}$$

Claim 2. *The sequence* (y_n) *is a G-Cauchy sequence.*

Proof of Claim 2. Let $t_0 = \delta(O(y_0, \infty))$, then $t_0 > 0$, so by the property of ϕ we have

$$\lim_{m\to\infty} \phi^m(t_0) = 0.$$

Since, $\delta(O(y_m, \infty)) \le \phi^m(\delta(O(y_0, \infty))) = \phi^m(t_0) \to 0$, as $m \to \infty$, then given $\epsilon > 0$, there exists $m_0 \in N$ such that $\phi^{m_0}(\delta(O(y_0, \infty))) < \epsilon$, so, for $q > r > l \ge m_0$ we have

$$G(y_q, y_r, y_l) \le \delta(O(y_{m_0}, q - m_0)) \le \delta(O(y_{m_0}, \infty)) \le \phi^{m_0}(\delta(O(y_0, \infty))) < \epsilon.$$

Hence, $(y_n) = (g(x_{n+1}))$ is a G-Cauchy sequence in $g(X)$. Since $g(X)$ is G-complete, then there exists $t \in g(X)$ such that $\lim_{n\to\infty} g(x_n) = t = \lim_{n\to\infty} f(x_n)$, therefore f and g satisfy the (E.A) property.

Having $t \in g(X)$, so there exists $p \in X$ such that $g(p) = t$, also

$$\lim_{n\to\infty} f(x_n) = g(p) = \lim_{n\to\infty} f(x_n).$$

We will show that $f(p) = g(p)$. We argue by contradiction and suppose that $f(p) \ne g(p)$, then condition (4) implies that

$$G(f(p), f(p), f(x_n)) \le \phi\left(\max\begin{Bmatrix} G(g(p), g(p), g(x_n)), G(g(p), f(p), g(p)), \\ G(g(p), f(x_n), g(p)), G(g(x_n), f(p), g(x_n)) \end{Bmatrix}\right). \tag{2.12}$$

Taking the limit as $n \to \infty$ and using that the function G is continuous we get

$$G(f(p), f(p), g(p)) \le \phi(G(g(p), f(p), g(p))) < G(g(p), f(p), g(p)). \tag{2.13}$$

Therefore,

$$G(f(p), f(p), g(p)) < G(g(p), f(p), g(p)). \tag{2.14}$$

Similarly, one can get

$$G(f(p), g(p), g(p)) < G(g(p), f(p), f(p)), \tag{2.15}$$

so, from (2.14) and (2.15), we have

$$G(f(p), f(p), g(p)) < G(f(p), g(p), g(p)) < G(g(p), f(p), f(p)),$$

which is contradiction, hence $f(p) = g(p)$.

Since the pair (f, g) is G-weakly commuting of type G_f, then

$$G(f(g(p)), g(f(p)), f(f(p))) \le G(f(p), g(p), f(p)) = 0.$$

Thus, $ff(p) = fg(p) = gf(p) = gg(p)$, then

$$f(t) = fg(p) = gf(p) = g(t).$$

Finally, we will show that $t =: g(p)$ is a common fixed point of f and g (that is, $t = g(t) = f(t)$). Suppose that $ft \ne t$, then

$$G(f(t), f(p), f(p)) \le \phi\left(\max\begin{Bmatrix} G(g(t), g(p), g(p)), G(g(t), f(p), g(t)), \\ G(g(p), f(t), g(p)), G(g(p), f(t), g(p)), \\ G(g(p), f(p), g(p)), G(g(t), f(p), g(t)), \end{Bmatrix}\right). \tag{2.16}$$

Since $g(t) = f(t)$ and $g(p) = f(p)$, therefore (2.16) implies that

$$\begin{aligned} G(f(t), f(p), f(p)) &\le \phi(\max\{G(f(t), f(t), f(p)), G(f(t), f(p), f(p))\}) \\ &< \max\{G(f(t), f(t), f(p)), G(f(t), f(p), f(p))\}. \end{aligned} \tag{2.17}$$

The same idea as the above gives that

$$G(f(t), f(p), f(p)) < G(f(t), f(t), f(p)).$$

Similarly, we have

$$G(f(t), f(p), f(t)) < G(f(t), f(p), f(p)).$$

We obtain a contradiction. This implies that $f(t) = f(p) = t$, then t is a common fixed point of f and g.

To prove the uniqueness, suppose we have u and v are such that $u \ne v$, $f(u) = g(u) = u$ and $f(v) = g(v) = v$, then again condition (4) implies that

$$\begin{aligned} G(u, v, v) &\le \phi(\max\{G(v, u, u), G(u, v, v)\}) \\ &< \max\{G(v, u, u), G(u, v, v)\}. \end{aligned} \tag{2.18}$$

Therefore,

$$G(u, v, v) < G(v, u, u).$$

Similarly, we get $G(v, u, u) < G(u, v, v)$, thus $G(u, v, v) < G(u, v, v)$, a contradiction, so $u = v$. Then, t is the unique common fixed point. □

Corollary 1. *Theorem 2.1 remains true if we replace G-weakly commuting of type G_f by G-weakly commuting of type G_g or G–R-weakly commuting of type G_f (retaining the rest of hypothesis).*

Now we give some examples to support Theorem 2.1.

Example 7. Let $X = [0, 1]$, $G(x, y, z) = \max\{|x-y|, |y-z|, |x-z|\}$, $f(x) = \frac{1}{4}x^2$, $g(x) = x^2$ and $\phi(t) = \frac{1}{3}t$ for all $x, y, z \in X$ and $t \geq 0$.

We have $f(X) = [0, \frac{1}{4}]$, $g(X) = [0, 1]$ and $g(X)$ is G-complete subspace of X.

It is easy to see that f and g are G-weakly commuting of type G_f. Also we have

$$\begin{aligned} G(fx, fy, fz) &= \frac{1}{4}\max\{|x^2-y^2|, |y^2-z^2|, |x^2-z^2|\} \\ &\leq \frac{1}{3}\max\{|x^2-y^2|, |y^2-z^2|, |x^2-z^2|\} \\ &= \phi(G(gx, gy, gz)) \\ &\leq \phi(M(x, y, z)). \end{aligned}$$

For $x_0 = 0$, we have $\delta(O(x_0, f, \infty)) < \infty$. Hence, all conditions of Theorem 2.1 are satisfied and $u = 0$ is the unique common fixed point of f and g.

Note that Theorem 1.1 is not applicable because f doesn't commute with g. Indeed, $f(g(x)) = \frac{1}{4}x^4 \neq g(f(x)) = \frac{1}{16}x^4$ for any $x \neq 0$ in X.

The following example shows that hypothesis of Theorem 2.1 is stronger than Theorem 1.1 for commuting maps (the idea of this example appeared in [2]).

Example 8. Let $X = [0, \infty]$ and $G(x, y, z) = \max\{|x-y|, |y-z|, |x-z|\}$ for all $x, y, z \in X$. Define the mappings $f, g : X \to X$ by

$$f(x) = \begin{cases} x - \dfrac{x^2}{2} & \text{if } 0 \leq x \leq 1 \\ \dfrac{1}{2} & \text{if } x > 1, \end{cases} \qquad g(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

$$\text{and} \quad \phi(t) = \begin{cases} t - \dfrac{t^2}{2} & \text{if } 0 \leq t \leq 1 \\ \dfrac{t}{2} & \text{if } t > 1. \end{cases}$$

We have $f(X) = [0, \frac{1}{2}]$, $g(X) = [0, 1]$ and $\phi(t) < t$, $\forall t > 0$. Also $g(X)$ is a G-complete subspace of X and $f(g(x)) = g(f(x))$ for every $x \in X$.

Now we shall show that f and g are G-weakly commuting of type G_g (Here we required to Corollary 1). First, we see that

$$f(g(x)) = g(f(x)) = \begin{cases} x - \dfrac{x^2}{2} & \text{if } 0 \leq x \leq 1 \\ \dfrac{1}{2} & \text{if } x > 1 \end{cases}$$

and

$$g(g(x)) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

Moreover

$$|f(x) - g(x)| = \begin{cases} \dfrac{x^2}{2} & \text{if } 0 \leq x \leq 1 \\ \dfrac{1}{2} & \text{if } x > 1, \end{cases}$$

$$|f(g(x)) - g(f(x))| = 0 \quad \forall x \in X$$

and

$$|f(g(x)) - g(g(x))| = |g(f(x)) - g(g(x))| = \begin{cases} \frac{x^2}{2} & \text{if } 0 \le x \le 1 \\ \frac{1}{2} & \text{if } x > 1. \end{cases}$$

If $0 \le x \le 1$, we have

$$\begin{aligned} G(f(g(x)), g(f(x)), g(g(x))) &= \max\{|f(g(x)) - g(f(x))|, |f(g(x)) - g(g(x))|, |g(f(x)) - g(g(x))|\} \\ &= \frac{x^2}{2} = |f(x) - g(x)| = G(f(x), g(x), f(x)). \end{aligned}$$

If $x > 1$, then

$$\begin{aligned} G(f(g(x)), g(f(x)), g(g(x))) &= \max\{|f(g(x)) - g(f(x))|, |f(g(x)) - g(g(x))|, |g(f(x)) - g(g(x))|\} \\ &= \frac{1}{2} = |f(x) - g(x)| = G(g(x), f(x), g(x)) \end{aligned}$$

Thus, f and g are G weakly commuting of type G_g.

To show that f and g satisfy condition (4), it suffices to prove

$$G(fx, fy, fz) \le \phi(G(gx, gy, gz)) \quad \text{for all } x, y, z \in X. \tag{2.19}$$

First, note that

- Let $0 \le x \le 1$ and $0 \le y \le 1$, then
$$|f(x) - f(y)| = |x - y|\left(1 - \frac{(x+y)}{2}\right) \le |x - y|\left(1 - \frac{|x-y|}{2}\right) = \phi(|x - y) = \phi(|gx - gy|).$$
- Let $0 \le x \le 1$ and $y > 1$, then
$$|f(x) - f(y)| = \frac{1}{2} - x + \frac{x^2}{2} \le \frac{1}{2} - \frac{x^2}{2} = (1 - x) - \frac{(1-x)^2}{2} = \phi(1 - x) = \phi(|gx - gy|).$$
- Let $x > 1$ and $y > 1$, then $|f(x) - f(y)| = 0 = \phi(|gx - gy|)$.

On the other hand, since ϕ is nondecreasing, so for all $a, b, c \ge 0$ we have

$$\max\{\phi(a), \phi(b), \phi(c)\} = \phi(\max\{a, b, c\}).$$

By symmetry of (2.19) and without loss of generality we take $x \le y \le z$. We distinguish the following cases:

Case 1. If $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$, then

$$\begin{aligned} G(fx, fy, fz) &= \max\{|fx - fy|, |fy - fz|, |fx - fz|\} \\ &\le \max\{\phi(|gx - gy|), \phi(|gy - gz|), \phi(|gx - gz|)\} \\ &= \phi(\max\{|gx - gy|, |gy - gz|, |gx - gz|\}) = \phi(G(gx, gy, gz)). \end{aligned}$$

Case 2. If $x > 1$, $y > 1$ and $z > 1$, then $G(fx, fy, fz) = 0 = \phi(G(gx, gy, gz))$.

Case 3. If $0 \le x \le 1$, $0 \le y \le 1$ and $z > 1$, we have

$$\begin{aligned} G(fx, fy, fz) &= \max\{|fx - fy|, |fy - fz|, |fx - fz|\} \\ &\le \max\left\{|x - y|\left(1 - \frac{|x-y|}{2}\right), \left((1 - y) - \frac{(1-y)^2}{2}\right), \left((1 - x) - \frac{(1-x)^2}{2}\right)\right\} \\ &= \max\{\phi(|gx - gy|), \phi(|gy - gz|), \phi(|gx - gz|)\} = \phi(G(gx, gy, gz)). \end{aligned}$$

Case 4. If $0 \le x \le 1$, $y > 1$ and $z > 1$, then

$$\begin{aligned} G(fx, fy, fz) &= \max\{|fx - fy|, |fy - fz|, |fx - fz|\} \\ &\le \max\left\{\left((1 - x) - \frac{(1-x)^2}{2}\right), 0, \left((1 - x) - \frac{(1-x)^2}{2}\right)\right\} \\ &= \phi(G(gx, gy, gz)). \end{aligned}$$

Thus, $G(fx, fy, fz) \le \phi(G(gx, gy, gz))$ for all $x, y, z \in X$. We deduce that

$$G(fx, fy, fz) \le \phi(M(x, y, z)).$$

Let $x_0 = 0$, then $\delta(O(x_0, f, \infty)) < \infty$, so $u = 0$ is the unique common fixed point of f and g.

We see that all hypothesis of Theorem 2.1 are satisfied, but this is not the case for Theorem 1.1. Indeed, if we suppose that the condition (1.2) holds, then for $x = 0$ and $0 < y \leq 1$ we have

$$d(fx, fy) = y - \frac{y^2}{2} \leq \alpha \max\left\{y, 0, \frac{y^2}{2}, y - \frac{y^2}{2}, y\right\} = \alpha y,$$

that is $1 - \frac{y}{2} \leq \alpha$ and as $y \to 0$, we may have $1 \leq \alpha$, which is a contradiction.

Theorem 2.2. *Let (X, G) be a G-metric space. Suppose the mappings $f, g : X \to X$ are G-weakly commuting of type G_f satisfying the following conditions:*

(1) *f and g satisfy the (E.A) property,*
(2) *$g(X)$ is a closed subspace of X,*
(3) *$G(f(x), f(y), f(z)) \leq \phi(M(x, y, z))$, where,*

$$M(x, y, z) = \max\left\{\begin{matrix} G(g(x), f(y), f(y)), G(g(x), f(z), f(z)), \\ G(g(y), f(x), f(x)), G(g(z), f(x), f(x)), \\ G(g(z), f(y), f(y)), G(g(y), f(z), f(z)) \end{matrix}\right\} \tag{2.20}$$

for all $x, y, z \in X$, then f and g have a unique common fixed point.

Proof. The mappings f and g satisfy the (E.A) property, then there exists in X a sequence (x_n) satisfying $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$ for some $t \in X$.

Since, $g(X)$ is a closed subspace of X and $\lim_{n\to\infty} g(x_n) = t$, hence there exists $p \in X$ such that $g(p) = t$. Also,

$$\lim_{n\to\infty} f(x_n) = g(p) = \lim_{n\to\infty} g(x_n).$$

We will show that $f(p) = g(p)$. Suppose to the contrary that $f(p) \neq g(p)$. The condition (3) implies that

$$G(f(p), f(p), f(x_n)) \leq \phi(M(p, p, x_n)) = \phi\left(\max\left\{\begin{matrix} G(g(p), f(x_n), f(x_n)), \\ G(g(x_n), f(p), f(p)), \\ G(g(p), f(p), f(p)) \end{matrix}\right\}\right). \tag{2.21}$$

Taking the limit as $n \to \infty$ and using the fact that the functions ϕ is continuous and G is jointly continuous, we get

$$0 < G(f(p), f(p), g(p)) \leq \phi(G(g(p), f(p), f(p))) < G(g(p), f(p), f(p)), \tag{2.22}$$

which is contradiction, so $f(p) = g(p)$. Since f and g are G-weakly commuting of type G_f, then

$$G(fg(p), gf(p), ff(p)) \leq G(f(p), g(p), f(p)) = 0,$$

therefore $ff(p) = fg(p) = gf(p) = gg(p)$, then

$$f(t) := fg(p) = gf(p) = gg(p) := g(t).$$

Now, we will show that $t = f(p)$ is a common fixed point of f and g. Suppose that $f(t) \neq t$, then

$$G(f(t), t, t) = G(f(t), f(p), f(p)) < \phi(M(t, p, p)) \tag{2.23}$$

where,

$$\begin{aligned} M(t, p, p) &= \max\{G(g(t), f(p), f(p)), G(g(p), f(t), f(t)), G(g(p), f(p), f(p)) = 0\} \\ &= \max\{G(g(t), f(p), f(p)), G(g(p), f(t), f(t))\} \\ &= \max\{G(f(t), t, t), G(t, f(t), f(t))\} \end{aligned}$$

Thus,

$$\begin{aligned} G(f(t), t, t) &\leq \phi\left(\max\{G(f(t), t, t), G(t, f(t), f(t))\}\right) \\ &< \max\{G(f(t), t, t), G(t, f(t), f(t))\}. \end{aligned}$$

We deduce that

$$G(f(t), t, t) < G(t, f(t), f(t)).$$

Adjusting similarly, we get that $G(t, f(t), f(t)) < G(f(t), t, t)$, therefore $G(f(t), t, t) < G(f(t), t, t)$, a contradiction, so that $t = f(t) = g(t)$, then t is a common fixed point of f and g.

To prove uniqueness, suppose we have u and v such that $u \neq v, f(u) = g(u) = u$ and $f(v) = g(v) = v$, then

$$G(u, v, v) = G(f(u), f(v), f(v)) \leq \phi\left(\max\begin{Bmatrix} G(g(u), f(v), f(v)), \\ G(g(v), f(v), f(v)), \\ G(g(v), f(u), f(u)) \end{Bmatrix}\right)$$
$$< \max\{G(u, v, v), G(v, u, u)\}. \tag{2.24}$$

Hence, $G(u, v, v) < G(v, u, u)$. Similarly, $G(v, u, u) < G(u, v, v)$, which is a contradiction, so $u = v$. Then t is a unique common fixed point. □

Theorem 2.3. *Let (X, G) be a G-metric space. Suppose the mappings $f, g : X \to X$ are weakly compatible satisfying the following conditions:*

(1) *f and g satisfy the (E.A) property,*
(2) *$g(X)$ is a closed subspace of X,*
(3) $G(f(x), f(y), f(z)) \leq$

$$\phi\left(\max\begin{Bmatrix} G(g(x), g(y), g(z)), G(g(x), f(x), g(z)), \\ G(g(z), f(z), g(z)), G(g(y), f(y), g(z)) \end{Bmatrix}\right) \tag{2.25}$$

for all $x, y, z \in X$, then f and g have a unique common fixed point.

Proof. Since f and g satisfy the (E.A) property, there exists in X a sequence (x_n) satisfying $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$ for some $t \in X$.

Since $g(X)$ is a closed subspace, then there exists $p \in X$ such that $g(p) = t$. Also, $\lim_{n\to\infty} f(x_n) = g(p) = \lim_{n\to\infty} g(x_n)$.

We will show that $f(p) = g(p)$. Suppose that $f(p) \neq g(p)$, then the condition (3) implies that

$$G(f(p), f(p), f(x_n)) \leq \phi\left(\max\begin{Bmatrix} G(g(p), g(p), g(x_n)), \\ G(g(p), f(p), g(x_n)), \\ G(g(x_n), f(x_n), g(x_n)) \end{Bmatrix}\right). \tag{2.26}$$

Taking the limit as $n \to \infty$, we get

$$G(f(p), f(p), g(p)) \leq \phi(\max\{G(g(p), g(p), g(p)), G(g(p), f(p), g(p))\})$$
$$= \phi(G(g(p), f(p), g(p))).$$

Therefore,

$$G(f(p), f(p), g(p)) \leq \phi(G(g(p), f(p), g(p))) < G(g(p), f(p), g(p)). \tag{2.27}$$

Similarly,

$$G(g(p), g(p), f(p)) < G(g(p), f(p), f(p)). \tag{2.28}$$

Hence, from (2.27) and (2.28), we get

$$G(f(p), f(p), g(p)) < G(g(p), f(p), f(p)),$$

a contradiction, hence $f(p) = g(p)$.

Since f and g are weakly compatible, then $gf(p) = fg(p)$, and therefore, $ff(p) = fg(p) = gf(p) = gg(p)$. It follows that $f(t) = fg(p) = gf(p) = gg(p) = g(t)$.

Finally, we will show that $t := g(p)$ is a common fixed point of f and g. Suppose that $f(t) \neq t$, then

$$G(f(t), t, t) = G(f(t), f(p), f(p)) \leq \phi\left(\max\begin{Bmatrix} G(g(t), g(p), g(p)), \\ G(g(t), f(t), g(p)), \\ G(g(p), f(p), g(t)) \end{Bmatrix}\right) = \phi\left(\max\begin{Bmatrix} G(f(t), t, t), \\ G(f(t), f(t), t) \end{Bmatrix}\right)$$
$$< \max\begin{Bmatrix} G(f(t), t, t), \\ G(f(t), f(t), t) \end{Bmatrix}.$$

Thus,

$$G(f(t), t, t) < G(f(t), f(t), t).$$

Similarly, $G(f(t), f(t), t) < G(f(t), t, t)$. It is a contradiction, so $t = f(t) = g(t)$. Then t is a common fixed point.

To prove uniqueness, suppose we have u and v such that $u \neq v, f(u) = g(u) = u$ and $f(v) = g(v) = v$, then an easy calculation leads to

$$G(u, v, v) < G(v, u, u).$$

Similarly, $G(v, u, u) < G(u, v, v)$, it is a contradiction. Hence, $u = v$. Then t is a unique common fixed point. □

Now we give some examples to support Theorem 2.3.

Example 9. Let $X = [0, \infty)$, $G(x, y, z) = |x - y| + |y - z| + |x - z|$, $f(x) = \frac{1}{8}x$, $g(x) = \frac{1}{2}x$ and $\phi(t) = \frac{2}{3}t$ for all $x, y, z \in X$ and $t \geq 0$.

Note that $x = 0$ is the only coincidence point of f and g. Also $f(g(0)) = g(f(0)) = 0$, therefore f and g are weakly compatible.

Let $x_n = \frac{1}{n}$, then $\lim_{n\to\infty} G(fx_n, fx_n, 0) = \lim_{n\to\infty} G(gx_n, gx_n, 0) = 0$, so f and g satisfy the (E.A.) property. Also

$$\begin{aligned} G(fx, fy, fz) &= \frac{1}{8}(|x - y| + |y - z| + |x - z|) \\ &\leq \frac{1}{3}(|x - y| + |y - z| + |x - z|) \\ &= \frac{2}{3}(\frac{1}{2}(|x - y| + |y - z| + |x - z|)) \\ &= \phi(G(gx, gy, gz)) \\ &\leq \phi\left(\max\{G(g(x), g(y), g(z)), G(g(x), f(x), g(z)), G(g(z), f(z), g(z)), G(g(y), f(y), g(z))\}\right). \end{aligned}$$

Hence, all conditions of Theorem 2.3 are satisfied and $u = 0$ is the unique common fixed point of f and g.

Example 10. We return to Example 7. It is easy to see that f and g are weakly compatible and verify the (E.A) property (by taking the sequence $x_n = \frac{1}{n}$). We have

$$\begin{aligned} G(fx, fy, fz) &= \frac{1}{4}\max\{|x^2 - y^2|, |y^2 - z^2|, |x^2 - z^2|\} \\ &\leq \frac{1}{3}\max\{|x^2 - y^2|, |y^2 - z^2|, |x^2 - z^2|\} \\ &= \phi(G(gx, gy, gz)) \\ &\leq \phi\left(\max\{G(g(x), g(y), g(z)), G(g(x), f(x), g(z)), G(g(z), f(z), g(z)), G(g(y), f(y), g(z))\}\right). \end{aligned}$$

Hence, all conditions of Theorem 2.3 are satisfied and $u = 0$ is the unique common fixed point of f and g.

Example 11. Let $X = [2, 20]$ and $G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$ for all $x, y, z \in X$. Define the mappings $f, g : X \to X$ by

$$f(x) = \begin{cases} 2 & \text{if } x = 2 \\ 6 & \text{if } 2 < x \leq 5 \\ 2 & \text{if } 5 < x \leq 20, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2 & \text{if } x = 2 \\ 14 & \text{if } 2 < x \leq 5 \\ \dfrac{4x + 10}{15} & \text{if } 5 < x \leq 20. \end{cases}$$

Also, suppose that $\phi(t) = \frac{t}{2}$ for all $t \geq 0$. Then, it is clear that $g(X)$ is a closed subspace of X and f and g are weakly compatible. If we consider the sequence $\{x_n\} = \{5 + \frac{1}{n}\}$, then $fx_n \to 2$ and $gx_n \to 2$ as $n \to \infty$. Thus, f and g satisfy the (E.A) property.

On the other hand, a simple calculation gives that,

$$G(fx, fy, fz) \leq \phi(G(gx, gy, gz)) \quad \text{for all } x, y, z \in X,$$

so in particular (2.25) holds.

Finally all hypotheses of Theorem 2.3 are satisfied and $u = 2$ is the unique common fixed point of f and g.

Note that the main result of Mustafa [25] is not applicable in this case. Indeed, for $y = z = \frac{5}{2}$ and $x = 2$

$$G\left(f(2), f\left(\frac{5}{2}\right), f\left(\frac{5}{2}\right)\right) = 4 > \frac{k}{2} = kG\left(2, \frac{5}{2}, \frac{5}{2}\right) \quad \text{for all } k \in [0, 1).$$

Theorem 2.4. *Let (X, G) be a G-metric space. Suppose the mappings $f, g : X \to X$ are G–R-weakly commuting of type G_f and satisfy the following conditions:*

(1) *f and g satisfy the (E.A) property,*
(2) *$g(X)$ is a closed subspace of X,*
(3) *there exist nonnegative real constants α and β with $0 \leq \alpha + 2\beta < 1$ such that for all $x, y, z \in X$,*

$$G(f(x), f(y), f(z)) \leq \alpha G(g(x), g(y), g(z)) + \beta \begin{Bmatrix} G(g(y), f(y), f(y)) \\ +G(g(z), f(z), f(z)) \\ +G(g(x), f(x), f(x)) \end{Bmatrix}, \tag{2.29}$$

then f and g have a unique common fixed point.

Proof. The mappings f and g satisfy the (E.A) property, there exists in X a sequence (x_n) satisfying $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$ for some $t \in X$.

Since $g(X)$ is a closed subspace, then there exists $p \in X$ such that $g(p) = t$, also $\lim_{n\to\infty} f(x_n) = g(p)$.

We will show that $f(p) = g(p)$. The condition (3) implies that

$$G(f(p), f(p), f(x_n)) \le \alpha G(g(p), g(p), g(x_n)) + \beta \begin{Bmatrix} G(g(p), f(p), f(p))+ \\ G(g(x_n), f(x_n), f(x_n)) \\ +G(g(p), f(p), f(p)) \end{Bmatrix}. \tag{2.30}$$

Taking the limit as $n \longrightarrow \infty$ and using the fact that G is jointly continuous, we get

$$G(f(p), f(p), g(p)) \le (2\beta)G(g(p), f(p), f(p)) \tag{2.31}$$

which is true unless $G(f(p), f(p), g(p)) = 0$, that is, $f(p) = g(p)$.

Since f and g are G–R-weakly commuting of type G_f, then there exists a real constant R such that

$$G(fg(p), gf(p), ff(p)) \le RG(f(p), g(p), f(p)) = 0,$$

so $ff(p) = fg(p) = gf(p) = gg(p)$. Then,

$$f(t) = fg(p) = gf(p) = g(t).$$

Finally, we will show that t is a common fixed point of f and g. We have

$$G(f(t), t, t) = G(f(t), f(p), f(p)) \le \alpha G(g(t), g(p), g(p)) + \beta \begin{Bmatrix} G(g(p), f(p), f(p)) \\ +G(g(p), f(p), f(p)) \\ +G(g(t), f(t), f(t)) \end{Bmatrix}. \tag{2.32}$$

Since $g(t) = f(t)$ and $g(p) = f(p) = t$, then the above equation becomes

$$G(f(t), t, t) \le \alpha G(f(t), f(p), f(p)) = \alpha G(f(t), t, t),$$

which holds unless $G(f(t), t, t) = 0$, so $t = f(t) = g(t)$. Then t is a common fixed point.

To prove uniqueness suppose we have u and v such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$, then (2.25) implies that

$$G(u, v, v) \le \alpha G(u, v, v),$$

which yields that $u = v$. Then, t is the unique common fixed point. □

Example 12. Let $X = [1, \infty)$ be endowed with the G-metric $G(x, y, z) = |x-y| + |y-z| + |x-z|$ for all $x, y, z \in X$. Define $f, g : X \to X$ by $f(x) = 2x - 1$ and $g(x) = 3x - 2$ for each $x \in X$. Set $\alpha = \frac{3}{4}$ and $\beta = 0$.

It is clear that the mappings f and g are G–R-weakly commuting of type G_f (with $R = 2$) and satisfy the following:

(i) f and g satisfy the (E.A) property (by taking $x_n = 1 + \frac{1}{n}$ and $t = 1$),
(ii) $g(X)$ is a closed subspace of X.

Moreover, for all $x, y, z \in X$ we have

$$\begin{aligned} G(f(x), f(y), f(z)) &= 2[|x-y| + |x-z| + |y-z|] \\ &\le \frac{9}{4}[|x-y| + |x-z| + |y-z|] \\ &= \alpha G(g(x), g(y), g(z)) + \beta \begin{Bmatrix} G(g(y), f(y), f(y)) \\ +G(g(z), f(z), f(z)) \\ +G(g(x), f(x), f(x)) \end{Bmatrix}. \end{aligned}$$

Thus, all conditions of Theorem 2.4 are satisfied and $u = 1$ is the unique common fixed point of f and g.

Note that the main result of Mustafa [25] is not applicable in this case. Indeed, for $y = z = 1$ and $x = 2$

$$G(f(2), f(1), f(1)) = 4 > 2k = kG(2, 1, 1) \quad \text{for all } k \in [0, 1).$$

Also, the Banach principle [26] is not applicable. Indeed, for $d(x, y) = |x - y|$ for all $x, y \in X$ we have for $x \ne y$

$$d(f(x), f(y)) = 2|x-y| > k|x-y| \quad \text{for all } k \in [0, 1).$$

Moreover, taking the partial metric p given by $p(x, y) = \max(x, y)$ for all $x, y \in X$, we have (in particular for $x < y$)

$$p(f(x), f(y)) = \max(2x-1, 2y-1) = 2y-1 \ge y > ky = kp(x, y) \quad \text{for all } k \in [0, 1),$$

that is, the main theorem of Matthews [27] could not be applicable.

Corollary 2. *Theorems 2.2–2.4 remain true if we replace respectively, G-weakly commuting of type G_f, weakly compatible and G–R-weakly commuting of type G_f by any one of them (retaining the rest of hypothesis).*

Corollary 3. *Some corollaries could be derived from Theorems 2.1–2.4 by taking $z = y$ or $g = Id_X$.*

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