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Journal of Computational and Applied Mathematics 136 (2001) 185–187

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

A note on normal matrices

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Received 15 May 2000; received in revised form 2 September 2000

Abstract

In this short note we present two simple necessary and sufficient conditions under which a matrix is normal.

Let us denote by $\lambda_1(A), \lambda_2(A), \ldots, \lambda_N(A)$ the eigenvalues of a matrix $A \in \mathbb{C}^{N\times N}$ and by $\sigma_1(A)$, $\sigma_2(A), \ldots, \sigma_N(A)$ its singular values. We assume that the eigenvalues are ordered such that

$$|\lambda_1(A)| = \cdots = |\lambda_{s_1}(A)| > |\lambda_{s_1+1}(A)| = \cdots = |\lambda_{s_1+s_2}(A)| > \cdots > |\lambda_{s_1+\cdots+s_{K-1}+1}(A)| = \cdots = |\lambda_{s_1+\cdots+s_K}(A)|$$

with

$$s_1 + s_2 + \cdots + s_K = N$$

and that

$$\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_N(A).$$

Recall that A is normal when it commutes with its adjoint A^*. Recall also the classical characterization of normal matrices

Proposition 1. *The matrix A is normal if and only if*

$$\sigma_j(A) = |\lambda_j(A)| \quad \forall j = 1, 2, \ldots, N. \tag{1}$$

Proof. See for example [1, p. 157]. □

We have the following proposition.

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0377-0427/01/$ - see front matter
PII: S0377-0427(00)00609-9

Proposition 2. *The matrix A is normal if and only if*

$$\sigma_j(A^{n+m}) = \sqrt{\sigma_j(A^{2n})\sigma_j(A^{2m})} \quad \forall j = 1,2,\ldots,N, \quad \forall n,m = 0,1,\ldots. \tag{2}$$

Proof. If A is normal, then (2) is clearly satisfied. Conversely, under the condition (2) we have

$$\sigma_j(A) = (\sigma_j(A^2))^{1/2} = (\sigma_j(A^4))^{1/4} = \cdots = (\sigma_j(A^{2^k}))^{1/2k} \quad \forall k = 0,1,\ldots.$$

And since (see [1, p. 180]) $\lim_{k\to\infty}(\sigma_j((A)^{2^k}))^{1/2k} = |\lambda_j(A)|$, we deduce from (1) that A is normal. □

The above proposition can be sharpened in the following way:

Theorem 3. *The matrix A is normal if and only if for all vectors* $x \in \mathbb{C}^N$

$$\|A^{n+m}x\|_2 \leqslant \sqrt{\|A^{2n}x\|_2\|A^{2m}x\|_2} \quad \forall n,m = 0,1,\ldots, \tag{3}$$

where $\|\|_2$ *denotes the Euclidean norm on* $\mathbb{C}^N$.

Proof. If A is normal, then A is unitarily diagonalizable: $A = Q\Lambda Q^*$ with $QQ^* = Q^*Q = I$(identity matrix) and $\Lambda = \operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_N)$. Then

$$\begin{aligned}\|A^{n+m}x\|_2^2 &= \|\Lambda^{n+m}y\|_2^2 \quad \text{with}\, y = Q^*x\\ &\leqslant \|\Lambda^{2n}y\|_2\|\Lambda^{2m}y\|_2\\ &= \|A^{2n}x\|_2\|A^{2m}x\|_2.\end{aligned}$$

Conversely, condition (3) implies that

$$\sigma_1(A^{n+m}) \equiv \max_{x\neq 0}\frac{\|A^{n+m}x\|_2}{\|x\|_2} \leqslant \sqrt{\sigma_1(A^{2n})\sigma_1(A^{2m})}$$

and in particular we have

$$|\lambda_1(A^n)| \leqslant \sigma_1(A^n) \leqslant (\sigma_1(A^{2n}))^{1/2} \leqslant (\sigma_1(A^{4n}))^{1/4} \leqslant \cdots \leqslant |\lambda_1(A^n)|.$$

The last inequality is derived as in the proof of (2). In particular, for $n=1$, we obtain that A has equal spectral radius and spectral norm. From [2, p. 428] there exists a unitary matrix $Q_1 = [Q_1^{(1)} \vdots Q_1^{(2)}] \in \mathbb{C}^{N\times N}$ with $Q_1^{(1)} \in \mathbb{C}^{N\times s_1}$ and $Q_1^{(2)} \in \mathbb{C}^{N\times(N-s_1)}$ that block-diagonalizes A in the following form:

$$A = Q_1\begin{pmatrix}\Lambda_1 & 0\\ 0 & A_1\end{pmatrix}Q_1^*, \tag{4}$$

where $\Lambda_1 = \operatorname{diag}(\lambda_1,\ldots,\lambda_{s_1})$ and $A_1 \in \mathbb{C}^{(N-s_1)\times(N-s_1)}$.

Now condition (3) applied with $x = Q_1^{(2)}x_1$, where $x_1 \in \mathbb{C}^{N-s_1}$, implies that

$$\|A_1^{n+m}x_1\|_2 \leqslant \sqrt{\|A_1^{2n}x_1\|_2\,\|A_1^{2m}x_1\|_2} \quad \forall n,m = 0,1,\ldots \tag{5}$$

from which we deduce that the matrix A_1 can in its turn be block-diagonalized as follows:

$$A_1 = Q_2\begin{pmatrix}\Lambda_2 & 0\\ 0 & A_2\end{pmatrix}Q_2^*, \tag{6}$$

where $Q_2 \in \mathbb{C}^{(N-s_1)\times(N-s_1)}$ is unitary, $\Lambda_2 = \operatorname{diag}(\lambda_{s_1+1},\ldots,\lambda_{s_1+s_2})$ and $A_2 \in \mathbb{C}^{(N-s_1-s_2)\times(N-s_1-s_2)}$.

We repeat the same argument on $A_2,\ldots,$ and conclude that A should be unitarily diagonalizable and hence normal. □

References

[1] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[2] M. Goldberg, G. Zwas, On matrices having equal spectral radius and spectral norm, Linear Algebra Appl. 8 (1974) 427–434.