Self-improving behaviour of inner functions as multipliers

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Abstract

Let $X$ and $Y$ be two spaces of analytic functions in the disk, with $X \subset Y$. For an inner function $\theta$, it is sometimes true that whenever $f \in X$ and $f\theta \in Y$, the latter product must actually be in $X$. We discuss this phenomenon for various pairs of (analytic) smoothness classes $X$ and $Y$.

Keywords: Inner functions; Lipschitz spaces; BMO; Bloch space

1. Introduction and results

A bounded analytic function $\theta$ on the disk $\mathbb{D} := \{z \in \mathbb{C}: |z| < 1\}$ is said to be inner if $\lim_{r \to 1^-} |\theta(r\zeta)| = 1$ for $m$-almost all $\zeta \in T$; here $T := \partial D$ is the unit circle and $m$ is the normalized arclength measure on $T$. Given an inner function $\theta$, we write $\sigma(\theta)$ for its singular set on $T$, also known as the boundary spectrum of $\theta$; thus $\sigma(\theta)$ is the smallest closed set $E \subset T$ for which $\theta$ is analytic across $T \setminus E$. Equivalently, $\sigma(\theta)$ is formed by the accumulation points of the zeros of $\theta$ and the (closed) support of the associated singular measure; see [17, Chapter II].

Because a nontrivial inner function $\theta$ is extremely oscillatory near $\sigma(\theta)$, the same should be expected of (and is “usually” true for) the product $f\theta$, where $f$ is a generic analytic function on...
that is smooth, in a sense, up to $T$. In some special cases, however, the product $f\theta$ happens to preserve the nice properties of $f$. At the same time, it is often true that multiplication by $\theta$ either destroys smoothness quite drastically or does not affect it at all. We may speak then of a certain automatic smoothness enhancement, in the sense that stronger smoothness properties of $f\theta$ can be derived from weaker ones, and this phenomenon will be our main theme.

To be more precise, suppose $X$ and $Y$ are two classes of analytic functions on $D$, with $X \subset Y$, and let $\theta$ be an inner function. We say that $\theta$ is $(X,Y)$-improving if every function $f \in X$ satisfying $f\theta \in Y$ must actually satisfy $f\theta \in X$. Thus, saying that $\theta$ is $(X,Y)$-improving amounts to the implication

$$(f \in X, f\theta \in Y) \Rightarrow f\theta \in X. \quad (1.1)$$

This implication, when true, reflects the self-improving behavior of the multiplication operator $f \mapsto f\theta$—or rather of the set $\{f\theta : f \in X\} \cap Y$—referred to in the title. In what follows, we shall focus on various specific pairs $(X,Y)$ of “smooth analytic spaces” and describe the $\theta$’s that enjoy the self-improving property (1.1).

Meanwhile, let us pause to make some immediate observations. First of all, given $X$ and $Y$ as above, every inner multiplier of $X$ (i.e., every inner function $\theta$ satisfying $\theta X \subset X$) is $(X,Y)$-improving. In particular, finite Blaschke products are always $(X,Y)$-improving for a reasonable choice of $X$ and $Y$. And, of course, all inner functions are $(H^p,Y)$-improving, where $H^p$ is the classical Hardy space and $Y$ is any space containing it. Secondly, if $X \subset Y_1 \subset Y_2$, then every $(X,Y_2)$-improving inner function is $(X,Y_1)$-improving. Thirdly, we mention a generalization of the preceding property: if $X,Y$ and $Z$ are three analytic function spaces with $X \subset Y$, then

$$\text{every } (X,Y)\text{-improving inner function is } (X \cap Z, Y \cap Z)\text{-improving.} \quad (1.2)$$

As regards our specific pairs $(X,Y)$, one source of these is the following string of inclusions:

$$A^\alpha \subset \mathcal{A} \subset \text{VMOA} \subset \text{BMOA} \subset \mathcal{B}. \quad (1.3)$$

Here, $\mathcal{A}$ stands for the disk algebra, i.e., the set of analytic functions on $D$ that admit a continuous extension to $D \cup T$. By $A^\alpha$, with $0 < \alpha < 1$, we denote the Lipschitz space consisting of those $f \in \mathcal{A}$ which satisfy

$$|f(z_1) - f(z_2)| \leq C|z_1 - z_2|^\alpha \quad (z_1, z_2 \in D)$$

with some $C = C(f) > 0$. Further, BMOA (respectively VMOA) is the analytic subspace of $\text{BMO}(T)$ (respectively $\text{VMO}(T)$), the space of functions with bounded (respectively vanishing) mean oscillation; see [17, Chapter VI]. Finally, $\mathcal{B}$ stands for the Bloch space, defined as the set of analytic functions $f$ with

$$\sup\{(1 - |z|)|f'(z)| : z \in D\} < \infty.$$ 

Some other pairs $(X,Y)$ to be studied are obtained by coupling the above classes with Besov spaces; these will be defined later on.

In fact, among the four inclusions in (1.3), only the first and the last—i.e., only the pairs $(A^\alpha, \mathcal{A})$ and $(\text{BMOA}, \mathcal{B})$—are of interest to us, since the other two lead to trivial classes of inner functions (representing the two extreme situations), as we shall now explain.
On the one hand, the only inner functions that happen to be (VMOA, BMOA)-improving are the trivial ones, i.e., finite Blaschke products. Indeed, the constant function 1 is in VMOA, while the product $1 \cdot \theta = \theta$ lies in BMOA \ VMOA whenever $\theta$ is a nontrivial inner function. (See, e.g., [17, Chapter IX, Section 2] for the fact that an inner function in VMOA is necessarily a finite Blaschke product.) On the other hand, we have the following result.

**Proposition 1.1.** Every inner function is $(A, VMOA)$-improving.

This can be derived from the following “Garsia-type” characterization of the space VMOA: for a function $g \in H^2$ to be in VMOA, it is necessary and sufficient that

$$P(|g|^2)(z) - |g(z)|^2 \to 0 \quad \text{as } |z| \to 1^-.$$  \hfill (1.4)

Here, $P(\cdot)$ denotes the Poisson integral of the function in question.

To prove Proposition 1.1, consider an inner function $\theta$ and suppose $f\theta \in VMOA$ for some $f \in A$. Applying (1.4) to $g := f\theta$ and noting that $|g| = |f|$ a.e. on $\mathbb{T}$, we get

$$P(|f|^2)(z) - |f(z)|^2 |\theta(z)|^2 \to 0 \quad \text{as } |z| \to 1^-.$$

And since

$$P(|f|^2)(z) \geq |f(z)|^2, \quad z \in \mathbb{D},$$

it follows that

$$|f(z)|^2 \left(1 - |\theta(z)|^2\right) \to 0 \quad \text{as } |z| \to 1^-.$$

This, in turn, implies that $\liminf_{z \to \zeta} |f(z)| = 0$ for all $\zeta \in \sigma(\theta)$, since every such $\zeta$ can be approached along a sequence $\{z_j\} \subset \mathbb{D}$ with sup $|\theta(z_j)| < 1$. Recalling that $f$ is continuous up to $\mathbb{T}$, we finally conclude that $f|_{\sigma(\theta)} = 0$. This clearly yields $f\theta \in A$ and completes the proof.

Now let us consider the pair $(A^\alpha, A)$. First of all, we single out an obvious class of $(A^\alpha, A)$-improving inner functions. Namely, we observe that if $\sigma(\theta)$ is a non-Carleson set, in the sense that

$$L_\theta := \int \log \text{dist}(\xi, \sigma(\theta)) \, dm(\xi) = -\infty,$$  \hfill (1.5)

then $\theta$ is $(A^\alpha, A)$-improving for the trivial reason that the only function $f \in A^\alpha$ satisfying $f\theta \in A$ is $f \equiv 0$. Indeed, the continuity of $f\theta$ on $\mathbb{T}$ implies that $f$ vanishes on $\sigma(\theta)$, while (1.5) tells us that $\sigma(\theta)$ is a uniqueness set for $A^\alpha$. (The latter is due to the “easy part” of Carleson’s theorem in [6] that goes back to earlier work of Beurling.)

It remains to study the case where $\sigma(\theta)$ is a Carleson set, so that $L_\theta > -\infty$. First we need to define the sublevel sets

$$\Omega(\theta, \varepsilon) := \{z \in \mathbb{D} : |\theta(z)| < \varepsilon\}$$

with $0 < \varepsilon < 1$. Next, let us look at the simplest situation where $\sigma(\theta)$ is a one-point set, say $\sigma(\theta) = \{1\}$. Under this assumption, our problem turns out to have the following solution.
Proposition 1.2. In order that an inner function $\theta$, with $\sigma(\theta) = \{1\}$, be $(A^\alpha, A)$-improving for some (or all) $\alpha \in (0, 1]$, it is necessary and sufficient that there exist a number $\varepsilon \in (0, 1)$ and a Stolz angle $\Gamma$ with vertex at $1$ so that $\Omega(\theta, \varepsilon) \subset \Gamma$.

(A Stolz angle with vertex at $\zeta \in \mathbb{T}$ is, by definition, the convex hull of the set $r\mathbb{D} \cup \{\zeta\}$ with some fixed $r$, $0 < r < 1$.) For example, an interpolating Blaschke product with zeros in a Stolz angle is $(A^\alpha, A)$-improving, while the “atomic” singular inner function

$$S(z) := \exp\left(\frac{z + 1}{z - 1}\right)$$  (1.6)

is not. (The sublevel sets $\Omega(S, \varepsilon)$ are disks tangent to $\mathbb{T}$.) In fact, an $(A^\alpha, A)$-improving inner function with one-point spectrum must be a Blaschke product.

While the criterion in Proposition 1.2 is independent of $\alpha$, as long as $0 < \alpha \leq 1$, the result actually fails for $\alpha > 1$ (the higher order $A^\alpha$-spaces being defined in terms of derivatives, as usual). This can be deduced from the author’s work in [10], but we omit the details.

Proposition 1.2 is a special case of Theorem 1.3; we hope the theorem will become clearer after the special case has been considered. Before stating the general result, we introduce some terminology. We shall say that an inner function $\theta$ satisfies the Stolz condition (or is a Stolz inner function) if, for some $\varepsilon \in (0, 1)$,

$$\sup \left\{ \frac{\text{dist}(z, \sigma(\theta))}{1 - |z|} : z \in \Omega(\theta, \varepsilon) \right\} < \infty.$$  (1.7)

Geometrically, (1.7) means that $\Omega(\theta, \varepsilon)$ is contained in the union of Stolz angles of fixed opening which have their vertices in $\sigma(\theta)$. Further, we need the notion of a porous set. A closed set $E \subset \mathbb{T}$ is said to be porous if there is a constant $c > 0$ such that

$$\sup \{ \text{dist}(\zeta, E) : \zeta \in I \} \geq c \cdot m(I)$$  (1.8)

for every arc $I \subset \mathbb{T}$. Originally, porous sets arose in Kotochigov’s paper [19], and then Dyn’kin identified them as interpolating sets for $A^\alpha$; see [15,16]. The last-mentioned paper also contains a brief discussion of the porosity condition (1.8) from the geometric point of view. Here, we only remark that finite subsets of $\mathbb{T}$ are porous, as is the Cantor “middle-thirds” set adjusted to $\mathbb{T}$.

(And another remark: in connection with (1.7), (1.8) and similar conditions below, let us agree that $\text{dist}(z, \theta)$ equals $2$, the diameter of $\mathbb{D}$, for all $z \in \mathbb{D} \cup \mathbb{T}$. With this convention, finite Blaschke products are Stolz, and the empty set is porous.)

Theorem 1.3. Let $\theta$ be an inner function.

(a) If $\theta$ satisfies the Stolz condition, then it is $(A^\alpha, A)$-improving for all $\alpha \in (0, 1]$.
(b) If $\sigma(\theta)$ is a porous set and if $\theta$ is $(A^\alpha, A)$-improving for some $\alpha \in (0, 1]$, then $\theta$ satisfies the Stolz condition; moreover, (1.7) then holds for each $\varepsilon \in (0, 1)$.

We believe that part (b) above should remain true when $\sigma(\theta)$ is a Carleson, not necessarily porous, set. If so, we would have a complete characterization of $(A^\alpha, A)$-improving inner functions at our disposal. In its current form, however, the theorem yields an “if and only if” result only for inner functions with porous boundary spectra.
Yet another characterization of \((A^\alpha, A)\)-improving inner functions in the same class can be given in terms of derivatives.

**Theorem 1.4.** Suppose \(\theta\) is an inner function such that \(\sigma(\theta)\) is a Carleson set.

(a) If
\[
\sup \{ |\theta'(z)| \cdot \text{dist}(z, \sigma(\theta)) : z \in \mathbb{T} \setminus \sigma(\theta) \} < \infty, \tag{1.9}
\]
then \(\theta\) is \((A^\alpha, A)\)-improving for all \(\alpha \in (0, 1]\).

(b) If \(\sigma(\theta)\) is a porous set and if \(\theta\) is \((A^\alpha, A)\)-improving for some \(\alpha \in (0, 1]\), then (1.9) holds.

Now let \(\alpha\) and \(\beta\) be two exponents with \(0 < \beta < \alpha \leq 1\). What about \((A^\alpha, A^\beta)\)-improving inner functions? Since \(A^\beta \subset A\), every \((A^\alpha, A)\)-improving inner function is \((A^\alpha, A^\beta)\)-improving, and one might ask whether the latter class is actually larger. It seems likely that the answer should be “no.” At least we can prove that a reasonably tame inner function (in a sense to be specified) will belong to neither class unless it belongs to both.

Given an inner function \(\theta\) and a number \(\gamma > 0\), we shall say that \(\theta\) is of order \(\leq \gamma\) if there exists an \(\epsilon \in (0, 1)\) such that
\[
\sup \left\{ \frac{\text{dist}(z, \sigma(\theta))^\gamma}{1 - |z|} : z \in \Omega(\theta, \epsilon) \right\} < \infty. \tag{1.10}
\]

Accordingly, we say that \(\theta\) is of finite order to mean that there are \(\gamma \in (0, \infty)\) and \(\epsilon \in (0, 1)\) making (1.10) true. Finally, \(\theta\) will be called tame if it is of finite order and if \(\sigma(\theta)\) is a porous set.

We remark that, for \(0 < \gamma < 1\), only finite Blaschke products are of order \(\leq \gamma\). On the other hand, the inner functions of order \(\leq 1\) are precisely the Stolz ones.

**Theorem 1.5.** Let \(\theta\) be a tame inner function, and let \(0 < \beta < \alpha \leq 1\). If \(\theta\) is \((A^\alpha, A^\beta)\)-improving, then it is Stolz.

Thus, we may speak of a certain rigidity phenomenon. For example, the atomic function (1.6) (which is tame, and in fact of order \(\leq 2\), but not Stolz) is never \((A^\alpha, A^\beta)\)-improving, no matter how close together \(\alpha\) and \(\beta\) may be. To gain more flexibility, we now consider subtler perturbations of \(A^\alpha\)-spaces; these will involve intersections with suitable Besov spaces.

For \(1 \leq p < \infty\) and \(0 < s < 1\), the Besov space \(B^s_p\) is defined as the set of those functions \(f \in L^p (\mathbb{T}, m)\) for which
\[
\int_{-\pi}^{\pi} \frac{dt}{|t|^{sp+1}} \int_{\mathbb{T}} |f(e^{it}\xi) - f(\xi)|^p \, dm(\xi) < \infty. \tag{1.11}
\]

The analytic subspace \(AB^s_p := B^s_p \cap H^p\) is known to consist of precisely those \(f \in H^p\) which satisfy
\[
\int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^{(1-s)p-1} \, dA(z) < \infty,
\]
where \(dA\) denotes area measure.
Theorem 1.6. Let $1 \leq p < \infty$, $0 < s < 1$ and $\max(0, s - 1/p) < \alpha < s$. If $\theta' \in H^{(s-\alpha)p}$, then $\theta$ is $(A^\alpha \cap B^\alpha_p, A^\alpha)$-improving.

The atomic function (1.6) satisfies the conclusion of the theorem when $(s - \alpha)p < 1/2$. In connection with inner functions that have $H^q$-derivatives, we cite [1,2,11].

In the endpoint case $s = \alpha$, the preceding theorem reduces to the following.

Proposition 1.7. For $1 \leq p < \infty$ and $0 < \alpha < 1$, every inner function is $(A^\alpha \cap B^\alpha_p, A^\alpha)$-improving.

Some more results involving Besov spaces will be given. The first of these, which treats the “diagonal” case $s = 1/p$, is quickly derived from what we already know.

Proposition 1.8. Let $1 < p < \infty$ and $1/p < \alpha \leq 1$.

(a) Every Stolz inner function is $(A^\alpha, B^{1/p}_p)$-improving.

(b) Every tame inner function which is $(A^\alpha, B^{1/p}_p)$-improving is Stolz.

(c) Every inner function is $(A \cap B^{1/p}_p, AB^{1/p}_p)$-improving.

Indeed, combining Theorem 1.3 with Proposition 1.1, we see that every Stolz inner function is $(A^\alpha, VMOA)$-improving and hence (thanks to the inclusion $AB^{1/p}_p \subset VMOA$) also $(A^\alpha, B^{1/p}_p)$-improving; this proves (a).

Now if $\theta$ is $(A^\alpha, B^{1/p}_p)$-improving, then it is a fortiori $(A^\alpha, A^\beta)$-improving for $1/p < \beta < \alpha$, since $A^\alpha \subset A^\beta \subset AB^{1/p}_p$. And if $\theta$ is also tame, then it must be Stolz by virtue of Theorem 1.5; this proves (b).

Finally, to verify (c), we combine Proposition 1.1 with property (1.2), applied to $X = A$, $Y = VMOA$ and $Z = AB^{1/p}_p$.

The next theorem concerns “subdiagonal” Besov spaces and their intersections with BMOA.

Theorem 1.9. Let $2 \leq p < \infty$ and $0 < s < 1/p$. For an inner function $\theta$, the following conditions are equivalent.

(i) $\theta$ is $(AB^s_p \cap BMOA, BMOA)$-improving.

(ii) $\theta \in AB^s_p$.

In fact, this last theorem has a natural extension to the three-parameter Besov spaces $AB^s_{pq}$ (see, e.g., [13] for the definition). We have restricted ourselves to $AB^s_p := AB^s_{pp}$ for the sake of simplicity, and also because condition (ii) above seems to have received more attention than its $AB^s_{pq}$-counterpart. Indeed, various $AB^s_p$ criteria for inner functions can be found in [1,3] and in a number of subsequent papers.

However, there is another special case of $AB^s_{pq}$ that deserves to be mentioned. This is the “critical case” where $s = 1/p$ and $q = \infty$. The space $B^{1/p}_{p\infty}$, often denoted by $A(p, 1/p)$, is then the mean Lipschitz space formed by the functions $f \in L^p(\mathbb{T}, m)$ with

$$\int_{\mathbb{T}} |f(e^{i\xi}) - f(\xi)|^p \, dm(\xi) = O(|h|), \quad h \in \mathbb{R},$$
and its analytic subspace is $AB_{p,∞}^{1/p} := H^p \cap B_{p,∞}^{1/p}$. The nice thing about $AB_{p,∞}^{1/p}$ is that this class is still large enough to contain nontrivial inner functions, and yet small enough to be contained in BMOA. (The inclusion $B_{p,∞}^{1/p} \subset BMO$ is established in [5]; see also [7] for a partial result.) Moreover, the inner functions in $AB_{p,∞}^{1/p}$ have been described by Verbitsky [22]. We have incorporated (a special case of) his result as the (ii) ⇔ (iii) part of the theorem below.

**Theorem 1.10.** For $2 < p < ∞$ and an inner function $θ$, the following conditions are equivalent.

(i) $θ$ is $(AB_{p,∞}^{1/p}, \text{BMOA})$-improving.

(ii) $θ \in AB_{p,∞}^{1/p}$.

(iii) $θ$ is a Blaschke product whose zero sequence, $\{z_k\}$, satisfies

$$\sum_{j: |z_j| \geq |z_k|} (1 - |z_j|) = O(1 - |z_k|).$$

(1.12)

Finally, we turn to the pair $(\text{BMOA}, B)$. We shall write $B_0$ for the little Bloch space, defined as the set of functions $f \in B$ with $\lim_{|z| \to 1^{-}} (1 - |z|)|f'(z)| = 0$.

**Theorem 1.11.** Let $θ$ be an inner function distinct from a finite Blaschke product.

(a) If there is a number $ε \in (0, 1)$ such that

$$\inf\{(1 - |z|)|θ'(z)| : z \in Ω(θ, ε) \setminus Ω(θ, ε_0)\} > 0$$

whenever $0 < ε_0 < ε$, then $θ$ is $(\text{BMOA}, B)$-improving.

(b) If $θ \in B_0$, then $θ$ is not $(\text{BMOA}, B)$-improving.

Even though there is a gap between the necessary and sufficient conditions stated as (a) and (b), the theorem makes it easy to produce examples of $(\text{BMOA}, B)$-improving inner functions, as well as of inner functions that fail to have this property. For instance, any interpolating Blaschke product $B$, any of its powers $B^n$, and the atomic function (1.6) satisfy the hypothesis of (a) and are, therefore, $(\text{BMOA}, B)$-improving. On the other hand, singular inner functions generated by suitably smooth measures are known to be in $B_0$. For a complete description of inner functions in $B_0$, we refer to [4].

The proofs of our results are given in Sections 2–5, except for the two short proofs (those of Propositions 1.1 and 1.8) that have already been included.

2. Proofs of Theorems 1.3 and 1.4

First we cite two preliminary results, to be employed later on. The following theorem can be found in [8], except for the case $α = 1$ which is covered by results of [9,14]. (See also [12,16] for alternative versions and approaches.)

**Theorem A.** Let $0 < α \leq 1$. Given $f \in A^α$ and an inner function $θ$, one has $fθ \in A^α$ if and only if

$$f(z) = O\left((1 - |z|)^α\right), \quad z \in Ω(θ, ε),$$

(2.1)

for some (or any) $ε \in (0, 1)$.
Next comes a theorem of Dyn’kin (cf. [15,16]) concerning some special outer functions associated with porous sets; we only need a restricted version of his result.

**Theorem B.** Given a porous set \( E \subset \mathbb{T} \) and an exponent \( \alpha > 0 \), there exists an outer function \( F_\alpha = F_{\alpha,E} \in A^\alpha \) satisfying

\[
|F_\alpha(z)| \asymp \text{dist}(z,E)^\alpha, \quad z \in \mathbb{D} \cup \mathbb{T}.
\]  

(We use the sign \( \asymp \) to mean that the ratio of the two quantities lies between two positive constants.)

In the two proofs below, it will be assumed that the inner function \( \theta \) is not a finite Blaschke product, so that \( \sigma(\theta) \neq \emptyset \). The discarded case is trivial.

**Proof of Theorem 1.3.** (a) Suppose \( f \in A^\alpha \) is a function satisfying \( f\theta \in \mathcal{A} \), so that \( f|_{\sigma(\theta)} = 0 \). Assuming that the Stolz condition (1.7) holds, we now fix \( z \in \Omega(\theta, \varepsilon) \) and let \( \zeta \) be the point (or one of the points) in \( \sigma(\theta) \) which is nearest to \( z \). Thus

\[
\zeta \in \sigma(\theta), \quad |z - \zeta| = \text{dist}(z, \sigma(\theta)).
\]  

Since \( f(\zeta) = 0 \), we find that

\[
|f(z)| = |f(z) - f(\zeta)| \leq C_1 |z - \zeta|^\alpha = C_1 \text{dist}(z, \sigma(\theta))^\alpha \leq C_2 (1 - |z|)^\alpha
\]

for some constants \( C_1, C_2 > 0 \); the last step relies on (1.7). We have arrived at condition (2.1), and Theorem A tells us that \( f\theta \in A^\alpha \). In sum, we have verified the implication

\[
(f \in A^\alpha, f\theta \in \mathcal{A}) \Rightarrow f\theta \in A^\alpha,
\]

as required.

(b) In accordance with Theorem B, we can find an outer function \( F = F_\alpha \in A^\alpha \) satisfying (2.2) with \( E = \sigma(\theta) \) (recall that the latter set is assumed to be porous). It follows, in particular, that \( F|_{\sigma(\theta)} = 0 \), whence \( F\theta \in \mathcal{A} \). Now if (1.7) fails for some \( \varepsilon > 0 \), then for a suitable sequence \( \{z_j\} \subset \Omega(\theta, \varepsilon) \) and some fixed \( c > 0 \), we have

\[
\frac{|F(z_j)|}{(1 - |z_j|)^{\alpha}} \geq c \left( \frac{\text{dist}(z_j, \sigma(\theta))}{1 - |z_j|} \right)^{\alpha} \to \infty.
\]

Therefore, Theorem A yields \( F\theta \notin A^\alpha \), and we conclude that \( \theta \) is not \( (A^\alpha, \mathcal{A}) \)-improving.

The next proof makes use of the following result, due to Shirokov [21].

**Lemma 2.1.** Let \( 0 < \alpha \leq 1 \). Given \( f \in A^\alpha \) and an inner function \( \theta \) with \( m(\sigma(\theta)) = 0 \), one has \( f\theta \in A^\alpha \) if and only if

\[
\sup \{ |f(z)||\theta'(z)|^\alpha : z \in \mathbb{T} \setminus \sigma(\theta) \} < \infty.
\]
Proof of Theorem 1.4. (a) Suppose \( f \in A^\alpha \) is a function for which \( f \theta \in \mathcal{A} \), so that \( f|_{\sigma(\theta)} = 0 \). Assuming that (1.9) holds, fix \( z \in \mathbb{T} \setminus \sigma(\theta) \) and let \( \zeta \) be as in (2.3). We have then

\[
|f(z)| = |f(z) - f(\zeta)| \leq C_1|z - \zeta|^{\alpha} = C_1 \text{dist}(z, \sigma(\theta))^{\alpha} \leq C_2|\theta'(z)|^{-\alpha},
\]

and the inclusion \( f \theta \in A^\alpha \) follows by Lemma 2.1.

(b) Once again, we employ the outer function \( F = F_\alpha \in A^\alpha \) associated, as in Theorem B, with the porous set \( E = \sigma(\theta) \). As before, (2.2) implies that \( F|_{\sigma(\theta)} = 0 \), and so \( F \theta \in A \). On the other hand, if (1.9) fails, then another application of (2.2) yields

\[
|F(z_j)| \cdot |\theta'(z_j)|^{\alpha} \geq c\{\text{dist}(z_j, \sigma(\theta)) \cdot |\theta'(z_j)|\}^{\alpha} \to \infty
\]

for a suitable sequence \( \{z_j\} \subset \mathbb{T} \setminus \sigma(\theta) \). Therefore, Lemma 2.1 enables us to conclude that \( F \theta \notin A^\alpha \). \( \square \)

3. Proof of Theorem 1.5

Once again, we may assume that \( \theta \) is not a finite Blaschke product—otherwise, there is nothing to prove. We want to deduce the Stolz condition (1.7) from the hypotheses that \( \theta \) is tame and \((A^\alpha, A^\beta)\)-improving for some \( \alpha \in (0, 1] \) and \( \beta \in (0, \alpha) \). To this end, we now put

\[
\tau := \inf\{\gamma > 0 : \theta \text{ is of order } \leq \gamma\}, \quad (3.1)
\]

and the first step will consist in proving that \( \tau = 1 \).

Assume, to the contrary, that \( \tau > 1 \). (Observe that \( \tau < \infty \) because \( \theta \) is of finite order, and that \( \tau \geq 1 \) because \( \theta \) is not a finite Blaschke product.) We are going to verify, under the current assumption, that \( \theta \) is not \((A^\alpha, A^\beta)\)-improving whenever \( 0 < \alpha \leq 1 \) and \( \alpha/\tau \leq \beta < \alpha \). This in turn will imply that \( \theta \) is not \((A^\alpha, A^\beta)\)-improving whenever \( 0 < \beta < \alpha \leq 1 \) (indeed, for smaller values of \( \beta \) there are fewer \((A^\alpha, A^\beta)\)-improving functions), and we shall arrive at a contradiction.

With this plan in mind, we let \( \alpha/\tau \leq \beta < \alpha \) and then pick an \( \varepsilon > 0 \) so that

\[
(\tau + \varepsilon)\beta < (\tau - \varepsilon)\alpha. \quad (3.2)
\]

Writing \( E := \sigma(\theta) \) (this notation will be fixed throughout the proof), we infer from the definition (3.1) that there is an \( \varepsilon_1 \in (0, 1) \) such that

\[
\text{dist}(z, E)^{\tau + \varepsilon} \leq C(1 - |z|), \quad z \in \Omega(\theta, \varepsilon_1), \quad (3.3)
\]

and

\[
\frac{\text{dist}(z_j, E)^{\tau - \varepsilon}}{1 - |z_j|} \to \infty \quad (3.4)
\]

for a suitable sequence \( \{z_j\} \subset \Omega(\theta, \varepsilon_1) \).

Now let \( F = F_{(\tau + \varepsilon)\beta} \) be the outer function associated, in the sense of Theorem B, with the exponent \( (\tau + \varepsilon)\beta \) and the current set \( E \). We have then \( F \in A^{(\tau + \varepsilon)\beta} \subset A^\alpha \), since \( (\tau + \varepsilon)\beta > \tau \beta \geq \alpha \), and

\[
|F(z)| \asymp \text{dist}(z, E)^{(\tau + \varepsilon)\beta}, \quad z \in \mathbb{D} \cup \mathbb{T}. \quad (3.5)
\]
When coupled with (3.3), this yields
\[ \frac{|F(z)|}{(1-|z|)^\beta} \leq C_1 \left( \frac{\text{dist}(z, E)^{\tau+\epsilon}}{1-|z|} \right)^\beta \leq C_2 < \infty, \quad z \in \Omega(\theta, \varepsilon_1). \tag{3.6} \]

On the other hand, letting \( \{z_j\} \subset \Omega(\theta, \varepsilon_1) \) be the sequence from (3.4), we use (3.5) to get
\[ \frac{|F(z_j)|}{(1-|z_j|)^\alpha} \geq c \cdot \text{dist}(z_j, E)^{(\tau+\epsilon-\tau-\epsilon)\alpha} \left( \frac{\text{dist}(z_j, E)^{\tau-\epsilon}}{1-|z_j|} \right)^\alpha. \]

Hence, in view of (3.2) and (3.4), we obtain
\[ \frac{|F(z_j)|}{(1-|z_j|)^\alpha} \to \infty. \tag{3.7} \]

Recalling Theorem A, we readily deduce from (3.6) and (3.7) that \( F \in A^\beta \setminus A^\alpha \). Thus, \( \theta \) is not \((A^\alpha, A^\beta)\)-improving, and this contradiction completes the proof of the first step.

Now we know that \( \tau = 1 \). Thus, \( \theta \) is of order \( \leq \gamma \) for every \( \gamma > 1 \). In particular, this is true for \( \gamma = \alpha/\beta \) (where \( \alpha \) and \( \beta \) come from the theorem’s statement), which means that
\[ \text{dist}(z, E)^{\alpha} = O\left( (1 - |z|)^{\beta} \right), \quad z \in \Omega(\theta, \varepsilon_2) \tag{3.8} \]
for a suitable \( \varepsilon_2 \in (0, 1) \). Further, let \( F_\alpha \) be the outer function from Theorem B, where the porous set in question is again \( E = \sigma(\theta) \). We have then \( F_\alpha \in A^\alpha \) and
\[ F_\alpha(z) = O\left( (1 - |z|)^{\beta} \right), \quad z \in \Omega(\theta, \varepsilon_2), \tag{3.9} \]
this last estimate being a consequence of (2.2) and (3.8). By Theorem A, (3.9) implies that \( F_\alpha \theta \in A^\beta \). Now since \( \theta \) is \((A^\alpha, A^\beta)\)-improving, we must also have \( F_\alpha \theta \in A^\alpha \), or equivalently,
\[ F_\alpha(z) = O\left( (1 - |z|)^{\alpha} \right), \quad z \in \Omega(\theta, \varepsilon_2) \tag{3.10} \]
(we have used Theorem A again). Finally, (2.2) and (3.10) together yield
\[ \text{dist}(z, E) = O\left( 1 - |z| \right), \quad z \in \Omega(\theta, \varepsilon_2). \]
This is precisely the Stolz condition (1.7), with \( \varepsilon_2 \) in place of \( \varepsilon \), so the proof is complete.

4. Proofs of Theorems 1.6, 1.9 and 1.10

A few lemmas will be needed. In what follows, we write \( P_+ \) and \( P_- \) for the orthogonal projections from \( L^2(\mathbb{T}) \) onto \( H^2 \) and \( \overline{z}H^2 \), respectively. When necessary, these operators are extended to \( L^1(\mathbb{T}) \) in the natural way (as Cauchy integrals), even though they do not act boundedly on \( L^1(\mathbb{T}) \).

Our first lemma is contained in [11]; though not explicitly stated there, it follows upon combining inequalities (9.2) and (10.6) that appear on pp. 830 and 832 of that paper.
Lemma 4.1. Let $1 \leq p < \infty$ and $0 < s < 1$. Suppose that $f \in H^p$ and $\theta$ is an inner function, whose derivative $\theta'$ lies in the Nevanlinna class $N$ and satisfies $\int_T |f|^p |\theta'|^s \alpha < \infty$. Then $P_-(f \bar{\theta}) \in B^s_p$.

The following two lemmas are also known, but we include short proofs for the reader’s convenience.

Lemma 4.2. Given $1 \leq p < \infty$, $0 < s < 1$ and a function $h \in H^\infty$, the Toeplitz operator $f \mapsto P_+(f \bar{h})$ maps $AB^s_p$ continuously into itself.

Proof. We use a duality argument that goes back to [18]. Namely, it suffices to check that the multiplication map $g \mapsto gh$, with $h \in H^\infty$, acts boundedly on the dual space $(AB^s_p)^\ast$. This dual space is known to coincide with $AB^{-s}_p$, the set of analytic functions $g$ satisfying

$$
\int_\mathbb{D} \left| g(z) \right|^p (1 - |z|)^{sp - 1} dA(z) < \infty, \quad p' := p/(p - 1)
$$

(with the natural modification for $p = 1$). The required fact is now obvious. □

Lemma 4.3. Let $f \in B^s_p$, where $1 \leq p < \infty$ and $0 < s < 1$, and let $u \in L^\infty(\mathbb{T})$. In order that $fu \in B^s_p$, it is necessary and sufficient that

$$
\int_{-\pi}^{\pi} \left| f(\xi) \right|^p \left| u(e^{it} \xi) - u(\xi) \right|^p dm(\xi) < \infty.
$$

In particular, $fu \in B^s_p$ if and only if $f \bar{u} \in B^s_p$.

Proof. To arrive at (4.1), one plugs the product $fu$ in place of $f$ into the definition (1.11) of the Besov space, noting that

$$
f(e^{it} \xi) u(e^{it} \xi) - f(\xi) u(\xi) = \left[ f(e^{it} \xi) - f(\xi) \right] u(e^{it} \xi) + f(\xi) \left[ u(e^{it} \xi) - u(\xi) \right].
$$

The rest follows from the fact that (4.1) remains unchanged when passing from $u$ to $\bar{u}$. □

Proof of Theorem 1.6. Assuming that the hypotheses of the theorem are fulfilled, we let $f \in A^\alpha \cap B^s_p$ be a function with $f \theta \in A^\alpha$, and we have to prove that $f \theta \in B^s_p$.

Since

$$
|f| |\theta'|^\alpha \leq C < \infty \quad \text{a.e. on } \mathbb{T}
$$

(recall Lemma 2.1 and note that $m(\sigma(\theta)) = 0$ unless $f \equiv 0$), we obtain

$$
\int |f|^p |\theta'|^sp \, dm = \int |f|^p |\theta'|^{\alpha p} |\theta'|^{(s-\alpha)p} \, dm \leq C^p \int |\theta'|^{(s-\alpha)p} \, dm < \infty.
$$
By Lemma 4.1, it follows that $P_-(f \tilde{\theta}) \in B_p^s$. On the other hand, Lemma 4.2 tells us that $P_+(f \tilde{\theta}) \in B_p^s$. Taken together, the two inclusions yield $f \tilde{\theta} \in B_p^s$, which in turn is equivalent (by virtue of Lemma 4.3) to saying that $f \theta \in B_p^s$.  

The same proof, with $s = \alpha$, works for Proposition 1.7.

Before proceeding with the proof of Theorem 1.9, we state two more preliminary results. The first of these, Lemma 4.4, is a simple consequence of the “Garsia norm” characterization of BMOA. It can be found in [8] (as part of Theorem 1) and possibly elsewhere.

**Lemma 4.4.** Given $f \in \text{BMOA}$ and an inner function $\theta$, one has $f \theta \in \text{BMOA}$ if and only if

$$
\sup \left\{ \left| f(z) \right|^2 \left( 1 - \left| \theta(z) \right|^2 \right): z \in \mathbb{D} \right\} < \infty.
$$

The next lemma is a special case of Theorem 3.2 in [13] (put $f \equiv 1$ and $q = p$ to arrive at the current version).

**Lemma 4.5.** Let $2 \leq p < \infty$, $0 < s < 1/2$, and let $\theta$ be an inner function. In order that $\theta \in AB_p^s$, it is necessary and sufficient that

$$
\int_{\mathbb{D}} \frac{(1 - |\theta(z)|^2)^{p/2}}{(1 - |z|)^{sp+1}} dA(z) < \infty.
$$

**Proof of Theorem 1.9.** (i) $\Rightarrow$ (ii). This is immediate, since for $\theta \notin AB_p^s$ the product $1 \cdot \theta = \theta$ lies in $\text{BMOA} \setminus AB_p^s$, while the constant function $1$ obviously belongs to $AB_p^s \cap \text{BMOA}$.

(ii) $\Rightarrow$ (i). Let $\theta \in AB_p^s$, and suppose $f$ is a function in $AB_p^s \cap \text{BMOA}$ such that $f \theta \in \text{BMOA}$. We want to prove that $f \theta \in AB_p^s$, or equivalently, that

$$
\int_{\mathbb{D}} \left| (f \theta)'(z) \right|^p dv_{p,s}(z) < \infty, \tag{4.2}
$$

where

$$
dv_{p,s}(z) := (1 - |z|)^{(1-s)p-1} dA(z).
$$

Since $f \in AB_p^s$, we have

$$
\int_{\mathbb{D}} \left| f'(z) \right|^p \left| \theta(z) \right|^p dv_{p,s}(z) \leq \int_{\mathbb{D}} \left| f'(z) \right|^p dv_{p,s}(z) < \infty;
$$

therefore, (4.2) will be established as soon as we check that the integral

$$
I := \int_{\mathbb{D}} \left| f(z) \right|^p |\theta'(z)|^p dv_{p,s}(z)
$$

...
converges. To this end, we use the inequality
\[ |\theta'(z)| \leq \frac{1 - |\theta(z)|^2}{1 - |z|^2} \]
(which comes from the Schwarz lemma) and the fact that
\[ |f(z)|^2 \left( 1 - |\theta(z)|^2 \right) \leq C < \infty, \]
as asserted by Lemma 4.4. Taking the two estimates into account, we obtain
\[ I \leq \int_{\mathbb{D}} |f(z)|^p \left( 1 - |\theta(z)|^2 \right)^{p/2} \frac{dA(z)}{(1 - |z|)^{sp+1}} \leq C^{p/2} \int_{\mathbb{D}} \frac{(1 - |\theta(z)|^2)^{p/2}}{(1 - |z|)^{sp+1}} dA(z). \]
This last quantity being finite by Lemma 4.5, we readily conclude that \( I < \infty \), as desired.

**Proof of Theorem 1.10.** The (i) \( \iff \) (ii) part is proved exactly as above (the appropriate version of Lemma 4.5 is again a special case of Theorem 3.2 in [13]), while the (ii) \( \iff \) (iii) part is borrowed from [22]. \( \square \)

**5. Proof of Theorem 1.11**

First we cite a result from [8].

**Lemma 5.1.** Let \( 0 < \varepsilon < 1 \). Given \( f \in \text{BMOA} \) and an inner function \( \theta \), one has \( f \theta \in \text{BMOA} \) if and only if
\[ \sup \{ |f(z)| : z \in \Omega(\theta, \varepsilon) \setminus \Omega(\theta, \varepsilon_0) \} < \infty \quad (5.1) \]
whenever \( 0 < \varepsilon_0 < \varepsilon \).

Of course, the “only if” part is immediate from Lemma 4.4, but the converse is harder to prove. In fact, Theorem 1 of [8] involves the formally stronger condition
\[ \sup \{ |f(z)| : z \in \Omega(\theta, \varepsilon) \} < \infty \]
in place of (5.1), but the proof given there establishes the sufficiency of (5.1) as well.

**Proof of Theorem 1.11.** (a) Suppose \( f \in \text{BMOA} \) and \( f \theta \in \mathcal{B} \). Since
\[ |f(z)| |\theta'(z)| \leq |f'(z)| |\theta(z)| + |(f \theta)'(z)| \]
and since these last two terms are both \( O((1 - |z|)^{-1}) \), as ensured by our current hypotheses, we have
\[ |f(z)| |\theta'(z)| \leq C(1 - |z|)^{-1}, \quad z \in \mathbb{D}, \]
for some $C > 0$. Combining this with the assumption that
\[
(1 - |z|)|\theta'(z)| \geq \delta, \quad z \in \Omega(\theta, \varepsilon) \setminus \Omega(\theta, \varepsilon_0),
\]
for each $\varepsilon_0 \in (0, \varepsilon)$ and a suitable $\delta = \delta(\varepsilon_0) > 0$, we deduce that
\[
|f(z)| \leq C\delta^{-1}, \quad z \in \Omega(\theta, \varepsilon) \setminus \Omega(\theta, \varepsilon_0).
\]
By Lemma 5.1, it follows that $f \theta \in \text{BMOA}$.

(b) Given an inner function $\theta \in B_0$ (not a finite Blaschke product), we want to find a function $f \in \text{BMOA}$ for which $f \theta \in B \setminus \text{BMOA}$.

In view of Rohde’s results, $\theta$ has radial limit 0 at “many” points of $\mathbb{T}$. (Precisely speaking, this happens on a subset of $\mathbb{T}$ that has Hausdorff dimension 1; see [20, Theorem 1.4].) In particular, there exists at least one point with this property, and we may assume that
\[
\lim_{x \to 1^-} \theta(x) = 0. \quad (5.2)
\]
Further, the hypothesis $\theta \in B_0$ means that the quantity
\[
\beta(t) := \sup\{(1 - |z|)|\theta'(z)|: 1 - t \leq |z| < 1\}
\]
tends to 0 as $t \to 0^+$. To complete the proof, it now suffices to construct a function $f \in \text{BMOA}$ with the properties that
\[
f(z) = O\left(\frac{1}{\beta(1 - |z|)}\right), \quad z \in \mathbb{D}, \quad (5.3)
\]
and
\[
\lim_{x \to 1^-} f(x) = \infty. \quad (5.4)
\]
Indeed, for such an $f$ one has $f \theta \in B$, because
\[
|(f \theta)'(z)| \leq |f'(z)| + |f(z)||\theta'(z)|,
\]
where both terms on the right are $O((1 - |z|)^{-1})$. (To see why, recall that $f \in \text{BMOA} \subset B$ and use (5.3) in conjunction with the obvious fact that $(1 - |z|)|\theta'(z)| \leq \beta(1 - |z|)$.) On the other hand, (5.2) and (5.4) together imply that
\[
\lim_{x \to 1^-} |f(x)|^2(1 - |\theta(x)|^2) = \infty.
\]
By Lemma 4.4, this yields $f \theta \notin \text{BMOA}$ (alternatively, invoke Lemma 5.1).

Finally, we explain how to construct a BMOA-function satisfying (5.3) and (5.4). To this end, we put
\[
f(z) := \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \mu(\xi) \, dm(\xi), \quad z \in \mathbb{D},
\]
where \( h \in C(T) \) is a function with some special properties. Namely, we require that the function 
\[ t \mapsto h(e^{it}) \]
be positive for \( 0 < t < \pi \) and odd for \( -\pi \leq t \leq \pi \), with
\[
\int_0^\pi \frac{h(e^{it})}{t} dt = \infty, \tag{5.5}
\]
and that
\[
\int_\varepsilon^\pi \frac{\omega_h(t)}{t} dt = O\left(\frac{1}{\beta(\varepsilon)}\right) \quad \text{as } \varepsilon \to 0^+; \tag{5.6}
\]
here \( \omega_h(\cdot) \) stands for the modulus of continuity of \( h \), that is,
\[
\omega_h(t) := \sup\{|h(\zeta_1) - h(\zeta_2)| : \zeta_1, \zeta_2 \in T, \ |\zeta_1 - \zeta_2| < t\}.
\]
We have then \( f \in \text{VMOA} \) (and hence \( f \in \text{BMOA} \)), because \( h \) is continuous. As to the properties
(5.3) and (5.4), they come out as consequences of (5.6) and (5.5), respectively, via standard
estimates of the conjugate Poisson integral in terms of the function’s modulus of continuity (see [17, Chapter III, Section 1]). The proof is complete. \( \square \)

References

