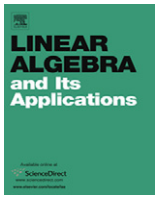
Linear Algebra and its Applications 435 (2011) 2837–2859

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa



A matrix pencil approach to the existence of compactly supported reconstruction functions in average sampling

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ARTICLE INFO

Article history:
Received 2 November 2009
Accepted 5 May 2011
Available online 21 June 2011

Submitted by V. Mehrmann

AMS classification:
15A21
15A22
42C15
42C40
94A20

Keywords:
Sampling in shift-invariant spaces
Smith canonical form
Matrix pencils
Kronecker canonical form
GUPTRI form

ABSTRACT

The aim of this work is to solve a question raised for average sampling in shift-invariant spaces by using the well-known matrix pencil theory. In many common situations in sampling theory, the available data are samples of some convolution operator acting on the function itself: this leads to the problem of average sampling, also known as generalized sampling. In this paper we deal with the existence of a sampling formula involving these samples and having reconstruction functions with compact support. Thus, low computational complexity is involved and truncation errors are avoided. In practice, it is accomplished by means of a FIR filter bank. An answer is given in the light of the generalized sampling theory by using the oversampling technique: more samples than strictly necessary are used. The original problem reduces to finding a polynomial left inverse of a polynomial matrix intimately related to the sampling problem which, for a suitable choice of the sampling period, becomes a matrix pencil. This matrix pencil approach allows us to obtain a practical method for computing the compactly supported reconstruction functions for the important case where the oversampling rate is minimum. Moreover, the optimality of the obtained solution is established.

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doi:10.1016/j.laa.2011.05.007

1. Statement of the problem

Let V_φ be a shift-invariant space in $L^2(\mathbb{R})$ with stable generator $\varphi \in L^2(\mathbb{R})$, i.e.,

$$V_\varphi := \left\{ f(t) = \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n) \ : \ \{a_n\} \in \ell^2(\mathbb{Z}) \right\} \subset L^2(\mathbb{R}),$$

where the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator (see [4]).

Nowadays, sampling theory in shift-invariant spaces is a very active research topic (see, for instance, [1–3,8] and the references therein) since an appropriate choice for the generator φ (for instance, a B-spline) eliminates some of the problems associated with the classical Shannon's sampling theory [17]. On the other hand, in many common situations the available data are samples of some filtered version $f * \mathsf{h}$ of the signal f itself. Suppose that a linear time-invariant system $\mathcal{L}$ of one of the following types (or a linear combination of both) is defined on V_φ:

(a) The impulse response h of $\mathcal{L}$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, for any $f \in V_\varphi$ we have

$$(\mathcal{L}f)\,(t) := [f*\mathsf{h}](t) = \int_{-\infty}^{\infty} f(x)\mathsf{h}(t-x)dx, \quad t \in \mathbb{R}.$$

(b) $\mathcal{L}$ involves samples of the function itself, i.e., $(\mathcal{L}f)(t) = f(t+d)$, $t \in \mathbb{R}$, for some constant $d \in \mathbb{R}$.

Under suitable conditions, Unser and Aldroubi [16] have derived sampling formulas allowing the recovering of any function $f \in V_\varphi$ from the sequence of samples $\{(\mathcal{L}f)\,(n)\}_{n\in\mathbb{Z}}$. Concretely, they proved that for any $f \in V_\varphi$,

$$f(t) = \sum_{n\in\mathbb{Z}} \mathcal{L}f(n) S_{\mathcal{L}}(t-n), \quad t \in \mathbb{R}, \tag{1}$$

where the sequence $\{S_{\mathcal{L}}(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. Notice that a reconstruction function $S_{\mathcal{L}}$ with compact support implies low computational complexity and avoids truncation errors. Even when the generator φ has compact support, rarely the same occurs with the reconstruction function $S_{\mathcal{L}}$ in formula (1). A way to overcome this difficulty is to use the oversampling technique, i.e., for fixed positive integers $s > r$, consider the sampling period $T := r/s < 1$. The goal is to recover any function $f \in V_\varphi$ by using a sampling expansion involving the samples $\{(\mathcal{L}f)(rn/s)\}_{n\in\mathbb{Z}}$. This can be done in the light of the generalized sampling theory developed in [10]. Indeed, since the sampling points rn/s, $n \in \mathbb{Z}$, can be expressed as $\{rn/s\}_{n\in\mathbb{Z}} = \{rm + (j-1)r/s\}_{m\in\mathbb{Z},\, j=1,2,\ldots,s}$, the initial problem is equivalent to the recovery of $f \in V_\varphi$ from the sequence of samples $\{\mathcal{L}_j f\,(rn)\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$, where the linear time-invariant systems $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, are given in terms of $\mathcal{L}$ by: $(\mathcal{L}_j f)(t) := (\mathcal{L}f)\,[t + (j-1)r/s]$, $t \in \mathbb{R}$. Following the notation introduced in [10], consider the functions $g_j \in L^2(0,1)$, $j = 1, 2, \ldots, s$, defined as:

$$g_j(w) := \sum_{n\in\mathbb{Z}} (\mathcal{L}\varphi)\,[n + (j-1)r/s]\, e^{-2\pi i n w} = \sum_{n\in\mathbb{Z}} (\mathcal{L}_j\varphi)(n) e^{-2\pi i n w}, \tag{2}$$

the $s \times r$ matrix of functions $\mathbf{G}(w)$ given by:

$$\mathbf{G}(w) := \begin{bmatrix} g_1(w) & g_1(w+\frac{1}{r}) & \cdots & g_1(w+\frac{r-1}{r}) \\ g_2(w) & g_2(w+\frac{1}{r}) & \cdots & g_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ g_s(w) & g_s(w+\frac{1}{r}) & \cdots & g_s(w+\frac{r-1}{r}) \end{bmatrix} = \left[g_j\left(w + \frac{k-1}{r}\right) \right]_{\substack{j=1,2,\ldots,s \\ k=1,2,\ldots,r}},$$

and its related constants

$$\alpha_{\mathbf{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbf{G}^*(w)\mathbf{G}(w)], \quad \beta_{\mathbf{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbf{G}^*(w)\mathbf{G}(w)],$$

where $\mathbf{G}^*(w)$ denotes the transpose conjugate of the matrix $\mathbf{G}(w)$, and $\lambda_{\min}$ and $\lambda_{\max}$ denote, respectively, the smallest and the largest eigenvalue of the positive semidefinite matrix $\mathbf{G}^*(w)\mathbf{G}(w)$. Notice that in the definition of the matrix $\mathbf{G}(w)$ we are considering the 1-periodic extensions of the involved functions g_j, $j = 1, 2, \ldots, s$. Thus, the generalized sampling theory in [10] can be summarized as:

Theorem 1. *Assume that the functions g_j, $j = 1, 2, \ldots, s$, defined in (2) belong to $L^\infty(0, 1)$ (this is equivalent to $\beta_{\mathbf{G}} < \infty$). Then the following statements are equivalent:*

(i) $\alpha_{\mathbf{G}} > 0$.
(ii) There exist functions a_j in $L^\infty(0, 1)$, $j = 1, 2, \ldots, s$, such that

$$[a_1(w), \ldots, a_s(w)]\ \mathbf{G}(w) = [1, 0, \ldots, 0] \quad a.e.\ in\ (0, 1). \tag{3}$$

(iii) There exists a frame for V_φ having the form $\{S_j(\cdot - rn)\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ such that, for any $f \in V_\varphi$, we have

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn)\, S_j(t - rn) \quad in\ L^2(\mathbb{R}). \tag{4}$$

In case the equivalent conditions are satisfied, the reconstruction functions S_j, $j = 1, 2, \ldots, s$, in (4) are given by:

$$S_j(t) = r \sum_{n\in\mathbb{Z}} \langle a_j, e^{-2\pi i n w} \rangle_{L^2(0,1)} \varphi(t - n), \tag{5}$$

where the functions a_j, $j = 1, 2, \ldots, s$, satisfy (3). The convergence of the series in (4) is also absolute and uniform on $\mathbb{R}$.

For the details on the frame theory see the superb monograph [4] and the references therein. Observing (5), in case the generator φ is compactly supported, we have reconstruction functions S_j of compact support whenever the functions a_j in (3) are trigonometric polynomials. Notice that compactly supported reconstruction functions S_j, $j = 1, 2, \ldots, s$, in formula (4) involve low computational complexity and it avoids truncation errors. On the other hand, a sampling formula as those in (4) can be seen as a filter bank, where $\mathbf{G}(w)$ is its *modulation matrix*. Indeed, denoting the reconstruction function in (5) as $S_j(t) = \sum_{n\in\mathbb{Z}} d_j(n)\varphi(t - n)$, $j = 1, 2, \ldots, s$, for any $f(t) = \sum_{m\in\mathbb{Z}} c_m\, \varphi(t - m)$ in V_φ one can easily deduce that

$$c_m = \sum_{j=1}^{s} \sum_{n\in\mathbb{Z}} (\mathcal{L}_j f)(rn) d_j(m - rn), \quad m \in \mathbb{Z}. \tag{6}$$

As a consequence, compactly supported reconstruction functions S_j entail a FIR (finite impulse response, i.e., only a finite number of coefficients $d_j(n)$ are nonzero) filter bank.

It is worth to mention that whenever the 1-periodic functions g_j, $j = 1, 2, \ldots, s$, are continuous on $\mathbb{R}$, the conditions in Theorem 1 are also equivalent to the condition recently introduced in [11, Corollary 1]: *(iv)* $\operatorname{rank} \mathbf{G}(w) = r$ for all $w \in \mathbb{R}$.

In order to find reconstruction functions S_j, $j = 1, 2, \ldots, s$, in formula (4) having compact support we assume in what follows that the generator φ and $\mathcal{L}\varphi$ are compactly supported. We introduce the

$s \times r$ matrix

$$\mathsf{G}(z) := \begin{bmatrix} \mathsf{g}_1(z) & \mathsf{g}_1(Wz) & \cdots & \mathsf{g}_1(W^{r-1}z) \\ \mathsf{g}_2(z) & \mathsf{g}_2(Wz) & \cdots & \mathsf{g}_2(W^{r-1}z) \\ \vdots & \vdots & & \vdots \\ \mathsf{g}_s(z) & \mathsf{g}_s(Wz) & \cdots & \mathsf{g}_s(W^{r-1}z) \end{bmatrix}, \tag{7}$$

where $W := e^{-2\pi i/r}$ and $\mathsf{g}_j(z) := \sum_{n\in\mathbb{Z}}(\mathcal{L}\varphi)[n + (j-1)r/s]\, z^n$, $j = 1, 2 \ldots, s$. Notice that the matrix $\mathsf{G}(z)$ has Laurent polynomials entries, and $\mathbf{G}(w) = \mathsf{G}(e^{-2\pi iw})$. On the other hand, if the functions $\mathsf{a}_j(z)$, $j = 1, 2 \ldots, s$, are Laurent polynomials satisfying

$$[\mathsf{a}_1(z), \ldots, \mathsf{a}_s(z)]\mathsf{G}(z) = [1, 0, \ldots, 0], \tag{8}$$

then, the trigonometric polynomials $a_j(w) = \mathsf{a}_j(e^{-2\pi iw})$, $j = 1, 2, \ldots, s$, satisfy (3) and give reconstruction functions S_j via formula (5).

The existence of polynomial solutions of (8) is equivalent to the existence of a left inverse of the matrix $\mathsf{G}(z)$ whose entries are polynomials. This problem has been studied in [5] by Cvetković and Vetterli in the filter banks setting. By using the Smith canonical form $\mathsf{S}(z)$ of the matrix $\mathsf{G}(z)$ (see [14] for the details), a characterization for the existence of polynomial solutions of (8) has been found in [12]. Namely, assuming that the generator φ and $\mathcal{L}\varphi$ have compact support, there exists a polynomial vector $[\mathsf{a}_1(z), \mathsf{a}_2(z), \cdots, \mathsf{a}_s(z)]$ satisfying (8) if and only if the polynomials $i_j(z)$, $j = 1, 2, \ldots, r$, on the diagonal of the Smith canonical form $\mathsf{S}(z)$ of the matrix $\mathsf{G}(z)$ are monomials. Assume that the $s \times r$ matrix

$$\mathsf{S}(z) = \begin{bmatrix} i_1(z) & 0 & \cdots & 0 \\ 0 & i_2(z) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & i_r(z) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{9}$$

is the Smith canonical form of the matrix $\mathsf{G}(z)$ (note that it is the case whenever $\alpha_{\mathbf{G}} > 0$) and consider the unimodular matrices $\mathbf{V}(z)$ and $\mathbf{W}(z)$, of dimension $s \times s$ and $r \times r$ respectively, such that $\mathsf{G}(z) = \mathbf{V}(z)\mathsf{S}(z)\mathbf{W}(z)$.

Observe that if $\mathsf{S}(z)$ is the Smith form of the matrix $\mathsf{G}(z)$ then, taking into account that $\mathbf{V}(z)$ and $\mathbf{W}(z)$ are unimodular matrices, we have

$$\operatorname{rank}\mathsf{S}(z) = \operatorname{rank}\mathsf{G}(z) \text{ for all } z \in \mathbb{C}.$$

Therefore, it is straightforward to deduce that the polynomial $i_j(z)$ is a monomial, for each $j = 1, 2, \ldots, r$, if and only if $\operatorname{rank}\mathsf{S}(z) = r$ for all $z \in \mathbb{C} \setminus \{0\}$. This condition, under the above hypotheses on φ and $\mathcal{L}\varphi$, is equivalent to saying that

$$\operatorname{rank}\mathsf{G}(z) = r \text{ for all } z \in \mathbb{C} \setminus \{0\}. \tag{10}$$

(See [12] for the details.) From a practical point of view, the decomposition $\mathsf{G}(z) = \mathbf{V}(z)\mathsf{S}(z)\mathbf{W}(z)$ has an important drawback: there is not a stable method for its computation. Nevertheless, there exists a finite algorithm to determine $\mathsf{S}(z)$, and consequently, for checking condition (10): see Ref. [19]. As

pointed out in (8), in order to obtain reconstruction functions with compact support we also need to compute a polynomial left inverse of matrix $\mathsf{G}(z)$.

Another algebraic approach is the following (see, for instance, [15]): Assume that $\mathsf{G}(z)$ is a $s \times r$ Laurent polynomial matrix ($r < s$); whenever the greatest common divisor of all minors of maximum order r is a monomial, then its Smith canonical form $\mathsf{S}(z)$ has monomials in its diagonal. Without loss of generality we can assume that the $\gamma := \binom{s}{r}$ minors of order r in $\mathsf{G}(z)$ are polynomials with positive powers in z. Invoking Euclides algorithm we can obtain $\binom{s}{r}$ polynomials, $f_1(z), \ldots, f_\gamma(z)$, such that

$$\sum_{n=1}^{\gamma} f_n(z) A_n(z) = m(z), \quad \text{for all } z \in \mathbb{C},$$

where A_n, $1 \leqslant n \leqslant \gamma$, are the minors of order r of $\mathsf{G}(z)$ and $m(z)$ is a monomial. Denote by $D'_n(z)$ the adjoint matrix corresponding to the minor A_n and $D_n(z)$ the matrix obtained from $D'_n(z)$ by adding $s - r$ zero columns. Thus, $D_n(z)\mathsf{G}(z) = A_n(z)\mathbf{I}_r$, and consequently

$$\left(\sum_{n=1}^{\gamma} f'_n(z) D_n(z)\right) \mathsf{G}(z) = \mathbf{I}_r,$$

where $f'_n(z) := f(z)/m(z)$ could be a Laurent polynomial, $1 \leqslant n \leqslant \gamma$. From a practical point of view the drawback here is the effective calculation of the $\binom{s}{r}$ minors of $\mathsf{G}(z)$ whenever r becomes larger.

In this paper, along with finding necessary and sufficient conditions assuring compactly supported reconstruction functions, we are also interested in obtaining these functions, and in proving the optimality of their supports. Taking advantage of the special structure of the matrix $\mathsf{G}(z)$ we reduce our problem to one solved by using the matrix pencil theory. Concretely, we use some information from the Kronecker canonical form of a matrix pencil associated with the matrix $\mathsf{G}(z)$ (see [9] for the details).

The paper is organized as follows: In Section 2, a suitable choice of the sampling period $T = r/s$ reduces our problem to a matrix pencil problem. This matrix pencil, related to the *polyphase matrix* of the filter bank given in (6), has proven to be useful in practice (see Ref. [13]). Thus, we give a necessary and sufficient condition for the existence of compactly supported reconstruction functions which involves the Kronecker canonical form of a singular matrix pencil. Section 3 is devoted to compute a polynomial left inverse of the matrix $\mathsf{G}(z)$ in the important case where the oversampling rate is minimum, i.e., $T = r/(r+1)$. Finally, we prove that the polynomial left inverse of the matrix $\mathsf{G}(z)$ previously calculated leads to reconstruction functions with minimal support.

2. Reducing the polynomial matrix $\mathsf{G}(z)$ to a matrix pencil

The first step is to reduce our polynomial matrix $\mathsf{G}(z)$ to a matrix pencil in order to use the well-established theory on matrix pencils. In so doing we need some preliminaries. Let $f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z^1 + a_0$ be an algebraic polynomial of order m, and let n be a positive integer. For each $j = 0, 1, \ldots, n-1$ let $\widehat{f}_j(z)$ denote the sum of the monomials $a_r z^r$ where $r \equiv j (\mathrm{mod}\ n)$. Obviously, $f(z) = \sum_{j=0}^{n-1} \widehat{f}_j(z)$. The polynomial $\widehat{f}_j$, $0 \leqslant j \leqslant n-1$, is the so-called n-harmonic of order j of the polynomial f; it satisfies:

$$\widehat{f}_j(e^{2\pi i/n} z) = e^{2\pi ij/n} \widehat{f}_j(z) \quad \text{for all } z \in \mathbb{C}.$$

Assume that $\mathrm{supp}\, \mathcal{L}\varphi$ is contained in an interval $[0, N]$, where $N \in \mathbb{N}$. Thus, the functions $\mathsf{g}_j(z)$ are Laurent polynomials in the variable z. Consider

$$p := \min\left\{ q \in \mathbb{N} : q\,\frac{r}{s} > 1 \right\}.$$

It is easy to check that $p = c + 1$ where c denotes the quotient in the euclidean division $s|r$. Hence, we can write the Laurent polynomials $g_i(z), j = 1, 2 \ldots, s$, as:

$$\begin{aligned}
g_1(z) &= \mathcal{L}\varphi(1)z + \mathcal{L}\varphi(2)z^2 + \cdots + \mathcal{L}\varphi(N-1)z^{N-1}\\
g_2(z) &= \mathcal{L}\varphi\left(\frac{r}{s}\right) + \mathcal{L}\varphi\left(1+\frac{r}{s}\right)z + \cdots + \mathcal{L}\varphi\left(N-1+\frac{r}{s}\right)z^{N-1}\\
&\vdots\\
g_p(z) &= \mathcal{L}\varphi\left((p-1)\frac{r}{s}\right) + \mathcal{L}\varphi\left(1+(p-1)\frac{r}{s}\right)z + \cdots + \mathcal{L}\varphi\left(N-1+(p-1)\frac{r}{s}\right)z^{N-1}\\
g_{p+1}(z) &= \mathcal{L}\varphi\left(p\frac{r}{s}-1\right)z^{-1} + \cdots + \mathcal{L}\varphi\left(N-2+p\frac{r}{s}\right)z^{N-2}\\
&\vdots\\
g_s(z) &= \mathcal{L}\varphi\left((s-1)\frac{r}{s}-r+1\right)z^{-(r-1)} + \cdots + \mathcal{L}\varphi\left(N-r+2+(s-1)\frac{r}{s}\right)z^{N-r+2}.
\end{aligned} \tag{11}$$

The polynomial $g_1(z)$ has at most $N-1$ nonzero terms; the rest of polynomials $g_j(z), 2 \leqslant j \leqslant s$, have at most N nonzero terms. In what follows, we use the new matrix $\mathbb{G}(z) = \mathsf{G}(z)\mathsf{U}(z)$, where

$$\mathsf{U}(z) = \operatorname{diag}\left[z^{r-1}, (Wz)^{r-1}, (W^2z)^{r-1}, \ldots, (W^{r-1}z)^{r-1}\right].$$

Thus, all entries of the polynomial matrix $\mathbb{G}(z)$ are algebraic polynomials in z and, moreover we have $\operatorname{rank}\mathbb{G}(z) = \operatorname{rank}\mathsf{G}(z)$ for all $z \in \mathbb{C} \setminus \{0\}$. We denote by $\widetilde{g}_j(z)$ the algebraic polynomial $z^{r-1}g_j(z)$, $1 \leqslant j \leqslant s$.

The strategy is to reduce the polynomial matrix $\mathbb{G}(z)$ into another simpler one having the same rank for all $z \in \mathbb{C} \setminus \{0\}$.

Lemma 1. *Consider the matrix $\widehat{\mathbb{G}}(z) = [\widehat{\mathbb{G}}_0(z)\ \widehat{\mathbb{G}}_2(z)\ \ldots\ \widehat{\mathbb{G}}_{(r-1)}(z)]$, where $\widehat{\mathbb{G}}_j(z)$, $0 \leqslant j \leqslant (r-1)$, denotes the column vector consisting of the r-harmonics of order j of the polynomials $\widetilde{g}_i(z)$ where $1 \leqslant i \leqslant s$. Then*

$$\mathbb{G}(z) = \widehat{\mathbb{G}}(z)\Omega_r,$$

where Ω_r denotes the Fourier matrix of order r.

Proof. For each $i = 1, 2, \ldots, s$, we have that $\widetilde{g}_i(z) = \sum_{j=0}^{r-1}\widehat{\widetilde{g}}_{ij}(z)$ where $\widehat{\widetilde{g}}_{ij}(z)$ denotes the r-harmonic of order j of $\widetilde{g}_i$. We can write the matrix $\mathbb{G}(z)$ as

$$\begin{aligned}
\mathbb{G}(z) = [&\widehat{\mathbb{G}}_0(z) + \widehat{\mathbb{G}}_1(z) + \cdots + \widehat{\mathbb{G}}_{r-1}(z)\\
&\widehat{\mathbb{G}}_0(z) + W\widehat{\mathbb{G}}_1(z) + \cdots + W^{r-1}\widehat{\mathbb{G}}_{r-1}(z)\\
&\cdots\cdots\\
&\widehat{\mathbb{G}}_0(z) + W^{r-1}\widehat{\mathbb{G}}_1(z) + \cdots + W^{(r-1)^2}\widehat{\mathbb{G}}_{r-1}(z)].
\end{aligned}$$

Hence, in matrix form we have

$$\mathbb{G}(z) = \left[\widehat{\mathbb{G}}_0(z)\ \widehat{\mathbb{G}}_1(z)\ \ldots\widehat{\mathbb{G}}_{r-1}(z)\right]\Omega_r = \widehat{\mathbb{G}}(z)\Omega_r,$$

where

$$\Omega_r = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{r-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(r-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W^{r-1} & W^{2(r-1)} & \cdots & W^{(r-1)^2} \end{bmatrix}$$

is the Fourier matrix of order r. □

Observe that $\operatorname{rank} \mathbb{G}(z) = \operatorname{rank} \widehat{\mathbb{G}}(z)$ for all $z \in \mathbb{C} \setminus \{0\}$.

In what follows, we assume that $\operatorname{supp} \mathcal{L}\varphi \subseteq [0, N]$ and, in addition, we also assume that $N \leqslant r$. In this case, having in mind the number of nonzero consecutive terms of the polynomial $\widetilde{\mathrm{g}}_j(z)$, we conclude that the r-harmonic of order q, $q = 0, 1 \ldots, r-1$, of the polynomial $\widetilde{\mathrm{g}}_i(z)$, $1 \leqslant i \leqslant s$, is a monomial having the form $c_{ip} z^{kr+q}$ where $c_{iq} \in \mathbb{C}$ and $k \in \{0, 1\}$. This choice of r and, consequently, of the sampling periods $T = r/s$, r, $s \in \mathbb{N}$ and $s > r$, simplifies the structure of the matrix $\widehat{\mathbb{G}}(z)$.

First, let us to give an illustrative example: Consider $N = 3$, $r = 4$ and $s = 5$; here $T = 4/5$, $p = 2$ and the polynomials $\widetilde{\mathrm{g}}_j(z)$, $1 \leq j \leq 5$, read:

$$\begin{aligned} &\widetilde{\mathrm{g}}_1(z) = \star z^4 + \star z^5, \quad \widetilde{\mathrm{g}}_2(z) = \star z^3 + \star z^4 + \star z^5, \\ &\widetilde{\mathrm{g}}_3(z) = \star z^2 + \star z^3 + \star z^4, \quad \widetilde{\mathrm{g}}_4(z) = \star z + \star z^2 + \star z^3, \\ &\widetilde{\mathrm{g}}_5(z) = \star + \star z + \star z^2. \end{aligned}$$

Hence, the matrix $\widehat{\mathbb{G}}(z)$ reads

$$\widehat{\mathbb{G}}(z) = \begin{bmatrix} \star z^4 & \star z^5 & 0 & 0 \\ \star z^4 & \star z^5 & 0 & \star z^3 \\ \star z^4 & 0 & \star z^2 & \star z^3 \\ 0 & \star z & \star z^2 & \star z^3 \\ \star & \star z & \star z^2 & 0 \end{bmatrix}. \tag{12}$$

This example shows that the 3rd and 4th columns have the form $z^2 C$ and $z^3 C'$ where C, $C' \in \mathbb{C}^{s\times 1}$. The first and second columns do not share this property. If we right multiply the matrix $\widehat{\mathbb{G}}(z)$ by $\operatorname{diag}[1, z^{-1}, z^{-2}, z^{-3}]$, we get the new matrix

$$\widetilde{\mathbb{G}}(z) := \begin{bmatrix} \star z^4 & \star z^5 & 0 & 0 \\ \star z^4 & \star z^5 & 0 & \star z^3 \\ \star z^4 & 0 & \star z^2 & \star z^3 \\ 0 & \star z & \star z^2 & \star z^3 \\ \star & \star z & \star z^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & z^{-1} & & \\ & & z^{-2} & \\ & & & z^{-3} \end{bmatrix} = \begin{bmatrix} \star z^4 & \star z^4 & 0 & 0 \\ \star z^4 & \star z^4 & 0 & \star \\ \star z^4 & 0 & \star & \star \\ 0 & \star & \star & \star \\ \star & \star & \star & 0 \end{bmatrix}.$$

Now we can go into the general case for the matrix $\widehat{\mathbb{G}}(z)$. Having in mind Eqs. (11) and that $\widetilde{\mathrm{g}}_i(z) = z^{r-1} \mathrm{g}_i(z)$ we obtain:

$$\max \{\operatorname{grad} \widetilde{\mathrm{g}}_j : 1 \leqslant j \leqslant s\} = (N-1) + (r-1) = N + r - 2 < 2r.$$

Hence, the matrix $\widehat{\mathbb{G}}(z)$ has the form

$$\widehat{\mathbb{G}}(z) = \begin{bmatrix} c_{11}z^{k_{11}r} & c_{12}z^{k_{12}r+1} & \cdots & c_{1r}z^{k_{1r}r+(r-1)} \\ \vdots & \vdots & \vdots & \vdots \\ c_{s1}z^{k_{s1}r} & c_{s2}z^{k_{s2}r+1} & \cdots & c_{sr}z^{k_{sr}r+(r-1)} \end{bmatrix},$$

where the coefficients $k_{ij} \in \{0, 1\}$. We can easily obtain the following result:

Lemma 2. *Assume that $N > 1$. Then, for each $1 \leqslant j \leqslant N-1$ there exist indices $i' \neq i$, $1 \leqslant i,\ i' \leqslant s$, such that $k_{ij} \neq k_{i'j}$. Otherwise, for each $N \leqslant j \leqslant r$ it holds that $k_{ij} = k_{i'j}$ for all $1 \leqslant i,\ i' \leqslant s$.*

Assume that $N > 1$ and recall that $N \leqslant r$. The entries of the jth column of the matrix $\widehat{\mathbb{G}}(z)$, where $N \leqslant j \leqslant r$, have the form $\star z^{j-1}$ ($\star \in \mathbb{C}$); they could have the form $\star z^{j-1}$ or $\star z^{r+(j-1)}$ whenever $1 \leqslant j \leqslant N-1$. Dividing the jth column by z^{j-1}, obviously we obtain a matrix with the same rank than $\widehat{\mathbb{G}}(z)$ for any $z \in \mathbb{C} \setminus \{0\}$. Thus, we introduce the new polynomial matrix $\widetilde{\mathbb{G}}(z)$:

$$\widetilde{\mathbb{G}}(z) := \widehat{\mathbb{G}}(z)Q(z) = [\mathbb{M}(z)\, \mathcal{G}],$$

where $\mathcal{G} \in \mathbb{C}^{s\times(r-N+1)}$ denotes a scalar matrix and $Q(z) := \operatorname{diag}[1, z^{-1}, \ldots, z^{1-r}]$. Whenever $\operatorname{rank}\mathcal{G} < r-N+1$, we have that $\operatorname{rank}\widetilde{\mathbb{G}}(z) = \operatorname{rank}\widehat{\mathbb{G}}(z) < r$ for all $z \in \mathbb{C} \setminus \{0\}$ and, hence, there is no polynomial left inverse for $\widehat{\mathbb{G}}(z)$. In the case $\operatorname{rank}\mathcal{G} = r-N+1$, there exists an invertible matrix $R \in \mathbb{C}^{s\times s}$ such that

$$R\,\mathcal{G} = \begin{bmatrix} \mathcal{G}' \\ 0 \end{bmatrix},$$

where $\mathcal{G}' \in \mathbb{C}^{(r-N+1)\times(r-N+1)}$ is invertible. Thus,

$$R\,\widetilde{\mathbb{G}}(z) = [R\mathbb{M}(z)\, R\mathcal{G}] = \begin{bmatrix} \mathbb{M}_1(z) & \mathcal{G}' \\ \mathbb{M}_2(z) & 0 \end{bmatrix}.$$

The entries of the polynomial matrix $\mathbb{M}(z) \in \mathbb{C}^{s\times(N-1)}$ are of the form $\star z^r$ or constants; denoting $\lambda = z^r$, the matrices $\mathbb{M}_i(z)$, $i = 1, 2$, can be expressed as

$$\mathbb{M}_i(\lambda) = M_{i1} - \lambda M_{i2},$$

where $M_{1i} \in \mathbb{C}^{(r-N+1)\times(N-1)}$ and $M_{2i} \in \mathbb{C}^{(s-r+N-1)\times(N-1)}$. As a consequence, we have the following result:

Lemma 3. *Assume that* $\operatorname{rank}\mathcal{G} = r-N+1$. *Then,* $\operatorname{rank}\mathsf{G}(z) = r$ *for all $z \in \mathbb{C} \setminus \{0\}$ if and only if* $\operatorname{rank}\mathbb{M}_2(\lambda) = N-1$ *for all $\lambda \in \mathbb{C} \setminus \{0\}$.*

The next step is to characterize when the rank of the matrix $M_{21} - \lambda M_{22}$ equals $N-1$ for any $\lambda \in \mathbb{C} \setminus \{0\}$. To this end, we use the Kronecker canonical form (KCF hereafter) of the matrix pencil $\mathbb{M}_2(\lambda)$ (see [9] for the details). By using the block structure notation $A \oplus B := \operatorname{diag}(A, B)$, consider the KCF of the matrix pencil $\mathbb{M}_2(\lambda)$, i.e.,

$$\mathbb{K}(\lambda) := S^{right}_{\mathbb{M}_2}(\lambda) \oplus J_{\mathbb{M}_2}(\lambda) \oplus N_{\mathbb{M}_2}(\lambda) \oplus S^{left}_{\mathbb{M}_2}(\lambda),$$

where $S^{right}_{\mathbb{M}_2}(\lambda)$ denotes the right singular part of $\mathbb{M}_2(\lambda)$, $S^{left}_{\mathbb{M}_2}(\lambda)$ denotes the left singular part, $J_{\mathbb{M}_2}(\lambda)$ is the block associated with the finite eigenvalues of the pencil and, finally, $N_{\mathbb{M}_2}(\lambda)$ is the block associated with the infinite eigenvalue. Having in mind the structure of the different blocks appearing in the KCF of the matrix pencil $\mathbb{M}_2(\lambda)$, we can derive that the rank of $\mathbb{K}(\lambda)$, and consequently of $\mathbb{M}_2(\lambda)$, is $N-1$ for all $\lambda \in \mathbb{C}\setminus\{0\}$ if and only if $\mathbb{K}(\lambda)$ has no right singular part and the only possibly finite eigenvalue is the zero one. In fact, we have the following result:

Lemma 4. *The rank of the matrix $\mathbb{M}_2(\lambda)$ is $N-1$ for each $\lambda \in \mathbb{C}\setminus\{0\}$ if and only if the following conditions hold:*

1. *The KCF of the matrix pencil $\mathbb{M}_2(\lambda)$ has no right singular part; and*
2. *If μ is a finite eigenvalue of the matrix pencil $\mathbb{M}_2(\lambda)$, then $\mu = 0$.*

Now, Lemma 4 allows us to decide when the rank of our initial polynomial matrix $\mathsf{G}(z)$ is r for all $z \in \mathbb{C}\setminus\{0\}$. Let us to remind all the given steps in reducing the initial polynomial matrix $\mathsf{G}(z)$. Namely:

$$\mathsf{G}(z) \rightsquigarrow \mathbb{G}(z) \rightsquigarrow \widehat{\mathbb{G}}(z) \rightsquigarrow \widetilde{\mathbb{G}}(z) \rightsquigarrow \begin{bmatrix} \mathbb{M}_1(z) & \mathcal{G}' \\ \mathbb{M}_2(z) & 0 \end{bmatrix},$$

where

1. $\mathbb{G}(z) = \mathsf{G}(z)\mathsf{U}(z)$.
2. $\widehat{\mathbb{G}}(z)\Omega_r = \mathbb{G}(z)$.
3. $\widetilde{\mathbb{G}}(z) = \widehat{\mathbb{G}}(z)Q(z) = [\mathbb{M}(z)\ \mathcal{G}]$, where $\mathcal{G} \in \mathbb{C}^{s\times(r-N+1)}$ and $Q(z) = \operatorname{diag}[1, z^{-1}, \ldots, z^{1-r}]$.
4. If $\operatorname{rank}\mathcal{G} = r-N+1$, there exists $R \in \mathbb{C}^{s\times s}$ invertible such that $R\,\widetilde{\mathbb{G}}(z) = \begin{bmatrix} \mathbb{M}_1(z) & \mathcal{G}' \\ \mathbb{M}_2(z) & 0 \end{bmatrix}$ where the matrix $\mathcal{G}' \in \mathbb{C}^{(r-N+1)\times(r-N+1)}$ is invertible.
5. The matrices $\mathbb{M}_i(z)$, $i = 1, 2$, can be expressed as $\mathbb{M}_i(\lambda) = M_{i1} - \lambda M_{i2}$ with $\lambda = z^r$.

As a consequence, we have proved the following result:

Theorem 2. *Assume that* $\operatorname{supp}\mathcal{L}\varphi \subseteq [0, N]$, *where $N \in \mathbb{N}$ with $N > 1$, and take $N \leqslant r < s$. Let $\mathsf{G}(z)$ be the corresponding $s \times r$ polynomial matrix given in (7). Then,* $\operatorname{rank}\mathsf{G}(z) = r$ *for any $z \in \mathbb{C}\setminus\{0\}$ if and only if the following statements hold:*

1. $\operatorname{rank}\mathcal{G} = r-N+1$; *and*
2. *the KCF of the matrix pencil $\mathbb{M}_2(\lambda)$ has no right singular part, and the only possible finite eigenvalue is $\mu = 0$.*

For practical purposes it is not necessary to compute the KCF of the matrix pencil $\mathbb{M}_2(\lambda)$ (if possible). The needed information about $\mathbb{M}_2(\lambda)$ can be retrieved from the GUPTRI (General UPper TRIangular) form of the matrix pencil. It is worth to mention that the GUPTRI form can be stably computed [6,7,18,20]. As the matrix $\widetilde{\mathbb{G}}(z)$ depends on z^r, in what follows we identify the matrix $\widetilde{\mathbb{G}}(z)$ with $\widetilde{\mathbb{G}}(\lambda)$ where $\lambda = z^r$.

2.1. A toy model involving the quadratic B-spline

The following example illustrates the result given in Theorem 2. Consider as generator φ the quadratic B-spline $N_3(t)$, i.e.,

$$N_3(t) = \frac{t^2}{2}\chi_{[0,1)}(t) + \left(-\frac{3}{2} + 3t - t^2\right)\chi_{[1,2)}(t) + \frac{1}{2}(3-t)^2\chi_{[2,3)}(t),$$

where $\chi_{[a,b)}$ denotes the characteristic function of the interval $[a, b)$. In this case, for the identity system, $\mathcal{L}f = f$ for all $f \in V_\varphi$, we have $\operatorname{supp} \mathcal{L}\varphi \subseteq [0, 3]$, i.e., $N = 3$. Taking the sampling period $T = 4/5$, i.e., $r = 4$ and $s = 5$, the Laurent polynomials $\mathrm{g}_i(z)$, $1 \le i \le 5$, given by (11) read:

$$\mathrm{g}_1(z) = \frac{1}{2}z + \frac{1}{2}z^2, \quad \mathrm{g}_2(z) = \frac{8}{25} + \frac{33}{50}z + \frac{1}{50}z^2,$$
$$\mathrm{g}_3(z) = \frac{9}{50}z^{-1} + \frac{37}{50} + \frac{2}{25}z, \quad \mathrm{g}_4(z) = \frac{2}{25}z^{-2} + \frac{37}{50}z^{-1} + \frac{9}{50},$$
$$\mathrm{g}_5(z) = \frac{1}{50}z^{-3} + \frac{33}{50}z^{-2} + \frac{8}{25}z^{-1}.$$

Following the above steps we obtain

$$\widehat{\mathbb{G}}(z) = \begin{bmatrix} \frac{1}{2}z^4 & \frac{1}{2}z^5 & 0 & 0 \\ \frac{33}{50}z^4 & \frac{1}{50}z^5 & 0 & \frac{8}{25}z^3 \\ \frac{2}{25}z^4 & 0 & \frac{9}{50}z^2 & \frac{37}{50}z^3 \\ 0 & \frac{2}{25}z & \frac{37}{50}z^2 & \frac{9}{50}z^3 \\ \frac{1}{50} & \frac{33}{50}z & \frac{8}{25}z^2 & 0 \end{bmatrix}.$$

Right multiplication by the matrix $\operatorname{diag}[1, z^{-1}, z^{-2}, z^{-3}]$ gives:

$$\widetilde{\mathbb{G}}(\lambda) = [\mathbb{M}(\lambda) \mid \mathcal{G}] = \left[\begin{array}{cc|cc} \frac{1}{2}\lambda & \frac{1}{2}\lambda & 0 & 0 \\ \frac{33}{50}\lambda & \frac{1}{50}\lambda & 0 & \frac{8}{25} \\ \frac{2}{25}\lambda & 0 & \frac{9}{50} & \frac{37}{50} \\ 0 & \frac{2}{25} & \frac{37}{50} & \frac{9}{50} \\ \frac{1}{50} & \frac{33}{50} & \frac{8}{25} & 0 \end{array}\right],$$

where $\lambda = z^4$. The matrix $\mathcal{G} \in \mathbb{C}^{5\times 2}$ has rank 2; performing some elementary operations on the rows of $\mathcal{G}$ we obtain

$$\mathcal{G}' = \begin{bmatrix} \frac{9}{50} & \frac{37}{50} \\ 0 & \frac{8}{25} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{37}{9} & -\frac{161}{18} & 0 & 1 & 0 \\ \frac{16}{9} & -\frac{37}{9} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{8}{25} \\ \frac{9}{50} & \frac{37}{50} \\ \frac{37}{50} & \frac{9}{50} \\ \frac{8}{25} & 0 \end{bmatrix} = R\mathcal{G}.$$

Therefore, $R\,\widetilde{\mathbb{G}}(\lambda) = [R\mathbb{M}(\lambda)\, R\mathcal{G}] = \begin{bmatrix} \mathbb{M}_1(\lambda) & \mathcal{G}' \\ \mathbb{M}_2(\lambda) & 0 \end{bmatrix}$ where

$$\mathbb{M}_2(\lambda) = \begin{bmatrix} \frac{1}{2}\lambda & \frac{1}{2}\lambda \\ \frac{5017}{900}\lambda & \frac{2}{25} + \frac{161}{900}\lambda \\ \frac{1}{50} + \frac{1157}{450}\lambda & \frac{33}{50} + \frac{37}{450}\lambda \end{bmatrix}.$$

In this case, a direct computation gives $\mathbb{K}_{\mathbb{M}_2}(\lambda) = L_2^\top(\lambda)$, where

$$L_2(\lambda) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

As a consequence, Theorem 2 ensures that the corresponding polynomial matrix $\mathsf{G}(z)$ possesses a polynomial left inverse.

Next we deal with the problem of computing a polynomial left inverse of $\mathsf{G}(z)$ in the case where it exists.

3. Computing a polynomial left inverse of the matrix $\mathsf{G}(z)$

First notice that if we compute a polynomial left inverse of the matrix $\widetilde{\mathbb{G}}(\lambda)$ then we obtain a polynomial left inverse of the matrix $\mathsf{G}(z)$. Indeed, remind that

$$\widetilde{\mathbb{G}}(z) = \mathsf{G}(z)\mathsf{U}(z)\Omega_r^{-1}Q(z),$$

where $\mathsf{U}(z) = \operatorname{diag}[z^{r-1}, (Wz)^{r-1}, \ldots, (W^{r-1}z)^{r-1}]$, Ω_r is the Fourier matrix of order r, and $Q(z) = \operatorname{diag}[1, z^{-1}, \ldots, z^{1-r}]$. Thus, if $L(z)$ is a polynomial left inverse of the matrix $\widetilde{\mathbb{G}}(z)$, then the matrix

$$L_{\mathsf{G}}(z) = \operatorname{diag}[z^{r-1}, (Wz)^{r-1}, \ldots, (W^{r-1}z)^{r-1}]\Omega_r^{-1} \operatorname{diag}[1, z^{-1}, \ldots, z^{1-r}]L(z)$$

will be a polynomial left inverse of the matrix $\mathsf{G}(z)$. As a consequence, we confine ourselves to the problem of computing a polynomial left inverse of the matrix $\widetilde{\mathbb{G}}(z)$. To this end, consider $\widetilde{\mathbb{G}}(\lambda) = \mathcal{A}^\top - \lambda\mathcal{B}^\top$ ($\lambda = z^r$); being $L(\lambda)$ a polynomial left inverse of the matrix $\widetilde{\mathbb{G}}(\lambda)$, we have $(\mathcal{A}-\lambda\mathcal{B})L^\top(\lambda) = \mathbf{I}_r$. Let us denote $\mathsf{L}(\lambda) := L^\top(\lambda)$. As we are searching for $s \times r$ matrices $\mathsf{L}(\lambda)$, whose entries are polynomials, such that $(\mathcal{A} - \lambda\mathcal{B})\mathsf{L}(\lambda) = \mathbf{I}_r$ we can use the following notation:

$$\begin{aligned} \mathsf{L}(\lambda) &= [\mathsf{L}_1(\lambda)\, \mathsf{L}_2(\lambda)\, \ldots\, \mathsf{L}_r(\lambda)], \quad \text{i.e., } \mathsf{L}_i(\lambda) \text{ denotes the } i\text{th column of } \mathsf{L}(\lambda), \\ \mathsf{L}_i(\lambda) &= \boldsymbol{\ell}_i^0 + \boldsymbol{\ell}_i^1\lambda + \cdots + \boldsymbol{\ell}_i^\nu\lambda^\nu, \quad i = 1, 2, \ldots, r, \text{ where } \boldsymbol{\ell}_i^k \in \mathbb{C}^s, \ k = 0, 1, \ldots, \nu. \end{aligned}$$

As a consequence, equation $(\mathcal{A} - \lambda\mathcal{B})\mathsf{L}(\lambda) = \mathbf{I}_r$ is equivalent to

$$\mathcal{A}\boldsymbol{\ell}_i^0 + (\mathcal{A}\boldsymbol{\ell}_i^1 - \mathcal{B}\boldsymbol{\ell}_i^0)\lambda + \cdots + (\mathcal{A}\boldsymbol{\ell}_i^\nu - \mathcal{B}\boldsymbol{\ell}_i^{\nu-1})\lambda^\nu - \mathcal{B}\boldsymbol{\ell}_i^\nu\lambda^{\nu+1} = I_r^i, \quad i = 1, 2, \ldots, r, \tag{13}$$

where I_r^i denotes the ith column of the identity matrix $\mathbf{I}_r$. Equating coefficients, for each $i = 1, 2, \ldots, r$, we obtain the set of linear equations

$$\mathcal{A}\boldsymbol{\ell}_i^0 = I_r^i, \quad \mathcal{A}\boldsymbol{\ell}_i^1 - \mathcal{B}\boldsymbol{\ell}_i^0 = 0, \ldots, \quad \mathcal{A}\boldsymbol{\ell}_i^\nu - \mathcal{B}\boldsymbol{\ell}_i^{\nu-1} = 0, \quad -\mathcal{B}\boldsymbol{\ell}_i^\nu = 0,$$

or in matrix form

$$\begin{bmatrix} -\mathcal{B} & & & & \\ \mathcal{A} & -\mathcal{B} & & & \\ & \mathcal{A} & -\mathcal{B} & & \\ & & \ddots & & \\ & & & \mathcal{A} & -\mathcal{B} \\ & & & & \mathcal{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{\ell}_i^\nu \\ \boldsymbol{\ell}_i^{\nu-1} \\ \vdots \\ \boldsymbol{\ell}_i^0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ I_r^i \end{bmatrix}, \quad i = 1, 2, \ldots, r, \tag{14}$$

where the resulting block matrix has order $(\nu+2)r \times (\nu+1)s$. The goal is to find $\nu \in \mathbb{N}$ such that the above r linear systems become consistent. Next, we come back to the example in Section 2.1.

The example revisited: Consider again the example involving the quadratic B-spline given in Section 2.1. In this case, $\widetilde{\mathbb{G}}(z) = \mathsf{G}(z)\mathsf{U}(z)\Omega_4^{-1}\,\mathrm{diag}[1, z^{-1}, z^{-2}, z^{-3}]$ and, taking $\lambda = z^4$ we have

$$\widetilde{\mathbb{G}}(\lambda) = \begin{bmatrix} \frac{1}{2}\lambda & \frac{1}{2}\lambda & 0 & 0 \\ \frac{33}{50}\lambda & \frac{1}{50}\lambda & 0 & \frac{8}{25} \\ \frac{2}{25}\lambda & 0 & \frac{9}{50} & \frac{37}{50} \\ 0 & \frac{2}{25} & \frac{37}{50} & \frac{9}{50} \\ \frac{1}{50} & \frac{33}{50} & \frac{8}{25} & 0 \end{bmatrix} = \mathcal{A}^\top - \lambda\mathcal{B}^\top,$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{50} \\ 0 & 0 & 0 & \frac{2}{25} & \frac{33}{50} \\ 0 & 0 & \frac{9}{50} & \frac{37}{50} & \frac{8}{25} \\ 0 & \frac{8}{25} & \frac{37}{50} & \frac{9}{50} & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} -\frac{1}{2} & -\frac{33}{50} & -\frac{2}{25} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{50} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, the matrix $S = \begin{bmatrix} -\mathcal{B} & & \\ \mathcal{A} & -\mathcal{B} & \\ & \mathcal{A} & \end{bmatrix}$ of size 12×10 has rank 10. Choosing the columns of $\mathsf{L}(\lambda)$ as $\mathsf{L}_i(\lambda) = \boldsymbol{\ell}_i^0 + \boldsymbol{\ell}_i^1\lambda \in \mathbb{C}^{5\times 1}$, the linear systems

$$\begin{bmatrix} -\mathcal{B} & \\ \mathcal{A} & -\mathcal{B} \\ & \mathcal{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{\ell}_i^1 \\ \boldsymbol{\ell}_i^0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ I_4^i \end{bmatrix}, \quad i = 1, 2, 3, 4 \tag{15}$$

have a unique solution. Observe that deleting the trivial equations 3 and 4, we have consistent square systems. By using Matlab we obtain the left inverse

$$\mathsf{L}(\lambda) = 10^3 \begin{bmatrix} 4.4812 & -0.1438 & 0.0166 & -0.0043 \\ -3.4840 & 0.1118 & -0.0128 & 0.0031 \\ 1.6069 & -0.0514 & 0.0056 & 0.0000 \\ -0.4125 & 0.0125 & 0.0000 & -0.0000 \\ 0.0500 & 0.0000 & -0.0000 & 0.0000 \end{bmatrix} + 10^3 \begin{bmatrix} -0.0021 & 0.0001 & -0.0000 & 0.0000 \\ 0.0517 & -0.0017 & 0.0002 & -0.0000 \\ -0.4133 & 0.0133 & -0.0015 & 0.0004 \\ 1.6071 & -0.0516 & 0.0059 & -0.0015 \\ -3.4841 & 0.1118 & -0.0129 & 0.0033 \end{bmatrix} \lambda.$$

At this point, the challenge problem is to give conditions on the matrix pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$ in order to obtain a left inverse with polynomial entries (having nonnegative powers) by solving the corresponding linear systems (15). The answer to this question is based on the KCF of the matrix pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$. In our example the corresponding KCF is $N_1(\lambda)\oplus N_1(\lambda)\oplus L_2^\top(\lambda)$, i.e., the pencil has not finite eigenvalues, all the blocks associated with the infinite eigenvalue have order 1, and the left singular part has a unique block. In what follows, we prove that these conditions for the KCF of the matrix pencil $\widetilde{\mathbb{G}}(\lambda)$ are sufficient to give a positive answer to the raised problem in a very important particular case:

3.1. The case where the oversampling rate is minimum for a fixed $r \geqslant N$

It corresponds to the case where $N \leqslant r$ and $s = r+1$, i.e., the sampling period is $T = r/(r+1)$. Here, the matrix pencil $\widetilde{\mathbb{G}}(\lambda) = \mathcal{A}^\top - \lambda\mathcal{B}^\top$ has the form

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & * & * \\ \vdots & & \vdots & \vdots & \iddots & \iddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & * & \cdots & * & * \\ 0 & \cdots & 0 & * & * & \cdots & * & * \\ 0 & \cdots & * & * & * & \cdots & * & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \iddots & & \vdots \\ * & \cdots & * & * & * & \cdots & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} * & \cdots & * & * & 0 & \cdots & 0 \\ * & \cdots & * & * & 0 & \cdots & 0 \\ * & \ldots & * & 0 & 0 & \cdots & 0 \\ \vdots & \iddots & \iddots & \vdots & \vdots & & \vdots \\ * & \iddots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{16}$$

i.e., denoting the entries of $\mathcal{A}^\top$ and $\mathcal{B}^\top$ by $\mathcal{A}_{ij}^\top$ and $\mathcal{B}_{ij}^\top$ respectively, we have $\mathcal{A}_{ij}^\top = 0$ if $i+j < 2+r$ or $i+j > r+N+1$, $\mathcal{B}_{1N}^\top = 0$ and $\mathcal{B}_{ij}^\top = 0$ if $i+j > N+1$. Having in mind the structure of the matrices $\mathcal{A}^\top$ and $\mathcal{B}^\top$ we have $\operatorname{rank}(\mathcal{A}^\top) \leqslant r$, $\operatorname{rank}(\mathcal{B}^\top) \leqslant N-1$ and $\operatorname{rank}\left(\begin{bmatrix} -\mathcal{B} & \\ \mathcal{A} & -\mathcal{B} \end{bmatrix}\right) \leqslant r+N-1$. Whenever these matrices have maximum rank, the following result holds:

Theorem 3. *Assume that the singular matrix pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$ of size $(r+1)\times r$ satisfies the following conditions:*

1. *The pencil has no finite eigenvalues,*
2. $\operatorname{rank}(\mathcal{A}^\top) = r$,
3. $\operatorname{rank}(\mathcal{B}^\top) = N-1$, *with $N \leqslant r$, and*
4. $\operatorname{rank}\left(\begin{bmatrix} -\mathcal{B} & \\ \mathcal{A} & -\mathcal{B} \end{bmatrix}\right) = r+N-1$.

Then, the $Nr \times (N-1)(r+1)$ matrix

$$\mathsf{G}_r := \begin{bmatrix} -\mathcal{B} & & & & \\ \mathcal{A} & -\mathcal{B} & & & \\ & \mathcal{A} & -\mathcal{B} & & \\ & & & \ddots & \\ & & & \mathcal{A} & -\mathcal{B} \\ & & & & \mathcal{A} \end{bmatrix}$$

has rank $(N-1)(r+1)$.

First note that $\operatorname{rank}(\mathcal{A}^\top) = r$ implies that the KCF of the matrix pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$ has not right singular part (and also that 0 is not an eigenvalue). Thus, by using Theorem 2, the pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$ has a polynomial left inverse. Before to prove Theorem 3, and in order to ease its proof, we first obtain, under the theorem hypotheses, the KCF of the matrix pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$:

Lemma 5. *The KCF of the matrix pencil* $\mathcal{A}^\top - \lambda\mathcal{B}^\top$ *is* $\left(\bigoplus_{i=1}^{r-N+1} N_1(\lambda)\right) \oplus L_{N-1}^\top(\lambda)$.

Proof of Lemma 5. Since the matrix pencil has neither finite eigenvalues nor a right singular part, we conclude that its KCF has the form $N(\lambda) \oplus L^{left}(\lambda)$, where $N(\lambda)$ denotes the blocks associated with the infinite eigenvalue and $L^{left}(\lambda)$ denotes the left singular part. Since $r+1$ is the number of rows of the matrix pencil, r the number of columns, and the rank of $\mathcal{B}$ is $N-1$ it cannot appear blocks of the form $L_i^\top(\lambda)$ for $i \geqslant N$. Each left singular block increases in one the number of rows with respect to the number of columns; hence, as the size of $\mathcal{A}^\top - \lambda\mathcal{B}^\top$ is $(r+1)\times r$, it can appear only one left singular block in its KCF. Furthermore, we prove that this only left singular block corresponds to $L_{N-1}^\top(\lambda)$. Indeed, let $\mathcal{K}_\mathcal{A}^\top - \lambda\mathcal{K}_\mathcal{B}^\top$ be the KCF of the matrix pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$. Obviously, we have that $\operatorname{rank}(\mathcal{A}^\top) = \operatorname{rank}(\mathcal{K}_\mathcal{A}^\top) = r$, $\operatorname{rank}(\mathcal{B}^\top) = \operatorname{rank}(\mathcal{K}_\mathcal{B}^\top) = N-1$ and

$$\operatorname{rank}\begin{bmatrix} -\mathcal{B} & \\ \mathcal{A} & -\mathcal{B} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -\mathcal{K}_\mathcal{B} & \\ \mathcal{K}_\mathcal{A} & -\mathcal{K}_\mathcal{B} \end{bmatrix} = r + N - 1.$$

The rank of the matrix $\begin{bmatrix} -\mathcal{K}_\mathcal{B} \\ \mathcal{K}_\mathcal{A} & -\mathcal{K}_\mathcal{B} \end{bmatrix}$ coincides with its number of nonzero rows because the number of null rows of $\mathcal{K}_\mathcal{B}$ is $r-N+1$, i.e., the number of blocks in $N(\lambda)$; the matrix $\mathcal{K}_\mathcal{A}$ has not null rows so that, the number of nonzero rows of $\begin{bmatrix} -\mathcal{K}_\mathcal{B} \\ \mathcal{K}_\mathcal{A} & -\mathcal{K}_\mathcal{B} \end{bmatrix}$ is $2r-(r-N+1) = r+N-1$.

Assume that in the KCF of the matrix pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$ appears a singular block $L_i^\top(\lambda)$ with $i < N-1$. Since the rank of $\mathcal{B}^\top$ is $N-1$, the regular part in the KCF has a block of the form $N_l(\lambda)$ with $l \geqslant 2$. By rearranging the blocks, we obtain that the KCF of $\mathcal{A}^\top - \lambda\mathcal{B}^\top$ is $N_l(\lambda) \oplus \cdots \oplus L_i^\top(\lambda)$; therefore

$$\begin{bmatrix} -\mathcal{K}_\mathcal{B} & \\ \mathcal{K}_\mathcal{A} & -\mathcal{K}_\mathcal{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & * & \cdots & * & * & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & \cdots & * & * & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ * & * & \cdots & * & * & * & * & \cdots & * & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & \cdots & * & * & * & * & \cdots & * & * \end{bmatrix}.$$

In this case, the rank of $\begin{bmatrix} -\mathcal{K}_\mathcal{B} \\ \mathcal{K}_\mathcal{A} & -\mathcal{K}_\mathcal{B} \end{bmatrix}$ is strictly smaller than $r+N-1$ because the second row and the $(r+1)$th row are linearly dependent. This contradicts the hypotheses and, hence, the only left singular block is $L_{N-1}^\top(\lambda)$. Having in mind that $\operatorname{rank}(\mathcal{B}^\top) = N-1$, we conclude that the KCF of the matrix pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$ is $\left(\bigoplus_{i=1}^{r-N+1} N_1(\lambda)\right) \oplus L_{N-1}^\top(\lambda)$. □

Proof of Theorem 3. Once we have determined the KCF of the matrix pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$ we compute the rank of the matrix G_r. If $\mathcal{K}_\mathcal{A} - \lambda\mathcal{K}_\mathcal{B}$ is the KCF of the matrix pencil $\mathcal{A} - \lambda\mathcal{B}$, it is obvious that

$$\operatorname{rank}(\mathsf{G}_r) = \operatorname{rank}\begin{bmatrix} -\mathcal{K}_\mathcal{B} & & & & & \\ \mathcal{K}_\mathcal{A} & -\mathcal{K}_\mathcal{B} & & & & \\ & \mathcal{K}_\mathcal{A} & -\mathcal{K}_\mathcal{B} & & & \\ & & & \ddots & & \\ & & & & \mathcal{K}_\mathcal{A} & -\mathcal{K}_\mathcal{B} \\ & & & & & \mathcal{K}_\mathcal{A} \end{bmatrix}.$$

As $\mathcal{K}_\mathcal{A}^\top - \lambda\mathcal{K}_\mathcal{B}^\top$ is the KCF of the matrix pencil $\mathcal{A}^\top - \lambda\mathcal{B}^\top$, Lemma 5 gives

$$\mathcal{K}_\mathcal{A}^\top = \begin{bmatrix} I & 0 \\ 0 & L_\mathcal{A}^\top \end{bmatrix}, \quad \mathcal{K}_\mathcal{B}^\top = \begin{bmatrix} 0 & 0 \\ 0 & L_\mathcal{B}^\top \end{bmatrix},$$

where $I = \mathbf{I}_{(r-N+1)}$ denotes the identity matrix of order $r - N + 1$, and

$$L_\mathcal{A}^\top = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{C}^{N\times(N-1)}; \qquad L_\mathcal{B}^\top = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{N\times(N-1)}.$$

As a consequence,

$$\operatorname{rank}(\mathsf{G}_r) = \operatorname{rank}\begin{bmatrix} 0 & 0 & & & & & & & & \\ 0 & -L_\mathcal{B} & & & & & & & & \\ I & 0 & 0 & 0 & & & & & & \\ 0 & L_\mathcal{A} & 0 & -L_\mathcal{B} & & & & & & \\ & & I & 0 & 0 & 0 & & & & \\ & & 0 & L_\mathcal{A} & 0 & -L_\mathcal{B} & & & & \\ & & & & \ddots & & \ddots & & & \\ & & & & & & I & 0 & 0 & 0 \\ & & & & & & 0 & L_\mathcal{A} & 0 & -L_\mathcal{B} \\ & & & & & & & & I & 0 \\ & & & & & & & & 0 & L_\mathcal{A} \end{bmatrix}.$$

A suitable interchange of rows and columns gives

$$\operatorname{rank}(\mathsf{G}_r) = \operatorname{rank}\begin{bmatrix} 0 & \cdots & 0 & & & & 0 & 0 \\ I & & & & & & & \\ & \ddots & & & & & & \\ & & I & & & & & \\ & & & -L_{\mathcal{B}} & 0 & & & \\ & & & L_{\mathcal{A}} & -L_{\mathcal{B}} & & & \\ & & & & & \ddots & & \\ & & & & & & L_{\mathcal{A}} & -L_{\mathcal{B}} \\ & & & & & & 0 & L_{\mathcal{A}} \end{bmatrix},$$

where the first $r - N + 1 = r - \operatorname{rank}(\mathcal{K}_{\mathcal{B}})$ are null rows; hence, the rank of G_r equals $(N-1)(r+1)$ if and only if the remaining $(N-1)(r+1)$ rows are linearly independent. This is equivalent to the matrix

$$\mathcal{L}_{\mathcal{A},\mathcal{B}} = \begin{bmatrix} -L_{\mathcal{B}} & 0 & & \\ L_{\mathcal{A}} & -L_{\mathcal{B}} & & \\ & \ddots & & \\ & & L_{\mathcal{A}} & -L_{\mathcal{B}} \\ & & 0 & L_{\mathcal{A}} \end{bmatrix} \in \mathbb{C}^{N(N-1)\times N(N-1)}$$

has full rank. To prove it, we use the following result in [9, p. 32]: Let $\mathbf{x}(\lambda)$ be a nonzero vector having the form

$$\mathbf{x}(\lambda) = \mathbf{x}_0 + \lambda\mathbf{x}_1 + \lambda^2\mathbf{x}_2 + \cdots + \lambda^{\varepsilon}\mathbf{x}_{\varepsilon}, \;\; \mathbf{x}_i \in \mathbb{C}^{N\times 1}$$

such that $(L_{\mathcal{A}} - \lambda L_{\mathcal{B}})\mathbf{x}(\lambda) = \mathbf{0}$. Then, necessarily, $\varepsilon \geqslant N-1$. Now, let us continue by contradiction, and assume that the matrix $\mathcal{L}_{\mathcal{A},\mathcal{B}}$ has not full rank. Then, there exists a nonzero vector $\mathbf{z} \in \mathbb{C}^{N(N-1)\times 1}$ such that $\mathcal{L}_{\mathcal{A},\mathcal{B}}\,\mathbf{z} = \mathbf{0}$. Denoting $\mathbf{z}^{\top} = [\mathbf{z}_{N-2}^{\top} \;\ldots\; \mathbf{z}_1^{\top}\; \mathbf{z}_0^{\top}]$ where $\mathbf{z}_i \in \mathbb{C}^{N\times 1}$, we obtain that

$$(L_{\mathcal{A}} - \lambda L_{\mathcal{B}})(\mathbf{z}_0 + \lambda\mathbf{z}_1 + \lambda\mathbf{z}_2 + \cdots + \lambda^{N-2}\mathbf{z}_{N-2}) = \mathbf{0},$$

which contradicts the minimal property for $N-1$. Therefore, the matrix $\mathcal{L}_{\mathcal{A},\mathcal{B}}$ has full rank and, finally, $\operatorname{rank}\mathsf{G}_r = (N-1)(r+1)$. □

Remark. Notice that Theorem 3 remains valid for any singular matrix pencil $\mathcal{A}^{\top} - \lambda\mathcal{B}^{\top}$ of size $(r+1)\times r$ substituting $N-1$ by $p \in \mathbb{N}$ which satisfies $0 < p < r$.

Consider the matrix pencil $\widetilde{\mathbb{G}}(\lambda) = \mathcal{A}^{\top} - \lambda\mathcal{B}^{\top}$ of size $(r+1)\times r$ with $N \leqslant r$. Assuming that the $\widetilde{\mathbb{G}}(\lambda)$ has polynomial left inverses, the following result gives sufficient conditions for computing one of such polynomial left inverses. Once we have got one solution, it is straightforward to derive the remaining solutions.

Corollary 1 (Computing a polynomial left inverse of $\widetilde{\mathbb{G}}(\lambda)$)**.** *Let $\widetilde{\mathbb{G}}(\lambda) = \mathcal{A}^{\top} - \lambda\mathcal{B}^{\top}$ be a singular matrix pencil of size $(r+1)\times r$ with $N \leqslant r$. Assume that $\widetilde{\mathbb{G}}(\lambda)$ admits polynomial left inverses, and that the following conditions hold:*

1. $\operatorname{rank}(\mathcal{A}^\top) = r$,
2. $\operatorname{rank}(\mathcal{B}^\top) = N-1$, *and*
3. $\operatorname{rank}\left(\begin{bmatrix} -\mathcal{B} & \\ \mathcal{A} & -\mathcal{B} \end{bmatrix}\right) = r+N-1$.

Consider the $Nr \times (N-1)(r+1)$ *matrix* G_r *in Theorem 3. Then, the linear systems*

$$\mathsf{G}_r \begin{bmatrix} \boldsymbol{\ell}_i^{N-2} \\ \boldsymbol{\ell}_i^{N-3} \\ \vdots \\ \boldsymbol{\ell}_i^{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ I_r^i \end{bmatrix}, \quad i = 1, 2, \ldots, r, \tag{17}$$

where I_r^i *denotes the ith column of the identity matrix* $\boldsymbol{I}_r$*, admit a unique solution. Moreover, let* $[\boldsymbol{\ell}_i^{N-2}\ \boldsymbol{\ell}_i^{N-3}\ \ldots\ \boldsymbol{\ell}_i^{0}]^\top \in \mathbb{C}^{(N-1)(r+1)}$ *be this solution for* $i = 1, 2, \ldots, r$*, and consider the polynomial vector* $\mathsf{L}_i(\lambda) = \boldsymbol{\ell}_i^0 + \boldsymbol{\ell}_i^1\lambda + \cdots + \boldsymbol{\ell}_i^{N-2}\lambda^{N-2}$*,* $i = 1, 2, \ldots, r$*. Then, the* $(r+1) \times r$ *polynomial matrix*

$$\mathsf{L}(\lambda) := [\mathsf{L}_1(\lambda)\, \mathsf{L}_2(\lambda)\ \ldots\ \mathsf{L}_r(\lambda)]$$

satisfies

$$\mathsf{L}^\top(\lambda)\widetilde{\mathbb{G}}(\lambda) = \boldsymbol{I}_r.$$

Proof. Theorem 3 implies that the rank of the coefficient matrix $\mathsf{G}_r \in \mathbb{C}^{Nr\times(N-1)(r+1)}$ is $(N-1)(r+1)$ in (17). Having in mind (16), the last $r-N+1$ rows of $\mathcal{B}$ are null. Deleting these rows in the first row block (which become trivial equations in (17)), we obtain an square invertible matrix, and consequently (17) has a unique solution for each $i = 1, 2, \ldots, r$. Recalling (14), we finally obtain that $\mathsf{L}^\top(\lambda)$ is a polynomial left inverse of $\widetilde{\mathbb{G}}(\lambda)$. □

Observe that any other polynomial left inverse $A(\lambda)$ of the matrix $\widetilde{\mathbb{G}}(\lambda)$ is given by

$$A(\lambda) = \mathsf{L}^\top(\lambda) + B(\lambda)\left[\mathbf{I}_{r+1} - \widetilde{\mathbb{G}}(\lambda)\mathsf{L}^\top(\lambda)\right],$$

where $B(\lambda)$ is an arbitrary $r \times (r+1)$ polynomial matrix.

For the matrix pencil $\widetilde{\mathbb{G}}(\lambda) = \mathcal{A}^\top - \lambda\mathcal{B}^\top$ of size $(r+1) \times r$ with $N \leqslant r$, it is easy to give sufficient conditions in order to satisfy the conditions 1–3 in Corollary 1. Namely:

Corollary 2. *Consider the singular matrix pencil* $\widetilde{\mathbb{G}}(\lambda) = \mathcal{A}^\top - \lambda\mathcal{B}^\top$ *of size* $(r+1) \times r$ *with* $N \leqslant r$*. Denoting* $\mathcal{A}^\top = [\mathcal{A}_{ij}^\top]$ *and* $\mathcal{B}^\top = [\mathcal{B}_{ij}^\top]$*, assume that the following conditions hold:*

$$\mathcal{A}_{ij}^\top \neq 0 \ \text{if } i+j = r+2 \text{ or } i+j = r+N+1, \tag{18}$$

$$\mathcal{B}_{ij}^\top \neq 0 \ \text{if } i+j = N+1 \text{ and } i \geqslant 2. \tag{19}$$

Then the conditions 1–3 in Corollary 1 are satisfied.

Proof. Conditions (18) and (19) say that the entries marked as $\bullet$ in the matrices below are nonzero

$$\mathcal{A} = \left[\begin{array}{cccccc|ccc} 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & \cdots & \bullet \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & \cdots & * \\ \vdots & & & \vdots & \vdots & & \vdots & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \bullet & * & \cdots & * \\ 0 & \cdots & \cdots & 0 & \bullet & * & * & \cdots & * \\ \hline 0 & \cdots & 0 & \bullet & * & * & * & \cdots & * \\ 0 & \cdots & \bullet & * & * & * & * & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots & \ddots & \vdots \\ 0 & \bullet & \cdots & * & * & * & \bullet & \cdots & 0 \end{array}\right] = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] \in \mathbb{C}^{r\times(r+1)},$$

$$\mathcal{B} = \left[\begin{array}{ccccc|ccc} * & * & \cdots & * & \bullet & 0 & \cdots & 0 \\ * & * & \cdots & \bullet & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ * & \bullet & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] = \left[\begin{array}{cc} B_{11} & 0 \\ 0 & 0 \end{array}\right] \in \mathbb{C}^{r\times(r+1)},$$

where $A_{22} \in \mathbb{C}^{(r-N+1)\times(r-N+1)}$ and $B_{11} \in \mathbb{C}^{(N-1)\times N}$. Trivially, $\operatorname{rank}(\mathcal{A}^\top) = r$ and $\operatorname{rank}(\mathcal{B}^\top) = N-1$. Condition 3 comes by observing the form of the matrix $\left[\begin{smallmatrix} -\mathcal{B} & \\ \mathcal{A} & -\mathcal{B} \end{smallmatrix}\right]$. Interchanging rows and columns we obtain that the matrix $\left[\begin{smallmatrix} -\mathcal{B} & \\ \mathcal{A} & -\mathcal{B} \end{smallmatrix}\right]$ has the same rank than the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ B_{11} & 0 & 0 & 0 \\ A_{11} & B_{11} & A_{12} & 0 \\ A_{21} & 0 & A_{22} & 0 \end{bmatrix}.$$

Since the matrix $A_{22} \in \mathbb{C}^{(r-N+1)\times(r-N+1)}$ is invertible, elementary row operations give the new matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ B_{11} & 0 & 0 & 0 \\ \widetilde{A}_{11} & B_{11} & 0 & 0 \\ A_{21} & 0 & A_{22} & 0 \end{bmatrix}.$$

Finally, the above matrix has rank $2(N-1) + r - N + 1 = r + N - 1$. □

Remark that this condition can be checked by using the algorithm guptri. In case that conditions 1–3 in Corollary 1 are satisfied, we could check directly the consistency of the linear systems (17); if they are not consistent, we derive that the pencil $\widetilde{\mathbb{G}}(\lambda)$ has not polynomial left inverses.

In Corollary 2 we allow the first column of $\mathcal{B}$ to be a zero column. Nevertheless, for the condition 1 in Theorem 3 to be satisfied, the first column of $\mathcal{B}$ should have at least one nonzero element.

Corollary 1 provides a method to obtain an algebraic matrix polynomial of degree $\nu = N-2$ which is a left inverse of $\widetilde{\mathbb{G}}(\lambda)$. In the next section we prove that all terms in this polynomial matrix are nonzero. The number of nonzero terms in a left inverse of $\widetilde{\mathbb{G}}(\lambda)$ and the support of the reconstruction functions are intimately related (see (5)): More zero terms implies a smaller support. Below we prove that the mentioned solution is optimal in the sense that every solution of the problem has, at least, $N-1$ nonzero terms.

3.2. Optimality of the solution

In the previous section we have found an algebraic polynomial matrix, $\mathsf{L}(\lambda) \in \mathbb{C}^{(r+1)\times r}$, which is a left inverse of $\widetilde{\mathbb{G}}(\lambda)$. This algebraic polynomial matrix can be written as:

$$\mathsf{L}(\lambda) = \mathcal{L}_0 + \mathcal{L}_1\lambda + \cdots + \mathcal{L}_{N-2}\lambda^{N-2},$$

where $\mathcal{L}_i = [\boldsymbol{\ell}_1^i \ \ldots \ \boldsymbol{\ell}_r^i] \in \mathbb{C}^{(r+1)\times r}$ and $\boldsymbol{\ell}_j^i \in \mathbb{C}^{(r+1)\times 1}$. Hence, each column of $\mathsf{L}(\lambda)$, $\mathsf{L}_j(\lambda)$, can be written as $\mathsf{L}_j(\lambda) = \boldsymbol{\ell}_j^0 + \boldsymbol{\ell}_j^1\lambda + \cdots + \boldsymbol{\ell}_j^{N-2}\lambda^{N-2}$.

The optimality problem involves finding a left inverse polynomial matrix of $\widetilde{\mathbb{G}}(\lambda)$ with the minimum number of nonzero terms. Let $p \in \mathbb{Z}$ denotes the smallest power of λ in the polynomial matrix $\mathsf{L}(\lambda)$ we are looking for. If $\mathsf{L}(\lambda) = \mathcal{L}_p\lambda^p + \mathcal{L}_{p+1}\lambda^{p+1} + \cdots + \mathcal{L}_{p+\nu}\lambda^{p+\nu}$, then each column of $\mathsf{L}(\lambda)$ can be written as:

$$\mathsf{L}_j(\lambda) = \boldsymbol{\ell}_j^p\lambda^p + \boldsymbol{\ell}_j^{p+1}\lambda^{p+1} + \cdots + \boldsymbol{\ell}_j^{p+\nu}\lambda^{p+\nu} = \sum_{k=0}^{\nu} \boldsymbol{\ell}_j^{p+k}\lambda^{p+k}.$$

As a consequence, equation $(\mathcal{A}-\lambda\mathcal{B})\mathsf{L}(\lambda) = \mathbf{I}_r$ is equivalent to

$$\mathcal{A}\boldsymbol{\ell}_j^p\lambda^p + \sum_{k=1}^{\nu}\left(\mathcal{A}\boldsymbol{\ell}_j^{p+k} - \mathcal{B}\boldsymbol{\ell}_j^{p+k-1}\right)\lambda^{p+k} - \mathcal{B}\boldsymbol{\ell}_j^{p+\nu}\lambda^{p+\nu+1} = I_r^j, \tag{20}$$

for $j = 1, 2, \ldots, r$, where I_r^j denotes the j-th column of $\mathbf{I}_r$, the identity matrix of order r.

Notice that the left-hand side of (20) should have a constant term because the right one is a constant. As a consequence, $0 \le -p \le \nu+1$. Moreover, since the last $r-N+1$ rows of $\mathcal{B}$ are null, if $-p = \nu+1$, the equation $\mathcal{B}\boldsymbol{\ell}_j^{p+\nu+1} = I_r^j$ has no solution for $j = N, N+1, \ldots, r-1, r$; consequently, $0 \le -p \le \nu$. Therefore, (20) is equivalent to the recursive scheme:

$$\left.\begin{array}{rcll} \mathcal{A}\boldsymbol{\ell}_j^p & = & & \mathcal{I}_p^j \\ \mathcal{A}\boldsymbol{\ell}_j^{p+1} & = & \mathcal{B}\boldsymbol{\ell}_j^p & +\mathcal{I}_{p+1}^j \\ \vdots & & \vdots & \vdots \\ \mathcal{A}\boldsymbol{\ell}_j^{p+k} & = & \mathcal{B}\boldsymbol{\ell}_j^{p+k-1} & +\mathcal{I}_{p+k}^j \\ \vdots & & \vdots & \vdots \end{array}\right\}, \tag{21}$$

until $k = \nu$, together with the equation:

$$\mathcal{B}\boldsymbol{\ell}_j^{p+\nu} = 0, \tag{22}$$

where $\mathcal{I}_{p+k}^j = \delta_{-p,k}I_r^j$ for $k \ge 0$ and $\delta_{-p,k}$ is the Kronecker delta.

In what follows, we assume that $\widetilde{\mathbb{G}}(\lambda)$ admits a polynomial left inverse and that the matrices $\mathcal{A}$ and $\mathcal{B}$ verifies the hypotheses in Corollary 2. Recall that under these hypotheses, in Section 3 a solution of (20) has been obtained for $p = 0$ and $\nu = N - 2$.

Next step is to prove that whenever $0 \leq -p \leq \nu < N - 2$ the system (21) and (22) has not a solution. Consequently, the matrix $\widetilde{\mathbb{G}}(\lambda)$ does not admit a polynomial left inverse.

3.2.1. Case $\nu < N - 2$

Since we are assuming the hypotheses in Corollary 2 we can write $\mathcal{A} = [\mathbf{0} \;\; A]$ and $\mathcal{B} = [b \;\; B]$, where $\mathbf{0}$, $b \in \mathbb{C}^{r\times 1}$, A, $B \in \mathbb{C}^{r\times r}$ and A is regular. The next result gives us the structure of the sequence $\{\boldsymbol{\ell}_j^{p+k}\}_{k=0}^{\infty}$ which solves the recursive scheme (21).

Theorem 4. *The solutions of* (21) *are of the form:*

$$\boldsymbol{\ell}_j^{p+k} = \begin{bmatrix} \boldsymbol{\ell}_{j,1}^{p+k} \\ \boldsymbol{\ell}_{j,2}^{p+k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_{j,1}^{p+k} \\ M^{[k]}\left[\boldsymbol{\ell}_{j,1}^{p+k-1}, \boldsymbol{\ell}_{j,1}^{p+k-2}, \ldots, \boldsymbol{\ell}_{j,1}^{p}\right]^{\top} + \sum_{i=0}^{k} (A^{-1}B)^i A^{-1} \mathcal{I}_{p+k-i}^{j} \end{bmatrix}, \tag{23}$$

where $\boldsymbol{\ell}_{j,1}^{p+k} \in \mathbb{C}$, $\boldsymbol{\ell}_{j,2}^{p+k} \in \mathbb{C}^{r\times 1}$, $M^{[0]} = 0$ and $M^{[k]} \in \mathbb{C}^{r\times k}$ for $k \in \mathbb{N} \cup \{0\}$. Moreover, the matrices $M^{[k]} \in \mathbb{C}^{r\times k}$ does not depend on $j \in \{1, 2 \ldots, r\}$.

Proof. We proceed by induction on k. For $k = 0$, we have to solve $\mathcal{A}\boldsymbol{\ell}_j^{p} = \mathcal{I}_p^j$. Therefore, $[\mathbf{0} \;\; A] \begin{bmatrix} \boldsymbol{\ell}_{j,1}^{p} \\ \boldsymbol{\ell}_{j,2}^{p} \end{bmatrix} = A\boldsymbol{\ell}_{j,2}^{p}$, and consequently $\boldsymbol{\ell}_{j,2}^{p} = M^{[0]} + A^{-1}\mathcal{I}_p^j$.

Suppose that (23) holds for $k \in \mathbb{N} \cup \{0\}$ and consider the equation $\mathcal{A}\boldsymbol{\ell}_j^{p+k+1} = \mathcal{B}\boldsymbol{\ell}_j^{p+k} + \mathcal{I}_{p+k+1}^{j}$, or equivalently,

$$[\mathbf{0} \;\; A] \begin{bmatrix} \boldsymbol{\ell}_{j,1}^{p+k+1} \\ \boldsymbol{\ell}_{j,2}^{p+k+1} \end{bmatrix} = [b \;\; B] \begin{bmatrix} \boldsymbol{\ell}_{j,1}^{p+k} \\ \boldsymbol{\ell}_{j,2}^{p+k} \end{bmatrix} + \mathcal{I}_{p+k+1}^{j}.$$

Hence, $\boldsymbol{\ell}_{j,2}^{p+k+1} = A^{-1}b\boldsymbol{\ell}_{j,1}^{p+k} + A^{-1}B\boldsymbol{\ell}_{j,2}^{p+k} + A^{-1}\mathcal{I}_{p+k+1}^{j}$. By using the induction hypothesis we obtain:

$$\begin{aligned} \boldsymbol{\ell}_{j,2}^{p+k+1} &= A^{-1}b\boldsymbol{\ell}_{j,1}^{p+k} + A^{-1}BM^{[k]}\left[\boldsymbol{\ell}_{j,1}^{p+k-1}, \boldsymbol{\ell}_{j,1}^{p+k-2}, \ldots, \boldsymbol{\ell}_{j,1}^{p}, \right]^{\top} \\ &\quad + A^{-1}B\sum_{i=0}^{k}(A^{-1}B)^i A^{-1}\mathcal{I}_{p+k-i}^{j} + A^{-1}\mathcal{I}_{p+k+1}^{j} \\ &= M^{[k+1]}\left(\boldsymbol{\ell}_{j,1}^{p+k}, \boldsymbol{\ell}_{j,1}^{p+k-1}, \ldots, \boldsymbol{\ell}_{j,1}^{p}, \right)^{\top} + \sum_{i=0}^{k+1}(A^{-1}B)^i A^{-1}\mathcal{I}_{p+(k+1)-i}^{j}. \end{aligned}$$

Notice that matrices $M^{[k]}$ depend on $\mathcal{A}$ and $\mathcal{B}$ and are independent of j. □

By using Theorem 4 and that there is only one nonzero $\mathcal{I}_{p+i}^{j}$, say $\mathcal{I}_{p+i_0}^{j} = I_r^j$, we have that

$$\boldsymbol{\ell}_j^{p+\nu} = \begin{bmatrix} \boldsymbol{\ell}_{j,1}^{p+\nu} \\ \boldsymbol{\ell}_{j,2}^{p+\nu} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_{p+\nu}^{1} \\ M^{[\nu]}\left[\boldsymbol{\ell}_{j,1}^{p+\nu-1}, \boldsymbol{\ell}_{j,1}^{p+\nu-2}, \ldots, \boldsymbol{\ell}_{j,1}^{p}\right]^{\top} + (A^{-1}B)^{\nu-i_0}A^{-1}I_r^j \end{bmatrix}$$

for $1 \leq j \leq r$. Moreover, $\boldsymbol{\ell}_j^{p+\nu}$ has to verify $\mathcal{B}\boldsymbol{\ell}_j^{p+\nu} = 0$. Equivalently,

$$[b\ B]\begin{bmatrix}\boldsymbol{\ell}_{j,1}^{p+\nu}\\ \boldsymbol{\ell}_{j,2}^{p+\nu}\end{bmatrix} = b\boldsymbol{\ell}_{p+\nu}^{1} + BM^{[\nu]}\left[\boldsymbol{\ell}_{j,1}^{p+\nu-1}, \boldsymbol{\ell}_{j,1}^{p+\nu-2}, \ldots, \boldsymbol{\ell}_{j,1}^{p}\right]^{\top} + B(A^{-1}B)^{\nu-i_0}A^{-1}I_r^j$$
$$= M^{[\nu+1]}\left[\boldsymbol{\ell}_{j,1}^{p+\nu}, \boldsymbol{\ell}_{j,1}^{p+\nu-1}, \ldots, \boldsymbol{\ell}_{j,1}^{p}\right]^{\top} + (BA^{-1})^{\nu-i_0+1}I_r^j = 0, \tag{24}$$

which is a linear system with r equations and $\nu+1$ unknowns, $\boldsymbol{\ell}_{j,1}^{p+\nu}, \boldsymbol{\ell}_{j,1}^{p+\nu-1}, \ldots, \boldsymbol{\ell}_{j,1}^{p}$, for each $j = 1, 2, \ldots, r$.

To deal with the system (24) we have to calculate the rank of $(BA^{-1})^k$ for $k \in \mathbb{N}\cup\{0\}$:

Lemma 6. *For all $k \in \mathbb{N}\cup\{0\}$, the rank of $(BA^{-1})^k$ is $N-1$.*

Proof. Denote

$$A = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix};\quad A^{-1} = \begin{bmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{bmatrix};\quad B = \begin{bmatrix}B_{11} & 0\\ 0 & 0\end{bmatrix},$$

where $A_{11}, V_{11}, B_{11} \in \mathbb{C}^{(N-1)\times(N-1)}$ and $A_{22}, V_{22} \in \mathbb{C}^{(r-N+1)\times(r-N+1)}$. Then, since $AA^{-1} = \mathbf{I}_r$ we obtain:

$$A_{11}V_{11} + A_{12}V_{21} = \mathbf{I}_{N-1}, \tag{25}$$
$$A_{21}V_{11} + A_{22}V_{21} = 0. \tag{26}$$

From (18), we know that A_{22} is a regular matrix, so $V_{21} = -A_{22}^{-1}A_{21}V_{11}$ and, substituting in (26), we obtain that

$$A_{11}V_{11} - A_{12}A_{22}^{-1}A_{21}V_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})V_{11} = I_{N-1}.$$

Thus, V_{11} is a regular matrix. On the other hand, $BA^{-1} = \begin{bmatrix}B_{11}V_{11} & B_{11}V_{12}\\ 0 & 0\end{bmatrix}$. It is a straightforward calculation to prove that

$$(BA^{-1})^k = \begin{bmatrix}(B_{11}V_{11})^k & (B_{11}V_{11})^{k-1}B_{11}V_{12}\\ 0 & 0\end{bmatrix}$$

and, since by (19) B_{11} is regular, we have that $\operatorname{rank}(BA^{-1})^k = N-1$. □

The system

$$M^{[\nu+1]}\left[\boldsymbol{\ell}_{j,1}^{p+\nu}, \boldsymbol{\ell}_{j,1}^{p+\nu-1}, \ldots, \boldsymbol{\ell}_{j,1}^{p}\right]^{\top} = -(BA^{-1})^{\nu-i_0+1}I_r^j,\quad j = 1, 2, \ldots, r \tag{27}$$

is compatible for all $j = 1, 2, \ldots, r$ if and only if $\operatorname{rank}[M^{[\nu+1]} \mid (BA^{-1})^{\nu-i_0+1}] = \operatorname{rank} M^{[\nu+1]}$. Since $\operatorname{rank}(BA^{-1})^{\nu-i_0+1}$ is $N-1$ and $M^{[\nu+1]} \in \mathbb{C}^{r\times(\nu+1)}$ depends only on $\mathcal{A}$ and $\mathcal{B}$ we deduce that (27) is compatible if $\nu+1 \geq N-1$. The following results holds:

Theorem 5. *If $\nu < N-2$ there is not a $(r+1)\times r$ polynomial matrix $\mathsf{L}(\lambda) = \mathcal{L}_p\lambda^p + \mathcal{L}_{p+1}\lambda^{p+1} + \cdots + \mathcal{L}_{p+\nu}\lambda^{p+\nu}$ satisfying Eq. (20).*

3.2.2. Case $\nu \geq N-2$

Theorem 3 in Section 3 ensures that (20) has a unique solution for $\nu = N-2$. However, when $\nu > N-2$ there are infinitely many polynomial matrices $\mathsf{L}(\lambda)$ satisfying Eq. (20). Having in mind the matrix G_r in Theorem 3, we introduce the new matrices $\mathsf{G}_r(k) \in \mathbb{C}^{(k+2)r\times(k+1)(r+1)}$, $k \in \mathbb{N}\cup\{0\}$,

defined recursively as:

$$\mathsf{G}_r(0) = \begin{bmatrix} -\mathcal{B} \\ \mathcal{A} \end{bmatrix},\ \mathsf{G}_r(1) = \begin{bmatrix} -\mathcal{B} & 0 \\ \mathcal{A} & -\mathcal{B} \\ 0 & \mathcal{A} \end{bmatrix},\ \ldots,\ \mathsf{G}_r(k) = \left[\begin{array}{ccc|c} & & & 0 \\ & \mathsf{G}_r(k-1) & & \vdots \\ & & & 0 \\ & & & -\mathcal{B} \\ \hline 0 & \cdots & 0 & \mathcal{A} \end{array}\right].$$

Since $\mathcal{B}$ has $r - N + 1$ null rows, $\mathsf{G}_r(k)$ has $(kr + r) + N - 1$ nonzero rows. So we have that rank $\mathsf{G}_r(k) \leq \min\{(kr + r) + N - 1, (kr + r) + k + 1\}$. It is straightforward to prove that previous inequality is, indeed, an equality. Hence, since we are assuming the hypotheses in Corollary 2, if $\nu \geq N - 2$, then the rank of $\mathsf{G}_r(\nu)$ is the number of its nonzero rows, i.e., rank $\mathsf{G}_r(k) = (k + 1)r + N - 1$. Thus, for $\nu \geq N - 2$ the system (21)–(22) is compatible for every $j = 1, 2, \ldots, r$.

Let $\mathsf{L}(\lambda) = \mathcal{L}_p\lambda^p + \mathcal{L}_{p+1}\lambda^{p+1} + \cdots + \mathcal{L}_{p+\nu}\lambda^{p+\nu}$ a solution of (20). Whatever $\nu \geq N-2$, the number of nonzero terms of $\mathsf{L}(\lambda)$ is greater or equal than $N-1$. On the contrary, let us suppose that the number of nonzero terms of $\mathsf{L}(\lambda)$ is less than $N - 1$. In this case, let $u = \min\{m \geq 0 : \mathcal{L}_{m+1} = 0\}$ and $v = \max\{m \leqslant 0 : \mathcal{L}_{m-1} = 0\}$. It is easy to check that the polynomial matrix $\mathcal{L}_v\lambda^v + \cdots + \mathcal{L}_0 + \cdots + \mathcal{L}_u\lambda^u$ is a solution of (20) whose terms are all nonzero ($\mathcal{L}_0 \neq 0$ because $\mathsf{L}(\lambda)$ is a solution of (20)). But this leads to a contradiction with Theorem 5. Therefore, the following result holds:

Theorem 6. *Assume $\nu \geq N - 2$ and let $\mathsf{L}(\lambda) = \mathcal{L}_p\lambda^p + \mathcal{L}_{p+1}\lambda^{p+1} + \cdots + \mathcal{L}_{p+\nu}\lambda^{p+\nu}$ a solution of Eq. (20). The number of nonzero terms of $\mathsf{L}(\lambda)$ is at least $N - 1$.*

Acknowledgments

The authors thank the anonymous referee for her/his valuable and constructive comments: they have helped to improve both the contain and the presentation of the paper. The authors gratefully acknowledge F.M. Dopico (Universidad Carlos III de Madrid) for the fruitful discussions on the proof of Theorem 3. This work has been supported by the grant MTM2009-08345 from the Spanish *Ministerio de Ciencia e Innovación (MICINN)*.

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