A nontrivial upper bound on the largest Laplacian eigenvalue of weighted graphs

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Received 11 May 2006; accepted 24 August 2006
Available online 10 October 2006
Submitted by R.A. Brualdi

Abstract

Let $G$ be a simple connected weighted graph on $n$ vertices in which the edge weights are positive numbers. Denote by $i \sim j$ if the vertices $i$ and $j$ are adjacent and by $w_{i,j}$ the weight of the edge $ij$. Let $w_i = \sum_{j=1}^{n} w_{i,j}$. Let $\lambda_1$ be the largest Laplacian eigenvalue of $G$. We first derive the upper bound

$$\lambda_1 \leq \sum_{j=1}^{n} \max_{k \sim j} w_{k,j}.$$ 

We call this bound the trivial upper bound for $\lambda_1$. Our main result is

$$\lambda_1 \leq \frac{1}{2} \max_{i \sim j} \left\{ \frac{w_i + w_j + \sum_{k \sim i, k \neq j} w_{i,k} + \sum_{k \sim j, k \neq i} w_{j,k}}{\sum_{k \sim i, k \neq j} |w_{i,k} - w_{j,k}|} \right\}.$$ 

For any $G$, this new bound does not exceed the trivial upper bound for $\lambda_1$.

AMS classification: 05C50

Keywords: Graph; Weighted graph; Laplacian matrix; Spectral radius; Upper bound

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Work supported by Fondecyt 1040218, Chile.

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doi:10.1016/j.laa.2006.08.022
1. Preliminaries

We consider a simple connected weighted graph in which the edge weights are positive numbers. Let $G$ be such a graph. Labelling the vertices of $G$ by $1, 2, \ldots, n$, denote by $w_{i,j}$ the weight of the edge $ij$. We write $i \sim j$ if the vertices $i$ and $j$ are adjacent. Let $w_i = \sum_{k \sim i} w_{k,i}$. The Laplacian matrix of $G$ is the $n \times n$ matrix $L(G) = (l_{i,j})$ defined by

$$l_{i,j} = \begin{cases} w_i & \text{if } i = j, \\ -w_{i,j} & \text{if } i \sim j, \\ 0 & \text{if } i \not\sim j, i \neq j. \end{cases}$$

$L(G)$ is a real symmetric matrix. From this fact and Geršgorin’s Theorem, it follows that the eigenvalues of $L(G)$ are nonnegative real numbers. Since each row sum is 0, $(0, e)$ is an eigenpair for $L(G)$ where $e$ is the all ones vector. Moreover, $G$ is a connected graph if and only if 0 is a simple eigenvalue. We assume that $G$ is a connected graph.

Throughout this paper we assume that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$$

are the eigenvalues of $L(G)$.

If $w_{i,j} = 1$ for all edge $ij$ then $G$ is an unweighted graph. In [1], some of the many results known for the Laplacian matrix of an unweighted graph are given. Upper bounds for $\lambda_1$ in the case of unweighted graphs have been obtained by several authors [2–10].

We recall the following result due to Brauer [11]:

**Theorem 1.** Let $A$ be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Let

$$v = [v_1, v_2, \ldots, v_n]^T$$

be an eigenvector of $A$ corresponding to the eigenvalue $\lambda_k$ and let $q$ be any $n$-dimensional column vector. Then the matrix $A + vq^T$ has eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k + v^T q, \lambda_{k+1}, \ldots, \lambda_n.$$

We define the vector

$$p = [p_1, p_2, \ldots, p_n]^T$$

where, for $j = 1, 2, \ldots, n$,

$$p_j = \max_{k \sim j} w_{k,j}.$$

Then $p_j - w_{i,j} \geq 0$ for all $i \sim j$. We now define the matrix

$$B(G) = L(G) + ep^T. \hspace{1cm} (1)$$

The entries of $B(G)$ are

$$b_{i,j}(G) = \begin{cases} w_i + p_i & \text{if } i = j, \\ -w_{i,j} + p_j & \text{if } i \sim j, \\ p_j & \text{if } i \not\sim j, i \neq j. \end{cases} \hspace{1cm} (2)$$
Lemma 1. If $G$ is a simple connected weighted graph of order $n$ then the spectrum of $B(G)$ is

$$
\lambda_{n-1}, \lambda_{n-2}, \ldots, \lambda_1, \sum_{j=1}^{n} \max_{k \sim j} w_{k,j}.
$$

The spectral radius of $B(G)$ is

$$
\rho(B(G)) = \sum_{j=1}^{n} \max_{k \sim j} w_{k,j}
$$

and the all ones vector is a corresponding eigenvector.

Proof. We easily see that $b_{i,j}(G) \geq 0$ for all $i, j$. Moreover

$$
B(G)e = L(G)e + e(p^T e) = (p^T e)e.
$$

Therefore $\rho(B(G)) = p^T e = \sum_{j=1}^{n} \max_{k \sim j} w_{k,j}$ with eigenvector the all ones vector. Since $(e, 0)$ is an eigenpair for the Laplacian matrix $L(G)$, using Theorem 1, we obtain that the eigenvalues of $B(G)$ are

$$
\lambda_{n-1}, \lambda_{n-2}, \ldots, \lambda_1, \sum_{j=1}^{n} \max_{k \sim j} w_{k,j}.
$$

Thus the lemma is proved. □

An immediate consequence of Lemma 1 is our first upper bound for $\lambda_1$:

Theorem 2. If $G$ is a simple connected weighted graph of order $n$ then

$$
\lambda_1 \leq \sum_{j=1}^{n} \max_{k \sim j} w_{k,j}. \quad (3)
$$

We call the bound in (3) the trivial upper bound for $\lambda_1$.

Our purpose is to find an upper bound for $\lambda_1$ such that for any $G$ it does not exceed the trivial upper bound for $\lambda_1$.

2. A nontrivial upper bound

If $B$ is a nonnegative matrix then, by the Perron–Frobenius’s Theorem, $B$ has an eigenvalue equal to its spectral radius $\rho(B)$. Special attention has been devoted to find upper bounds for the second largest modulus $\xi(B)$ of the eigenvalues of $B$. We recall the result [12, p. 295]:

Theorem 3. If $B = (b_{i,j}) \geq 0$ of order $n \times n$ has a positive eigenvector

$$
v = [v_1, v_2, \ldots, v_n]^T
$$

corresponding to $\rho(B)$ then

$$
\xi(B) \leq \frac{1}{2} \max_{1 \leq i < j \leq n} \sum_{k=1}^{n} v_k \left| \frac{b_{i,k}}{v_i} - \frac{b_{j,k}}{v_j} \right|.
$$
Corollary 1. If $B = (b_{i,j}) \geq 0$ of order $n \times n$ has a positive eigenvector $v = [v_1, v_2, \ldots, v_n]^T$ corresponding to $\rho(B)$ then

$$\frac{1}{2} \max_{1 \leq i, j \leq n} \sum_{k=1}^{n} v_k \left| \frac{b_{i,k}}{v_i} - \frac{b_{j,k}}{v_j} \right| \leq \rho(B). \quad (4)$$

Proof. We have $Bv = \rho(B)v$. Then

$$\sum_{k=1}^{n} b_{i,k} \frac{v_k}{v_i} = \rho(B) \quad \text{and} \quad \sum_{k=1}^{n} b_{j,k} \frac{v_k}{v_j} = \rho(B)$$

for all $i, j$. Hence

$$\sum_{k=1}^{n} v_k \left| \frac{b_{i,k}}{v_i} - \frac{b_{j,k}}{v_j} \right| \leq \sum_{k=1}^{n} b_{i,k} \frac{v_k}{v_i} + \sum_{k=1}^{n} b_{j,k} \frac{v_k}{v_j} = 2 \rho(B).$$

From this inequality it follows (4). \hfill \Box

Our next upper bound for $\lambda_1$ is given in the following theorem.

Theorem 4. If $\mathcal{G}$ is a simple connected weighted graph of order $n$ then

$$\lambda_1 \leq \frac{1}{2} \max_{1 \leq i < j \leq n} \left\{ w_i + w_j + \sum_{k \sim i, k \sim j} w_{i,k} + \sum_{k \sim j, k \sim i} w_{j,k} \right\} + \sum_{k \sim i, k \sim j} |w_{i,k} - w_{j,k}| \quad (5)$$

and this upper bound does not exceed $\sum_{j=1}^{n} \max_{k \sim j} w_{k,j}$.

Proof. We apply Theorem 3 to $B(\mathcal{G}) = L(\mathcal{G}) + e^T p$ defined in (1). Then

$$\lambda_1 = \xi(B(\mathcal{G})) \leq \frac{1}{2} \max_{1 \leq i < j \leq n} \sum_{k=1}^{n} |b_{i,k}(\mathcal{G}) - b_{j,k}(\mathcal{G})|. \quad (6)$$

From Corollary 1, the right hand side of (6) does not exceed $\rho(B(\mathcal{G})) = \sum_{j=1}^{n} \max_{k \sim j} w_{k,j}$. Using (2), the proof is completed by showing that the right hand side of (6) is given by the right hand side of (5). \hfill \Box

Let $N_i$ and $d_i$ be the set of neighbours and the degree of the vertex $i$, respectively. Let $|S|$ be the cardinality of $S$. Then $|N_i| = d_i$.

Remark 1. If $\mathcal{G}$ is an unweighted graph then (5) becomes

$$\lambda_1 \leq \max_{1 \leq i < j \leq n} \{ d_i + d_j - |N_i \cap N_j| \}.$$ 

This is the main result in [9].

In fact

$$w_i + w_j + \sum_{k \sim i, k \sim j} w_{i,k} + \sum_{k \sim j, k \sim i} w_{j,k} + \sum_{k \sim i, k \sim j} |w_{i,k} - w_{j,k}|$$
\[ d_i + d_j + |N_i - N_j| + |N_j - N_i| = d_i + d_j + d_i - |N_i \cap N_j| + d_j - |N_i \cap N_j| = 2(d_i + d_j - |N_i \cap N_j|). \]

Our next theorem gives an improved version of (5).

**Theorem 5.** If \( G \) is a simple connected weighted graph of order \( n \) then

\[ \lambda_1 \leq \frac{1}{2} \max_{i \sim j} \left\{ w_i + w_j + \sum_{k \sim i, k \neq j} w_{i,k} + \sum_{k \sim j, k \neq i} w_{j,k} \right\} \]

and this upper bound for \( \lambda_1 \) does not exceed \( \sum_{j=1}^{n} \max_{k \sim j} w_{k,j} \).

**Proof.** Let

\[ \mathbf{x} = [x_1, x_2, \ldots, x_n]^T \]

be such that

\[ L\mathbf{x} = \lambda_1 \mathbf{x}. \]  

(8)

Then \( \mathbf{x} \neq 0 \) and \( \sum_{k=1}^{n} x_k = 0 \). Let

\[ \max\{x_1, x_2, \ldots, x_n\} = x_i. \]

Then \( x_i > 0 \). Let

\[ x_j = \min\{x_k : k \sim i\}. \]

Thus \( x_j \leq x_k \) for all \( k \sim i \). From (8)

\[ \lambda_1 x_i = w_i x_i - \sum_{k \sim i} w_{i,k} x_k \]

(9)

and

\[ \lambda_1 x_j = w_j x_j - \sum_{k \sim j} w_{i,k} x_k. \]

(10)

Subtracting (10) from (9) gives

\[ \lambda_1 (x_i - x_j) = w_i x_i - w_j x_j + \sum_{k \sim j} w_{j,k} x_k - \sum_{k \sim i} w_{i,k} x_k. \]

Hence

\[ \lambda_1 (x_i - x_j) = w_i x_i - w_j x_j + \sum_{k \sim j, k \sim i} w_{j,k} x_k + \sum_{k \sim j, k \sim i} w_{j,k} x_k - \sum_{k \sim i, k \sim j} w_{i,k} x_k - \sum_{k \sim i, k \sim j} w_{i,k} x_k. \]

(11)

Since \( x_j \leq x_k \) for all \( k \sim i \) and \( x_k \leq x_i \), we have

\[ \sum_{k \sim j, k \sim i} w_{j,k} x_k \leq x_i \sum_{k \sim j, k \sim i} w_{j,k} \]
and
\[-\sum_{k\sim i, k\neq j} w_{i,k}x_k \leq -x_j \sum_{k\sim i, k\neq j} w_{i,k}.\]

Replacing these inequalities in (11), we obtain
\[\lambda_1(x_i - x_j) \leq w_i x_i - w_j x_j + x_i \sum_{k\sim j, k\neq i} w_{j,k} - x_j \sum_{k\sim i, k\neq j} w_{i,k} + \sum_{k\sim i, k\neq j} (w_{j,k} - w_{i,k})x_k.\]

Let us denote the right hand side of (12) by \(S_{i,j}\). We have
\[S_{i,j} = w_i (x_i - x_j) + w_j (x_i - x_j) + x_i \sum_{k\sim j, k\neq i} w_{j,k} - x_j \sum_{k\sim i, k\neq j} w_{i,k} + \sum_{k\sim i, k\neq j} (w_{j,k} - w_{i,k})x_k.\]

Then
\[S_{i,j} = \frac{1}{2} w_i (x_i - x_j) + \frac{1}{2} w_j (x_i - x_j) + \frac{1}{2} (x_i - x_j) \sum_{k\sim i, k\neq j} w_{i,k} + \frac{1}{2} (x_i + x_j) \left( w_i - \sum_{k\sim i, k\neq j} w_{i,k} \right) \]
\[+ \frac{1}{2} (x_i - x_j) \sum_{k\sim j, k\neq i} w_{j,k} + \sum_{k\sim i, k\neq j} (w_{j,k} - w_{i,k})x_k.\]

Hence
\[S_{i,j} = \frac{1}{2} w_i (x_i - x_j) + \frac{1}{2} w_j (x_i - x_j) + \frac{1}{2} (x_i - x_j) \sum_{k\sim i, k\neq j} w_{i,k} + \frac{1}{2} (x_i + x_j) \sum_{k\sim i, k\neq j} w_{i,k} \]
\[+ \frac{1}{2} (x_i - x_j) \sum_{k\sim j, k\neq i} w_{j,k} + \sum_{k\sim i, k\neq j} (w_{j,k} - w_{i,k})x_k.\]
Clearly, $S_{i,j}$ becomes

$$S_{i,j} = \frac{1}{2} w_i (x_i - x_j) + \frac{1}{2} w_j (x_i - x_j) + \frac{1}{2} (x_i - x_j) \sum_{k \sim i, k \sim j} w_{i,k}$$

$$+ \frac{1}{2} (x_i - x_j) \sum_{k \sim j, k \sim i} w_{j,k} + \frac{1}{2} \sum_{k \sim i, k \sim j} (w_{i,k} - w_{j,k})(x_i - x_k + x_j - x_k).$$

Consider the last term in the right hand side. We have

$$\sum_{k \sim i, k \sim j} (w_{k,i} - w_{k,j})(x_i - x_k + x_j - x_k)$$

$$= \sum_{k \sim i, k \sim j} (w_{k,i} - w_{k,j})(x_i - x_k) + \sum_{k \sim i, k \sim j} (w_{k,i} - w_{k,j})(x_j - x_k)$$

$$\leq \sum_{k \sim i, k \sim j} |w_{i,k} - w_{j,k}|(x_i - x_k) + \sum_{k \sim i, k \sim j} |w_{i,k} - w_{j,k}|(x_k - x_j)$$

$$= \sum_{k \sim i, k \sim j} |w_{i,k} - w_{j,k}|(x_i - x_j) = (x_i - x_j) \sum_{k \sim i, k \sim j} |w_{i,k} - w_{j,k}|.$$

Therefore

$$S_{i,j} \leq \frac{1}{2} w_i (x_i - x_j) + \frac{1}{2} w_j (x_i - x_j) + \frac{1}{2} (x_i - x_j) \sum_{k \sim i, k \sim j} w_{k,i}$$

$$+ \frac{1}{2} (x_i - x_j) \sum_{k \sim j, k \sim i} w_{j,k} + \frac{1}{2} (x_i - x_j) \sum_{k \sim i, k \sim j} |w_{i,k} - w_{j,k}|.$$ 

Then (12) becomes

$$\lambda_1 (x_i - x_j) \leq \frac{1}{2} w_i (x_i - x_j) + \frac{1}{2} w_j (x_i - x_j) + \frac{1}{2} (x_i - x_j) \sum_{k \sim i, k \sim j} w_{k,i}$$

$$+ \frac{1}{2} (x_i - x_j) \sum_{k \sim j, k \sim i} w_{j,k} + \frac{1}{2} (x_i - x_j) \sum_{k \sim i, k \sim j} |w_{i,k} - w_{j,k}|. \quad (13)$$

If $x_j = x_i$ then $x_k = x_i$ for all $k \sim i$. Consequently, from $L \mathbf{x} = \lambda_1 \mathbf{x}$, we have

$$\lambda_1 x_1 = w_i x_i - \sum_{k \sim i} w_{k,i} x_k = w_i x_i - w_i x_i = 0.$$

Thus $\lambda_1 = 0$ which is a contradiction for a graph with at least one edge. Hence $x_i - x_j > 0$. Dividing both sides of (13) by $(x_i - x_j)$ the upper bound (7) is obtained. Since the second right hand of (7) does not exceed the second right hand of (5), the upper bound (7) does not exceed $\sum_{j=1}^{n} \max_{k \sim j} w_{k,j}$. \qed

**Remark 2.** If $G$ is an unweighted graph then (7) becomes

$$\lambda_1 \leq \max_{i, j} \{d_i + d_j - |N_i \cap N_j|\}.$$ 

This is the main result in [3].
In [13] an upper bound on the largest Laplacian eigenvalue for weighted graphs in which the edge weights are positive definite matrices is obtained. In particular, if the edge weights are positive numbers this upper bound becomes [13, Corollary 3.2]:

$$\lambda_1 \leq \max_{i \sim j} \{ w_i + w_j \}. \quad (14)$$

Remark 3. For any $\mathcal{G}$

$$\frac{1}{2} \max_{i \sim j} \left\{ w_i + w_j + \sum_{k \sim i, k \sim j} w_{i,k} + \sum_{k \sim i, k \sim j} w_{j,k} \right\} \leq \max_{i \sim j} \{ w_i + w_j \}.$$

In fact

$$w_i + w_j + \sum_{k \sim i, k \sim j} w_{i,k} + \sum_{k \sim i, k \sim j} w_{j,k} \leq w_i + w_j + \sum_{k \sim i, k \sim j} w_{i,k} + \sum_{k \sim i, k \sim j} w_{j,k} + \sum_{k \sim i, k \sim j} |w_{i,k} - w_{j,k}|$$

$$= w_i + w_j + \sum_{k \sim i} w_{i,k} + \sum_{k \sim j} w_{j,k} = 2w_i + 2w_j.$$

Therefore the upper bound (7) improves the upper bound (14). Below we give two examples.

Example 1. For the graph

the largest Laplacian eigenvalue, rounded to two decimal places, is $\lambda_1 = 17.93$. For this graph the bounds (3), (7) and (14) give

$$\begin{align*}
(3) & \quad 31.6 \\
(7) & \quad 21.4 \\
(14) & \quad 23.6
\end{align*}$$

The following example corresponds to a graph in which the bound (14) exceeds the trivial upper bound (3).

Example 2. For the graph
For this graph $\lambda_1 = 11.60$ and the bounds (3), (7) and (14) give

\[
\begin{align*}
(3) & \quad (7) & \quad (14) \\
16.3 & \quad 13.7 & \quad 17.9
\end{align*}
\]

Acknowledgments

The author wishes to express his thanks to the referee for valuable comments which led to an improved version of the paper.

References