Bounds on the locating-total domination number of a tree

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ABSTRACT

In this paper, we continue the study of locating-total domination in graphs, introduced by Haynes et al. [T.W. Haynes, M.A. Henning, J. Howard, Locating and total dominating sets in trees, Discrete Applied Mathematics 154 (8) (2006) 1293–1300]. A total dominating set $S$ in a graph $G = (V, E)$ is a locating-total dominating set of $G$ if, for every pair of distinct vertices $u$ and $v$ in $V - S$, $N_c(u) \cap S \neq N_c(v) \cap S$. The minimum cardinality of a locating-total dominating set is the locating-total domination number $\gamma^T_1(G)$. We show that, for a tree $T$ of order $n \geq 3$ with $l$ leaves and $s$ support vertices, $\frac{n+1}{2} - s \leq \gamma^T_1(T) \leq \frac{n+1}{2}$. Moreover, we constructively characterize the extremal trees achieving these bounds.

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1. Introduction

The concept of a locating-total domination in graph was introduced in [2,5]. The location of monitoring devices, such as surveillance cameras or fire alarms, to safeguard a system serves as the motivation for this work. The problem of placing monitoring devices in a system in such a way that every site in the system (including the monitors themselves) is adjacent to a monitor site can be modeled by total domination in graphs. Applications where it is also important that, if there is a problem at a facility, its location can be uniquely identified by the set of monitors, can be modeled by a combination of total domination sets and locating sets.

Graph theory terminology not presented here can be found in [3,4]. Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. For any vertex $v \in G$, the open neighborhood of $v$ is the set $N(v) = \{u \in V | uv \in E\}$, and its closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. We denote the degree of a vertex $v$ in $G$ by $d_G(v)$, or simply by $d(v)$ if the graph $G$ is clear from text. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and minimum degree of the graph $G$. For any $S \subseteq V$, $N_S(S) = \bigcup_{v \in S} N_G(v)$. Let $G[S]$ denote the graph induced by $S$. If $v \in S$ and $w \in V - S$, then the vertex $w$ is an external private neighbor of $v$ (with respect to $S$) if $N(w) \cap S = \{v\}$. Let $PN(v, S) = \{w | w \in V - S, N(v) \cap S = \{v\}\}$. Let $C_n$ and $P_n$ denote the cycle and the path of order $n$. A vertex of degree $1$ is called a leaf and its neighbor is called a support vertex. The eccentricity of a vertex $v$ in a connected graph $G$ is the maximum graph distance between $v$ and any other vertex $u$ of $G$.

A subset $S \subseteq V$ is a total dominating set if every vertex in $V$ has a neighbor in $S$. The total domination number of $G$, denoted by $\gamma_1(G)$, is the minimum cardinality of a total dominating set of $G$. Total domination was introduced by Cockayne et al. [1]. For a comprehensive survey of domination in graphs and its variations, see [3,4].

A total dominating set $S$ in a graph $G = (V, E)$ is a locating-total dominating set of $G$ if, for every pair of distinct vertices $u$ and $v$ in $V - S$, $N_c(u) \cap S \neq N_c(v) \cap S$. The minimum cardinality of a locating-total dominating set is the locating-total domination number $\gamma^T_1(G)$. We call a locating-total dominating set in $G$ of cardinality $\gamma^T_1(G)$ a $\gamma^T_1(G)$-set.
A total dominating set $S$ in a graph $G = (V, E)$ is a differentiating-total dominating set of $G$ if, for every pair of distinct vertices $u$ and $v$ in $V$, $N_G[u] \cap S \neq N_G[v] \cap S$. The minimum cardinality of a differentiating-total dominating set is the differentiating-total domination number $\gamma'_t(G)$.

Locating-total domination and differentiating-total domination were introduced by Haynes et al. [5]. They established bounds on these parameters in a tree and investigated the ratio of the two parameters in trees. In this paper, we show that, for a tree $T$ of order $n \geq 3$ with $l$ leaves and $s$ support vertices, $\frac{n+1}{2} - s \leq \gamma'_t(G) \leq \frac{n+1}{2}$. Moreover, we constructively characterize the extremal trees achieving these bounds.

2. Lower bound on the locating-total domination number of a tree

For any tree $T$, let $L(T)$ and $S(T)$ denote the set of leaves and support vertices, respectively. Let $\xi_1$ be the family of trees that can be obtained from $k$ disjoint copies of $P_k$ by first adding $k - 1$ edges in such a manner that they are incident only with support vertices and the resulting graphs is connected, and then subdividing each new edge exactly once. Let $\xi_2$ be the family of trees $T$ that can be obtained from any tree $T'$ by attaching at least two leaves to each vertex of $T'$ and, if $T'$ is nontrivial, subdividing each edge of $T'$ exactly once.

**Lemma 1** (Haynes et al. [5]). If $T$ is a tree of order $n \geq 2$, then $\gamma'_t(T) \geq \frac{2(n+1)}{5}$, with equality if and only if $T \in \xi_1$.

**Lemma 2** (Haynes et al. [5]). If $T$ is a tree of order $n \geq 3$ with $l$ leaves and $s$ support vertices, then $\gamma'_t(T) \geq \frac{n+2l-s+1}{3}$, with equality if and only if $T \in \xi_2$.

In the following, we give a lower bound on the locating-total domination number of a tree. Moreover, we give a characterization of the trees achieving the lower bound. In particular, the characterization of the trees achieving the lower bound is same as the characterization of Lemma 2.

**Theorem 3.** If $T$ is a tree of order $n \geq 3$ with $l$ leaves and $s$ support vertices, then $\gamma'_t(T) \geq \frac{n+1}{2} - s$, with equality if and only if $T \in \xi_2$.

**Proof.** Let $D$ be a $\gamma'_t(T)$-set that contains a minimum number of leaves. Then, for every support vertex $v$, exactly one leaf neighbor of $v$ is not in $D$. Let $P$ be the set of all external private neighbors of vertices in $D$. Let $L_1 = L(T) \cap D$, $L_2 = L(T) - L_1$, $A = P - L_2$, $B = V - D - P$, and $C = D - L_1 - S(T)$. Then $D = L_1 \cup S(T) \cup C$ and $V - D = L_2 \cup A \cup B$. Furthermore, $|L_1| = l - s$, $|L_2| = |S(T)| = s$, and $|A| \leq |C|$.

Let $T_1, \ldots, T_{n-1}$ be the components of $T[A \cup B]$. Since $T[A \cup B]$ is a forest, $|E(T[A \cup B])| = |A| + |B| - \omega_1$. Note that each vertex in $A$ is adjacent to at least one vertex in $A \cup B$. So, $|E(T[A \cup B])| = \frac{1}{2} \sum_{v \in T[A \cup B]} d_{T[A \cup B]}(v) \geq \frac{|A|}{2}$. Hence, $\omega_1 \leq \frac{|A|}{2} + |B|$.

Let $D_1, \ldots, D_{n-1}$ be the components of $T[S(T) \cup C]$. Since $T[S(T) \cup C]$ is a forest, $|E(T[S(T) \cup C])| = |S(T)| + |C| - \omega_2$. Note that each vertex in $C$ is adjacent to at least one vertex in $S(T) \cup C$. So, $|E(T[S(T) \cup C])| \geq \frac{|C|}{2}$. Hence, $\omega_2 \leq \frac{|C|}{2} + |S(T)|$.

Let $K$ be a set of $\omega_1$ vertices corresponding to the $\omega_1$ components of $T[A \cup B]$, and let $R$ be a set of $\omega_2$ vertices corresponding to the $\omega_2$ components of $T[S(T) \cup C]$. Say $K = \{k_1, \ldots, k_{\omega_1}\}$ and $R = \{r_1, \ldots, r_{\omega_2}\}$. Let $L$ be the graph of order $\omega_1 + \omega_2$ with $V(L) = K \cup R$. Moreover, $t_{ij} \in E(L)$ if and only if there exist $u \in V(T_i)$ and $v \in V(D_j)$ such that $uv \in E(T_i)$, where $1 \leq i \leq \omega_1$ and $1 \leq j \leq \omega_2$. Since $T[A \cup B] \cup S(T) \cup C$ is a tree, by the construction of $F$, it follows that $F$ is a tree. By the definition of $A$ and $B$, each vertex of $V$ is adjacent to exactly one vertex of $S(T) \cup C$ and each vertex of $B$ is adjacent at least two vertices of $S(T) \cup C$. So, $|E(F)| \geq |A| + 2|B|$. Since $|E(F)| = \omega_1 + \omega_2 - 1 \leq \frac{|A|}{2} + |B| + \frac{|C|}{2} + |S(T)| - 1$, it follows that $\frac{|A|}{2} + |B| \leq \frac{|C|}{2} + |S(T)| - 1$.

Hence, $n - |D| = |V - D| = |L_2| + |A| + |B| \leq |S(T)| + \frac{|C|}{2} + |S(T)| - 1 + \frac{|A|}{2} \leq 2|S(T)| + \frac{|C|}{2} - 1 + \frac{|C|}{2} = 2|S(T)| + |C| - 1 = |D| + |S(T)| - 1 - |L_1|$. So, $|D| \geq \frac{n+1}{2} - |S(T)| - 1 - |L_1|$. Hence, $\gamma'_t(T) \geq \frac{n+1}{2} - s$. That is, $\gamma'_t(T) \geq \frac{n+1}{2} - s$.

This bound is sharp if and only if equality is achieved in each of the above inequalities. In particular, $|A| = |C|$, $\gamma(T[A \cup B]) = \frac{|A|}{2}$, and $\gamma(T[S(T) \cup C]) = \frac{|C|}{2}$. Hence, $d_{T[A \cup B]}(u) = 1$ for any $u \in A$ and $d_{T[A \cup B]}(v) = 0$ for any $v \in B$. Similarly, $d_{T[S(T) \cup C]}(u) = 0$ for any $u \in S(T)$ and $d_{T[S(T) \cup C]}(v) = 1$ for any $v \in C$. If $A \neq \emptyset$, then, for any $u \in A \cup C$, $d_{T[A \cup C]}(u) \geq 2$. So, $T[A \cup C]$ contains a cycle, which is a contradiction. So, $A = C = \emptyset$. Furthermore, $S(T)$ and $B$ are two independent sets of $T$. Since $|A| + 2|B| = |E(F)|$, $d_{T}(u) = 2$ for any $u \in B$. Thus, $T$ can be obtained from a tree $T'$ of order $s$ by adding at least two leaves adjacent to each vertex in $T'$ and subdividing each edge of $T'$ exactly once. Hence, $T \in \xi_2$.

**Remark.** It is obvious that, if $n > \max\{10s - 5l - 1, l + 2s - 1\}$, then the lower bound is better than the lower bounds in Lemmas 1 and 2.

3. Upper bound on the locating-total domination number of a tree

In [5], Haynes et al. provided an upper bound on the differentiating-total domination number of a tree in terms of its order and number of support vertices.
Lemma 4 (Haynes et al. [5]). If \( T \neq P_4 \) is a tree of order \( n \geq 4 \) with \( s \) support vertices, then \( \gamma^D(T) \leq n - s \).

For any tree \( T \), \( \gamma^L(T) \leq \gamma^D(T) \). As an immediate consequence of Lemma 4, we have the following result.

Corollary 5. If \( T \) is a tree of order \( n \geq 3 \) with \( s \) support vertices, then \( \gamma^L(T) \leq n - s \).

Now, we show that, if \( T \) is a tree of order \( n \) with \( l \) leaves, then \( \gamma^L(T) \leq \frac{n + l}{2} \). For the purpose of characterizing the trees attaining this bound, we describe a procedure to build a family \( \Gamma \) of labeled trees. The label of a vertex \( v \) is called its status, denoted by \( \text{sta}(v) \). There are three kinds of status, say \( A \), \( B \), and \( C \), used to label the tree. Let \( \Gamma \) be the family of labeled trees \( T = T_k \) that can be obtained as follows. Let \( T_0 \) be a \( P_6 \) in which two leaves have status \( C \), the two support vertices have status \( A \), and the other vertices have status \( B \). If \( k \geq 1 \), then \( T_k \) can be obtained recursively from \( T_{k-1} \) by one of the following operations.

- **Operation \( \tau_1 \).** For any \( y \in V(T_{k-1}) \), if \( \text{sta}(y) = C \) and \( d_{T_{k-1}}(y) = 1 \), then add a path \( x, w, v, z \) and edge \( xy \). Let \( \text{sta}(x) = \text{sta}(w) = B \), \( \text{sta}(v) = A \) and \( \text{sta}(z) = C \).

- **Operation \( \tau_2 \).** For any \( y \in V(T_{k-1}) \), if \( \text{sta}(y) = B \), then add a path \( x, w, v \) and edge \( xy \). Let \( \text{sta}(x) = B \), \( \text{sta}(w) = A \) and \( \text{sta}(v) = C \).

The two operations are illustrated in the figure above. Suppose that \( T \in \Gamma \), and let \( A(T) = \{ v \in V(T) | \text{sta}(v) = A \} \), \( B(T) = \{ v \in V(T) | \text{sta}(v) = B \} \), and \( C(T) = \{ v \in V(T) | \text{sta}(v) = C \} \).

Lemma 6. If \( T \in \Gamma \), then \( \gamma^L(T) = 2|A(T)| \).

**Proof.** Let \( R = \{(u, v)| u \in A(T), v \in C(T), uw \in E(T)\} \). Let \( D \) be a \( \gamma^L(T) \)-set. For any \((u, v), (u_1, v_1) \in R\) by the construction for the tree \( T, N_T[u, v] \cap N_T[u_1, v_1] = \emptyset \). In order to total dominate the vertices \( u \) and \( v \), it follows that \( |D \cap N_T[u, v]| \geq 2 \). So, \( |D| \geq 2|A(T)| \). That is, \( \gamma^L(T) \geq 2|A(T)| \). It is obvious that since \( A(T) \cup (N(A(T)) \cap B(T)) \) is a totally dominating set of \( T \), \( \gamma^L(T) \leq |A(T) \cup (N(A(T)) \cap B(T))| = 2|A(T)| \). Hence, \( \gamma^L(T) = 2|A(T)| \).

Theorem 7. If \( T \in \Gamma \), then \( \gamma^L(T) = \frac{n + l}{2} \).

**Proof.** Since \( A(T) \cup (N(A(T)) \cap B(T)) \) is a locating-total dominating set of \( T \), \( \gamma^L(T) \leq |A(T) \cup (N(A(T)) \cap B(T))| = 2|A(T)| \). By Lemma 6, \( 2|A(T)| = \gamma^L(T) \leq \gamma^L(T) \leq 2|A(T)| \). Hence, \( \gamma^L(T) = 2|A(T)| \). Suppose that \( T \) is obtained from \( P_6 \) by applying \( k_1 \tau_1 \) operations and \( k_2 \tau_2 \) operations. Then \( n = 6 + 4k_1 + 3k_2, l = 2 + k_2, \) and \( |A(T)| = 2 + k_1 + k_2 \). Then, \( \gamma^L(T) = 2|A(T)| = \frac{n + l}{2} \).

Lemma 8. Let \( T \in \Gamma \). For any \( g \in C(T) \) and \( f \in N_T[g] \cap A(T) \), there exists a \( \gamma^L(T) \)-set \( D \) such that \( g, f \in D \) and \( \text{PN}(g, D) = \text{PN}(f, D) = \emptyset \).

**Proof.** If \( T = T_0 = P_6 \), it is obvious that the result holds. Without loss of generality, we can assume that \( T \) is obtained from \( P_6 \) by successive operations \( \tau_1 \), \( \ldots \), \( \tau_m \), respectively, where \( \tau^i \in \{ \tau_1, \tau_2 \} \) for \( i = 1, \ldots, m \) and \( m \geq 1 \). The proof is by induction on \( m \). If \( m = 1 \), it is easy to prove that the result holds. Assume that \( m \geq 2 \) and that the statement holds for all trees which are obtained from \( P_6 \) by applying at most \( m - 1 \) \( \tau \) operations.

Suppose that \( T = T_m \) is obtained from \( T_{m-1} \) by operation \( \tau_1 \). For any \( g \in C(T) \) and \( f \in N_T[g] \cap A(T) \), if \( g \in V(T_{m-1}) \), by inductive hypothesis, there exists a \( \gamma^L(T_{m-1}) \)-set \( D' \) such that \( g, f \in D' \) and \( \text{PN}(g, D') = \text{PN}(f, D') = \emptyset \). Let \( D = D' \cup \{w, v\} \). Then \( D \) is a \( \gamma^L(T) \)-set such that \( g, f \in D \) and \( \text{PN}(g, D) = \text{PN}(f, D) = \emptyset \).

Suppose that \( T = T_m \) is obtained from \( T_{m-1} \) by operation \( \tau_2 \). For any \( g \in C(T) \) and \( f \in N_T[g] \cap A(T) \), if \( g \in V(T_{m-1}) \), by inductive hypothesis, there exists a \( \gamma^L(T_{m-1}) \)-set \( D' \) such that \( g, f \in D' \) and \( \text{PN}(g, D') = \text{PN}(f, D') = \emptyset \). Let \( D = D' \cup \{x, w\} \). Then \( D \) is a \( \gamma^L(T) \)-set such that \( g, f \in D \) and \( \text{PN}(g, D) = \text{PN}(f, D) = \emptyset \).

If \( g = v \) and \( f = w \). If \( v \in N(A(T_{m-1})) \), then \( D' = A(T_{m-1}) \cup (N(A(T_{m-1})) \cap B(T_{m-1}))) \) is a \( \gamma^L(T_{m-1}) \)-set. Let \( D = \{x, w\} \cup \{v, w\} \). Then \( D \) is a \( \gamma^L(T) \)-set such that \( v, w \in D \) and \( \text{PN}(v, D) = \text{PN}(w, D) = \emptyset \).

If \( y \in N(C(T_{m-1})) \), then we say \( yu \in E(T_{m-1}) \), \( u \in C(T_{m-1}) \) and \( t \in N_{T_{m-1}}(u) \cap A(T_{m-1}) \). By inductive hypothesis, there exists a \( \gamma^L(T_{m-1}) \)-set \( D' \) such that \( u, t \in D' \) and \( \text{PN}(u, D') = \text{PN}(t, D') = \emptyset \). Let \( D = (D' \setminus \{t\}) \cup \{y, w, v\} \). Then \( D \) is a \( \gamma^L(T) \)-set such that \( w, v, D \in D \) and \( \text{PN}(w, D) = \text{PN}(v, D) = \emptyset \).
Theorem 9. If $T$ is a tree of order $n \geq 3$ with $l$ leaves, then $\gamma(T) \leq \frac{n+l}{2}$.

Proof. We proceed by induction on the order $n$. If $n \geq 3$, it follows that $\text{diam}(T) \geq 2$. If $\text{diam}(T) = 2$, then $\gamma(T) = n - 1 < \frac{n+l}{2}$. If $\text{diam}(T) = 3$, then $\gamma(T) = n - 2 < \frac{n+l}{2}$. This establishes the base cases.

Assume that every tree $T'$ of order $3 \leq n' \leq n$ and with $l'$ leaves satisfies $\gamma(T') \leq \frac{n'+l'}{2}$. Let $T$ be a tree of order $n$ and diameter at least 4 having $l$ leaves.

If a support vertex, say $x$, of $T$ is adjacent to two or more leaves, then let $T'$ be the tree obtained from $T$ by removing a leaf $y$ adjacent to $x$. Then $n' = n - 1$ and $l' = l - 1$. Applying the inductive hypothesis to $T'$, $\gamma(T') \leq \frac{n' + l'}{2} + 1 \leq \frac{n+l}{2}$. Thus, we can assume that every support vertex of $T$ is adjacent to exactly one leaf. We now root $T$ at a vertex of maximum eccentricity. Let $v$ be a support vertex at maximum distance from $u$, $u$ be the parent of $v$, and $w$ be the parent of $u$ in the rooted tree. For any vertex $x \in V(T)$, let $T_x$ denote the subtree induced by the vertex $x$ and its descendants in the rooted tree $T$. We have the following three cases.

Case 1: $d_T(u) \geq 3$. Then either $u$ has a child $b \neq v$ that is a support vertex or every child of $u$ except $v$ is a leaf.

Suppose first that $u$ has a child $b \neq v$ that is a support vertex. Let $T' = T - T_u$. Then $n' = n - 2$ and $l' = l - 1$. Let $D'$ be a $\gamma(T')$-set that contains a minimum number of leaves. Then $u, b \in D'$ and $D' \cup \{v\}$ is a locating-total dominating set of $T$. Hence, $\gamma(T) \leq \gamma(T') + 1 \leq \frac{n'+l'}{2} + 1 < \frac{n+l}{2}$.

Now assume that every child of $u$ except $v$ is a leaf. Since $u$ is adjacent to exactly one leaf, $d_T(u) = 3$. Let $T' = T - T_u$. Then $n' = n - 4$ and $l' = l - 1$. If $n' = 2$, then it is obvious that $\gamma(T) = 3 < \frac{n+l}{2}$. Suppose that $n' \geq 3$. Let $D'$ be a $\gamma(T')$-set.

Then $D' \cup \{u, v\}$ is a locating-total dominating set of $T$. Hence, $\gamma(T) \leq \gamma(T') + 2 \leq \frac{n'+l'}{2} + 2 < \frac{n+l}{2}$.

Case 2: $d_T(u) = 2$ and $d_T(w) \geq 3$. Let $T' = T - T_u$. Then $n' = n - 3$ and $l' = l - 1$. Let $D'$ be a $\gamma(T')$-set. Then $D' \cup \{u, v\}$ is a locating-total dominating set of $T$. Hence, $\gamma(T) \leq \gamma(T') + 2 \leq \frac{n'+l'}{2} + 2 < \frac{n+l}{2}$.

Case 3: $d_T(u) = 2$ and $d_T(w) = 2$. Let $T' = T - T_u$. Then $n' = n - 4$ and $l' = l$. If $n' = 1$, then $T' = P_3$ and $\gamma(T') = 3 < \frac{n+l}{2}$. If $n' = 2$, then $T' = P_3$ and $\gamma(T') = 4 = \frac{n+l}{2}$. Suppose that $n' \geq 3$. Let $D'$ be a $\gamma(T')$-set. Then $D' \cup \{u, v\}$ is a locating-total dominating set of $T$. Hence, $\gamma(T) \leq \gamma(T') + 2 \leq \frac{n'+l'}{2} + 2 < \frac{n+l}{2}$. □

Theorem 10. If $T$ is a tree of order $n \geq 3$ with $l$ leaves, then $\gamma(T) = \frac{n+l}{2}$ if and only if $T \in \Gamma$.

Proof. If $T \in \Gamma$, by Theorem 7, $\gamma(T) = \frac{n+l}{2}$. Conversely, let $T$ be a tree of order $n \geq 3$ with $\gamma(T) = \frac{n+l}{2}$. Then $\text{diam}(T) \geq 4$. In order to prove that $T \in \Gamma$, we proceed by induction on the order $n$. If $n \leq 6$, then $T = P_6$. So, $T \in \Gamma$. This establishes the base cases. Assume that every tree $T'$ of order 6 $\leq n' < n$ and with $l'$ leaves satisfies $\gamma(T') = \frac{n'+l'}{2}$ only if $T' \in \Gamma$. Let $T$ be a tree of order $n > 6$ and diameter at least 4 having $l$ leaves, and let $\gamma(T) = \frac{n+l}{2}$.

If a support vertex, say $x$, of $T$ is adjacent to two or more leaves, then let $T'$ be the tree obtained from $T$ by removing a leaf $y$ adjacent to $x$. Then $n' = n - 1$ and $l' = l - 1$. Then $\gamma(T) \leq \gamma(T') + 1 \leq \frac{n'+l'}{2} + 1 = \frac{n+l}{2}$. Since $\gamma(T) = \frac{n+l}{2}$, it follows that $\gamma(T') = \frac{n'+l'}{2}$ and $\gamma(T) = \frac{n+l}{2}$. Then $T \in \Gamma$. Since $x$ is a support vertex of $T'$, $x \in A(T')$. By Lemma 8, there exists a $\gamma(T')$-set $D'$ such that $x, z \in D'$ and $\text{PN}(x, D') = \emptyset$, where $z \in N_{T'}(x) \cap C(T')$. Then $D'$ is a locating-total dominating set of $T$. So, $\gamma(T) = \gamma(T')$, which is a contradiction.

Thus, we can assume that every support vertex of $T$ is adjacent to exactly one leaf. We now root $T$ at a vertex $x$ of maximum eccentricity. Let $v$ be a support vertex at maximum distance from $u$, $u$ be the parent of $v$, and $w$ be the parent of $u$ in the rooted tree. For any vertex $x \in V(T)$, let $T_x$ denote the subtree induced by the vertex $x$ and its descendants in the rooted tree $T$. By a similar proof as Case 1 of Theorem 9, it follows that $d_T(u) = 2$. We have the following two cases.

Case 1: $d_T(u) = 2$ and $d_T(w) \geq 3$. Let $T' = T - T_u$. Then $n' = n - 3$ and $l' = l - 1$. Let $D'$ be a $\gamma(T')$-set. Then $D' \cup \{u, v\}$ is a locating-total dominating set of $T$. Hence, $\gamma(T) \leq \gamma(T') + 2 \leq \frac{n'+l'}{2} + 2 < \frac{n+l}{2}$.

Since $\gamma(T) = \frac{n+l}{2}$, it follows that $\gamma(T') = \frac{n'+l'}{2}$ and $\gamma(T) = \gamma(T') + 2$. By the inductive hypothesis, $T' \in \Gamma$.

By Lemma 8, if $w \in A(T')$, there exists a $\gamma(T')$-set $D'$ such that $w, y \in D'$ and $\text{PN}(w, D') = \emptyset$, where $y \in N_{T'}(w) \cap C(T')$. Then $(D' - \{y\}) \cup \{u, v\}$ is a locating-total dominating set of $T$. So, $\gamma(T) \leq \gamma(T') + 1$, which is a contradiction.

If $w \in C(T') - C(T') \cap L(T')$, there exists a $\gamma(T')$-set $D'$ such that $w, y \in D'$ and $\text{PN}(w, D') = \emptyset$, where $y \in N_{T'}(w) \cap A(T')$. Then $(D' - \{y\}) \cup \{u, v\}$ is a locating-total dominating set of $T$. So, $\gamma(T) \leq \gamma(T') + 1$, which is a contradiction. Hence, $w \in B(T')$. Then $T$ is obtained from $T'$ by using operation $t_2$. So, $T \in \Gamma$.

Case 2: $d_T(u) = 2$ and $d_T(w) = 2$. Let $T' = T - T_u$, and let $y \in N_{T'}(w) \setminus \{u\}$. Then $n' = n - 4$ and $l' \leq l$. Let $D'$ be a $\gamma(T')$-set. Then $D' \cup \{u, v\}$ is a locating-total dominating set of $T$. Hence, $\gamma(T) = \gamma(T') + 2 \leq \frac{n'+l'}{2} + 2 \leq \frac{n+l}{2}$.

Since $\gamma(T) = \frac{n+l}{2}$, it follows that $\gamma(T') = \frac{n'+l'}{2}$ and $l' = l$. That is, $y$ is a leaf in $T'$ and $y \in C(T')$. By the inductive hypothesis, $T' \in \Gamma$. Then $T$ is obtained from $T'$ by using operation $t_1$. So, $T \in \Gamma$. □

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References