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Some structural properties of planar graphs and their applications to 3-choosability[☆]

Min Chen [1], Mickaël Montassier [*], André Raspaud

LaBRI UMR CNRS 5800, University of Bordeaux, 33405 Talence Cedex, France

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ABSTRACT

In this article, we consider planar graphs in which each vertex is not incident to some cycles of given lengths, but all vertices can have different restrictions. This generalizes the approach based on forbidden cycles which corresponds to the case where all vertices have the same restrictions on the incident cycles. We prove that a planar graph G is 3-choosable if it is satisfied one of the following conditions:

(1) G has no cycles of length 4 or 9 and no 6-cycle is adjacent to a 3-cycle. Moreover, for each vertex x, there exists an integer $i_x \in \{5, 7, 8\}$ such that x is not incident to cycles of length i_x.
(2) G has no cycles of length 4, 7, or 9, and for each vertex x, there exists an integer $i_x \in \{5, 6, 8\}$ such that x is not incident to cycles of length i_x.

This result generalizes several previously published results (Zhang and Wu, 2005 [12], Chen et al., 2008 [3], Shen and Wang, 2007 [6], Zhang and Wu, 2004 [13], Shen et al., 2011 [7]).

1. Introduction

Only simple graphs are considered in this paper unless otherwise stated. A *plane graph* is a particular drawing of a planar graph in the euclidean plane. For a plane graph G, we denote its vertex set, edge set, face set and minimum degree by $V(G)$, $E(G)$, $F(G)$ and $\delta(G)$, respectively. A *proper vertex coloring* of G is an assignment c of integers (or labels) to the vertices of G such that $c(u) \neq c(v)$ if the vertices u and v are adjacent in G. A graph G is *L-list colorable* if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there is a proper coloring c of the vertices such that $\forall v \in V(G)$, $c(v) \in L(v)$. If G is L-list colorable for every list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then G is said to be *k-choosable*.

Thomassen [8] proved that every planar graph is 5-choosable, whereas Voigt [9] proved that there exist planar graphs which are not 4-choosable. On the other hand, in 1976, Steinberg conjectured that every planar graph without cycles of lengths 4 and 5 is 3-colorable (see Problem 2.9 [5]). This conjecture remains widely open. In 1990, Erdős suggested the following relaxation of Steinberg's conjecture: what is the smallest integer i such that every graph without j-cycles for $4 \leq j \leq i$ is 3-colorable. The best known upper bound is $i \leq 7$ [2]. It is natural to ask the same question for choosability:

Problem 1. What is the smallest integer i such that every graph without j-cycles for $4 \leq j \leq i$ is 3-choosable?

Voigt [10] proved that it is not possible to extend Steinberg's conjecture to list coloring: she gave a planar graph without 4-cycles and 5-cycles which is not 3-choosable; hence $i \geq 6$. The best known upper bound is $i \leq 9$: this bound is obtained by using a structural lemma of Borodin [1].

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* Corresponding author.
E-mail addresses: chen@labri.fr (M. Chen), mickael.montassier@labri.fr, montassi@labri.fr (M. Montassier), andre.raspaud@labri.fr (A. Raspaud).
1 Present address: School of Mathematical Sciences, Soochow University, Suzhou 215006, China.

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Lemma 1 ([1]). *Let G be a planar graph with minimum degree at least* 3*. If G does not contain cycles of lengths* 4*to* 9*, then G contains a* 10*-face incident to ten* 3*-vertices and adjacent to five* 3*-faces.*

It follows by Erdős et al. [4] that every planar graph without cycles of lengths 4 to 9 is 3-choosable. Zhang and Wu [12] improved Borodin's result by proving that:

Lemma 2 ([12]). *Let G be a planar graph with minimum degree at least* 3*. If G does not contain cycles of lengths* 4, 5, 6*, and* 9*, then G contains a* 10*-face incident to ten* 3*-vertices and adjacent to five* 3*-faces.*

It implies that every planar graph without cycles of lengths 4, 5, 6, 9 is 3-choosable. Chen et al. [3] proved that every planar graph without cycles of lengths 4, 6, 7, 9 is 3-choosable. Their result is based on the following lemma:

Lemma 3 ([3]). *Let G be a planar graph with minimum degree at least* 3*. If G contains neither cycles of lengths* 4, 7, 9*, nor* 6*-cycle with a chord, then G contains a* 10*-face incident to ten* 3*-vertices or an* 8*-face incident to eight* 3*-vertices.*

Shen and Wang [6] proved that every planar graph without cycles of lengths 4, 6, 8, 9 is 3-choosable by showing that:

Lemma 4 ([6]). *Let G be a planar graph with minimum degree at least* 3*. If G does not contain cycles of lengths* 4, 6, 8*, and* 9*, then G contains a* 10*-face incident to ten* 3*-vertices.*

Moreover every planar graph without cycles of lengths 4, 5, 7, 9 (resp. 4, 5, 8, 9, and 4, 7, 8, 9) is 3-choosable [13] (resp. [11,7]).

In this article, we consider planar graphs in which each vertex is not incident to some cycles of given lengths, but all vertices can have different restrictions. This generalizes the approach based on forbidden cycles which corresponds to the case where all vertices have the same restrictions on the incident cycles. Let us introduce some notations which will allow to present our main result.

Some notations. For $x \in V(G) \cup F(G)$, let $d_G(x)$, or simply $d(x)$, denote the degree of x in G. A *k-vertex*, *k^--vertex*, or *k^+-vertex* is a vertex of degree k, at most k, or at least k. Similarly, we can define k-face, k^--face, k^+-face, etc. We say that two cycles (or faces) are *incident* if they share at least one common vertex. Suppose that f and f' are two adjacent faces by sharing a common edge e. We say that f and f' are *normally adjacent* if $|V(f) \cap V(f')| = 2$. For a face $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $b(f)$ appearing in a boundary walk of f. The degree of a face is the number of edge-steps in its boundary walk. Note that each cut-edge is counted twice. A face f is *simple* if $b(f)$ forms a cycle. A *triangle* is synonymous with a 3-face.

A vertex or edge is called *triangular* if it is contained in a 3-face. A cycle C or a face f is called *nontriangular* if it is not adjacent to any 3-cycles. We say that an i-face f is an *i^*-face* if f is adjacent to exactly one 3-face and they are normally adjacent. Moreover, we call such an i^*-face *heavy*. Similarly, we say that an i-cycle C is an *i^*-cycle* if C is adjacent to exactly one 3-cycle and they are normally adjacent. Similarly, we call such an i^*-cycle *heavy*. Two i^*-cycles (or i^*-faces) are *normally adjacent* if these two i-cycles (or i-faces) are normally adjacent. Suppose that v is a 4-vertex incident to two non-adjacent cycles C_1 and C_2 (or faces f_1 and f_2). We say that C_1 and C_2 (or f_1 and f_2) are *opposite* by the vertex v.

An *orchid* is a simple 6-face incident to six 3-vertices and normally adjacent to a 3-face. A *sunflower* is a simple 8-face incident to eight 3-vertices and adjacent to at least seven 5-faces. A *lotus* is a simple 10-face f incident to ten 3-vertices and adjacent to five clusters that are mutually disjoint with respect to f, where a cluster is either a 3-face, or a 5-face, or a 6^*-face (see Fig. 1). Here we say that two clusters, i.e., f_1, f_2, are *mutually disjoint* with respect to f if $b(f)$ does not contain two consecutive edges e_1, e_2 such that $e_i \in b(f_i)$ for each $i = 1, 2$. We should point out that none of orchids, sunflowers and lotus has an external chords in our definition. The following theorem is our main result which implies Lemmas 1–4.

Theorem 1. *Let G be a plane graph with minimum degree at least* 3 *and G does not contain* 4*-cycles and* 9*-cycles. If G further satisfies the following structural properties:*

(C1) *a* 5*-cycle or* 6*-cycle is adjacent to at most one* 3*-cycle;*
(C2) *a* 5^**-cycle is neither adjacent to a* 5^**-cycle normally, nor adjacent to an i-cycle with* $i \in \{7, 8\}$*;*
(C3) *a* 6^**-cycle is neither adjacent to a* 6*-cycle, nor incident to an i-cycle C with* $i \in \{3, 5\}$*, where C is opposite to such a* 6^**-cycle by a* 4*-vertex;*
(C4) *a nontriangular* 7*-cycle is not adjacent to two* 5*-cycles which are normally adjacent;*
(C5) *a* 7^**-cycle is neither adjacent to a* 5*-cycle nor a* 6^**-cycle.*

Then G contains an orchid or a sunflower or a lotus.

We obtain the following Corollaries 1 and 2 by Theorem 1.

Corollary 1. *Let G be a planar graph. Suppose G has no cycles of length* 4 *or* 9 *and no* 6*-cycle is adjacent to a* 3*-cycle. Moreover, for each vertex x, there exists an integer* $i_x \in \{5, 7, 8\}$ *such that x is not incident to cycles of length* i_x*. Then G is* 3*-choosable.*

Corollary 2. *Let G be a planar graph. Suppose G has no cycles of length* 4, 7*, or* 9*, and for each vertex x, there exists an integer* $i_x \in \{5, 6, 8\}$ *such that x is not incident to cycles of length* i_x*. Then G is* 3*-choosable.*

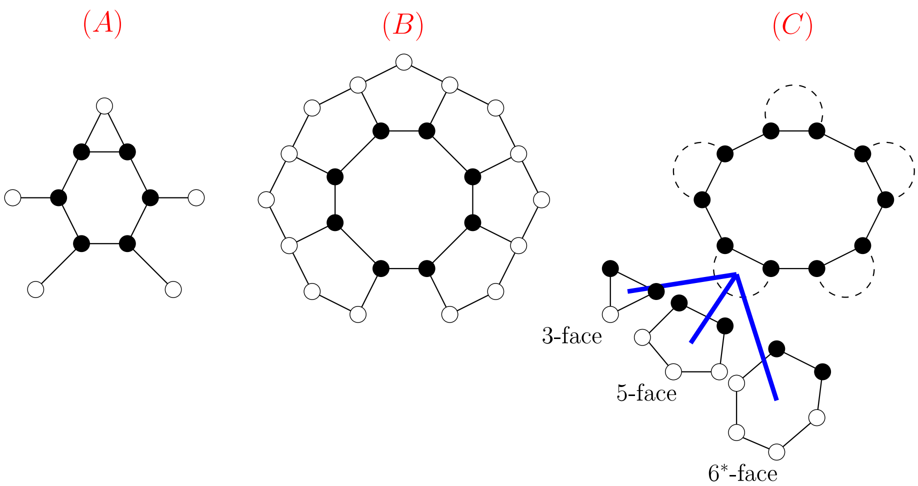

Fig. 1. (A) Orchid, (B) sunflower, and (C) lotus.

Assuming Theorem 1, we can easily prove Corollaries 1 and 2.

Proofs of Corollaries 1 and 2. Suppose that G_1, G_2 are plane presentations of the counterexamples to Corollary 1, Corollary 2 with the smallest number of vertices, respectively. Thus, G_i is connected ($i = 1, 2$). Obviously, for each $i \in \{1, 2\}$, we observe that $\delta(G_i) \geq 3$. Otherwise, let u_i be a vertex of minimum degree in G_i. By the minimality of G_i, $G_i - u_i$ is 3-choosable. Obviously, we can extend any L-coloring such that $\forall x \in V(G) : |L(x)| \geq 3$ of $G_i - u_i$ to G_i and ensure that G_i is 3-choosable. Next, in each case, we will show that each G_i contains either an orchid, or a sunflower, or a lotus. Denote N_A, N_B, N_C be the set of black vertices of (A)–(C) in Fig. 1, respectively. Since even cycles are 2-choosable, for each $j \in \{A, B, C\}$, one can easily observe that we can extend any L-coloring such that for all $x \in V(G) : |L(x)| \geq 3$ of $G_i - N_j$ to N_j and make sure that G_i is 3-choosable. Thus, G_1 and G_2 are both 3-choosable, which are contradictions.

Since G_i does not contain 4-cycles and 9-cycles, we only need to verify if G_i satisfies all the structural properties (C1)–(C5), where $i \in \{1, 2\}$.

(1) For G_1, since each vertex x is not incident to 6-cycles adjacent to a 3-cycle, each 5-cycle or 6-cycle only can be nontriangular cycles. This implies that there is neither 5^*-face nor 6^*-face in G_1. Thus, (C1)–(C3) are satisfied. If one of (C4) or (C5) is not satisfied, then in both cases there appears a vertex x which is incident to an i-cycle for all $i \in \{5, 7, 8\}$, which contradicts the assumption on G_1.

(2) For G_2, because it does not contain 7-cycles, we confirm that there is no 6^*-cycle and 7^*-cycle in G_2. Thus, we only need to check properties (C1) and (C2). It is easy to establish a 7-cycle or a 4-cycle if a 5-cycle or 6-cycle is adjacent to at least two 3-cycles. Thus, (C1) is satisfied. Let us check (C2). Suppose a 5^*-cycle is normally adjacent to another 5^*-cycle or is adjacent to an i-cycle with $i \in \{7, 8\}$. Since G_2 has no 7-cycles, in both cases there exists a vertex incident to a 5-cycle, a 6-cycle and an 8-cycle, which is a contradiction.

This completes the proofs of Corollaries 1 and 2. □

By Corollary 1, it is easy to deduce Corollary 3:

Corollary 3. *Every planar graph G in which every vertex v is not incident to cycles of lengths 4, 6, 9, i_x with $i_x \in \{5, 7, 8\}$ is 3-choosable.*

Thus, by Corollaries 2 and 3, we deduce Corollary 4 which covers five results mentioned before [12,3,6,13,7].

Corollary 4. *Every planar graph G without $\{4, i, j, 9\}$-cycles with $5 \leq i < j \leq 8$ and $(i, j) \neq (5, 8)$ is 3-choosable.*

Section 2 is dedicated to the proof of Theorem 1.

2. Proof of Theorem 1

Let G be a counterexample to Theorem 1, i.e., an embedded plane graph G with $\delta(G) \geq 3$, no cycles of lengths 4 and 9, satisfying the structural properties (C1)–(C5), and containing no orchid, no sunflower, and no lotus (i.e., none of the configurations depicted by Fig. 1).

2.1. The case G is 2-connected

First, we suppose that G is 2-connected. Thus, every face in G is simple. Besides, the following assertions (O1)–(O7) hold naturally by the assumption of G.

(O1) A 5-face or a 6-face is adjacent to at most one 3-face;
(O2) A 5^*-face is neither adjacent to a 5^*-face normally, nor adjacent to an i-face with $i \in \{7, 8\}$;

(O3) A 6^*-face is neither adjacent to a 6-face, nor incident to an i-face f with $i \in \{3, 5\}$, where f is opposite to such a 6^*-face by a 4-vertex;
(O4) A nontriangular 7-face is not adjacent to two 5-faces which are normally adjacent (there is no 3-vertex incident to a nontriangular 7-face and to two 5-faces);
(O5) A 7^*-face is neither adjacent to a 5-face nor a 6^*-face;
(O6) G does not contain 4-faces and 9-faces;
(O7) Each vertex v is incident to at most $\left\lfloor \frac{d(v)}{2} \right\rfloor$ 3-faces.

2.1.1. Structural properties of G

In this section, we will show that the following additional properties hold:

Claim 1. *For some fixed $i \in \{5, 6, 7, 8\}$, if an i-face is adjacent to a 3-face, then they are normally adjacent.*

Proof. Suppose the claim is false. Let $f_i = [v_1 v_2 \cdots v_i]$ be an i-face and $f_2 = [v_1 v_2 u]$ be a 3-face such that f_1 is adjacent to f_2 and $|V(f_1) \cap V(f_2)| = 3$. It means that u is equal to some v_j with $j \in \{3, 4, \ldots, i\}$. According to the value of i, one can easily observe that if u is a vertex v_j with $3 \le j \le i$, then G contains either a 2-vertex or a 4-cycle, a contradiction. This completes the proof of Claim 1. □

Since G does not contain 9-cycles, we obtain the following Claims 2 and 3 easily by Claim 1:

Claim 2. *Each 7-face is adjacent to at most one 3-face.*

Claim 3. *No 8-face is adjacent to a 3-face.*

Claim 4. *If two 5-faces are adjacent to each other, then they can only be normally adjacent.*

Proof. Suppose that there are two adjacent 5-faces $f_1 = [v_1 v_2 \cdots v_5]$ and $f_2 = [v_1 v_2 uvw]$ with $v_1 v_2$ as a common edge. If $|V(f_1) \cap V(f_2)| = 2$, then Claim 4 follows. Otherwise, by symmetry, we only need to consider the following cases. If $w = v_5$, then $d(v_1) = 2$ which is impossible. If $w = v_4$, then G contains a 4-cycle $v_1 v_2 v_3 v_4 v_1$, a contradiction. This implies $u \notin b(f_1)$ and $w \notin b(f_1)$. If $v = v_5$ or $v = v_4$, then a 4-cycle $uv_2 v_1 v_5 u$ or $wv_1 v_5 v_4 w$ can be easily established, a contradiction, that completes the proof of Claim 4. □

Claim 5. *A nontriangular 5-face cannot be adjacent to a 5^*-face in G.*

Proof. Suppose on the contrary that a nontriangular 5-face $f_1 = [v_1 v_2 \cdots v_5]$ is adjacent to a 5^*-face $f_2 = [v_1 v_2 u_3 u_4 u_5]$ by a common edge $v_1 v_2$. By definition, f_1 is not adjacent to any 3-face. By Claim 4, each u_i cannot be equal to some v_j with $i, j \in \{3, 4, 5\}$. By symmetry, we have to handle the following two properties:

- Assume that $v_1 u_5 u$ is a 3-face. By Claim 1, $u \neq v_2, u_3, u_4$. Moreover, $u \neq v_5$ by choice of f_1. If $u = v_4$ or $u = v_3$, then G contains a 4-cycle, which is impossible. Thus, u does not belong to $b(f_1) \cup b(f_2)$ and G contains a 9-cycle $uv_1 v_5 v_4 v_3 v_2 u_3 u_4 u_5 u$, a contradiction.
- Assume that $u_5 u_4 u$ is a 3-face. Notice that $u \neq v_1, v_2, u_3$ by Claim 1. If $u = v_3$ or v_4 or v_5, then G contains a 4-cycle which is impossible. Thus, u does not belong to $b(f_1) \cup b(f_2)$ and G contains a 9-cycle $uu_5 v_1 v_5 v_4 v_3 v_2 u_3 u_4 u$, a contradiction, that completes the proof of Claim 5. □

By Claim 4 and assertion (O2), we have:

Claim 6. *There is no adjacent two 5^*-faces in G.*

Claim 7. *No 3-vertex is incident to three 5-faces.*

Proof. Assume to the contrary that G contains a 3-vertex u adjacent to three vertices v_1, v_2, v_3 and incident to three 5-faces $f_1 = [uv_1 x_1 x_2 v_2]$, $f_2 = [uv_2 y_1 y_2 v_3]$, and $f_3 = [uv_3 z_1 z_2 v_1]$. By Claim 4, f_i and f_j are normally adjacent for each pair $\{i, j\} \subset \{1, 2, 3\}$. It implies that all vertices in $(V(f_1) \cup V(f_2) \cup V(f_3)) \setminus \{u\}$ are mutually distinct. However, a 9-cycle $v_1 x_1 x_2 v_2 y_1 y_2 v_3 z_1 z_2 v_1$ is established, contradicting the assumption on G. Thus, we complete the proof of Claim 7. □

Claim 8. *Under isomorphism, a 6-face can be adjacent to a 5-face in an unique way as depicted by Fig. 2.*

Proof. Assume that a 6-face $f_1 = [v_1 v_2 \cdots v_6]$ is adjacent to a 5-face $f_2 = [v_1 v_2 uvw]$ with $v_1 v_2$ as a common edge. We first suppose that $u, w \notin V(f_1)$. By the absence of 4-cycles in G, we deduce that $v \neq v_3$ and $v \neq v_4$. Otherwise, there is a 4-cycle either $wv_1 v_2 v_3 w$ or $uv_4 v_3 v_2 u$. So by symmetry, we have that $v \notin \{v_5, v_6\}$. However, one can easily check that a 9-cycle $v_2 v_3 v_4 v_5 v_6 v_1 wvuv_2$ is established, which is a contradiction.

Now, w.l.o.g., we may suppose that $w \in V(f_1)$. The following argument is divided into four cases.

- Assume that $w = v_6$. Then v_1 is a 2-vertex, which is a contradiction.
- Assume that $w = v_5$. Obviously, $u \neq v_3$ and $u \neq v_4$. Otherwise, either $d(v_2) = 2$ or a 4-cycle $v_1 wuv_2 v_1$ is established, which are both contradictions. So we may suppose that $u \notin V(f_1)$. If $v = v_3$, then a 4-cycle $wv_1 v_2 vw$ is formed. If $v = v_4$, then a 4-cycle $vv_3 v_2 uv$ is formed. A contradiction is always obtained, which implies that $v \notin V(f_1)$ and thus we are done, see Fig. 2.

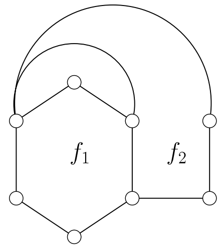

Fig. 2. A 6-face f_1 is adjacent to a 5-face f_2.

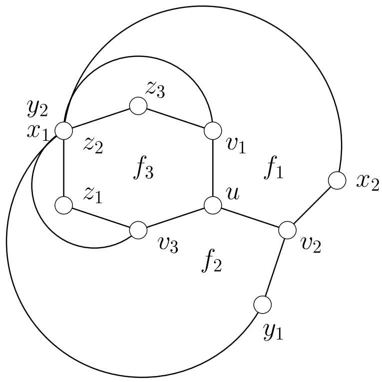

Fig. 3. A 3-vertex u incident to two 5-faces f_1 and f_2 and to one 6-face f_3.

- Assume that $w = v_4$. Then a 4-cycle $v_1v_6v_5wv_1$ is constructed, which is impossible.
- Assume that $w = v_3$. Since G is the plane graph, we see that $u, v \notin V(f_1)$. However, v_2uvwv_2 is a 4-cycle, which is a contradiction.

This completes the proof of Claim 8. □

Claim 9. *No 3-vertex is incident to two 5-faces and one 6-face.*

Proof. Suppose the claim is not true. We assume that there exists a 3-vertex u adjacent to three vertices v_1, v_2, v_3 and incident to two 5-faces $f_1 = [uv_1x_1x_2v_2]$, $f_2 = [uv_2y_1y_2v_3]$, and one 6-face $f_3 = [uv_3z_1z_2z_3v_1]$, see Fig. 3.

By Claim 8, $z_2 = y_2 = x_1$. Hence a 4-cycle $z_2v_1uv_3z_2$ exists which is a contradiction. Thus, we complete the proof of Claim 9. □

Claim 10. *No 3-vertex is incident to one 5-face and two 6-faces.*

Proof. Suppose to the contrary that there exists a 3-vertex u adjacent to three vertices v_1, v_2, v_3 and incident to two 6-faces $f_1 = [uv_3y_1y_2y_3v_1]$, $f_2 = [uv_2z_1z_2z_3v_3]$, and one 5-face $f_3 = [uv_1x_1x_2v_2]$, see Fig. 4. By Claim 8, we see that f_1 and f_3 can only be adjacent to each other in an unique way as depicted by Fig. 2. One can easily observe that $x_1 = y_2$ or $v_2 = y_1$. Next, we will make use of contradictions to show that f_2 cannot exist in G. We have to deal with the following two cases.

Case 1. $x_1 = y_2$.

For simplicity, denote $x^* = x_1 = y_2$. By Claim 8, we see that $x_2 = z_2$. It is easy to see that a 5-face $x^*v_1uv_2x_2x^*$ adjacent to two 3-cycles $x^*y_3v_1x^*$ and $v_2z_1x_2v_2$ is produced. This contradicts (C1).

Case 2. $v_2 = y_1$.

Clearly, uv_3y_1 is a 3-cycle which is not a 3-face. For simpleness, let $y^* = v_2 = y_1$. Obviously, $\{z_1, z_2, z_3\} \cap \{y_2, y_3, x_1, x_2\} = \emptyset$ since G is a plane graph. However, a 9-cycle $y^*z_1z_2z_3v_3uv_1x_1x_2y^*$ is easily established, which is impossible. This completes the proof of Claim 10. □

Claim 11. *No 6^*-face is adjacent to a 5-face in G.*

Proof. Suppose on the contrary that there exists a 6-face $f_1 = [v_1v_2 \cdots v_6]$ adjacent to a 5-face $f_2 = [v_1v_2uvw]$ by a common edge v_1v_2. By Claim 8, f_1 and f_2 can only be adjacent in an unique way depicted by Fig. 2, which means that $w = v_5$. Note that f_1 is adjacent to a 3-cycle $v_1v_5v_6v_1$ which is not a 3-face. Thus, f_1 cannot be adjacent to any other 3-face by (C1), which means that f_1 cannot be a 6^*-face. This completes the proof of Claim 11. □

By (C1), similarly as the proof of Claim 11 we have:

Claim 12. *No 5^*-face is adjacent to a 6-face in G.*

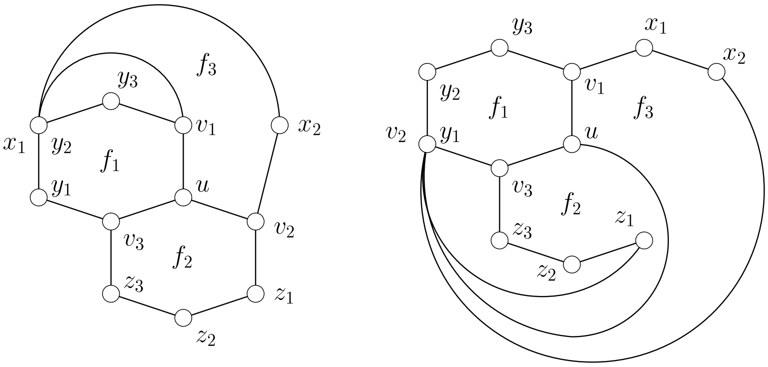

Fig. 4. A 3-vertex u incident to one 5-face f_3 and two 6-faces f_1 and f_2.

Furthermore, (O3) implies the following claim:

Claim 13. *There is no adjacent* 6^**-faces in G.*

By Claims 5, 6, 12, (O2) and (O6), we easily obtain the following claim:

Claim 14. *No* 5^**-face is adjacent to an i-face in G, where* $i \in \{4, \ldots, 9\}$.

2.1.2. Discharging argument

We now use a discharging procedure. First we define a weight function ω on the vertices and faces of G by letting $\omega(v) = 2d(v) - 6$ if $v \in V(G)$ and $\omega(f) = d(f) - 6$ if $f \in F(G)$. It follows from Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the relation $\sum_{v\in V(G)} d(v) = \sum_{f\in F(G)} d(f) = 2|E(G)|$ that the total sum of weights of the vertices and faces is equal to

$$\sum_{v\in V(G)} (2d(v) - 6) + \sum_{f\in F(G)} (d(f) - 6) = -12. \tag{1}$$

We shall design appropriate discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. The total sum of weights is kept fixed when the discharging is in process. Nevertheless, after the discharging is complete, the new weight function satisfies $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to the following obvious contradiction,

$$-12 = \sum_{x\in V(G)\cup F(G)} \omega(x) = \sum_{x\in V(G)\cup F(G)} \omega^*(x) \geq 0$$

and hence demonstrates that no such counterexample G exists.

Before stating the discharging rules, we first give some notations which will be used frequently in the following argument. For $x, y \in V(G) \cup F(G)$, let $\tau(x \to y)$ denote the amount of weights transferred from x to y. For a vertex $v \in V(G)$ and for an integer $i \geq 5$, let $m_3(v)$, $m_i(v)$, and $m_{i^*}(v)$ denote the number of 3-faces, nontriangular i-faces, and heavy i-faces incident to v, respectively. Furthermore, we denote $M_i(v) = m_i(v) + m_{i^*}(v)$ and call a face f a *non-3-face* if $d(f) \neq 3$.

For simplicity, we say an edge uv a (b_1, b_2)*-edge* if $d(u) = b_1$ and $d(v) = b_2$. Let $f_1 = [xuvy\cdots]$ and $f_2 = [zuvt\cdots]$ be two faces adjacent to each other by a common edge uv, where f_1 is a 7^+-face while f_2 is a 5- or 5^*- or 6^*-face. If both zu and vt are non-triangular edges of f_2, then we call uv a *good common edge*. We further say such uv a *good common* (b_1, b_2)*-edge* if uv is a (b_1, b_2)-edge.

The discharging rules are defined as follows (see Fig. 5):

(R1) Each 5^+-face sends 1 to each adjacent 3-face.
(R2) Let v be a 4-vertex.
- (R2a) If $m_3(v) = 2$, then for each non-3-face f, $\tau(v \to f) = 1$.
- (R2b) If $m_3(v) = 1$, then let f_1 denote the incident 3-face and f' be the opposite face of f_1.
  - (R2b1) If f' is a nontriangular 5-face, then v sends $\frac{2}{3}$ to each incident face different of f_1.
  - (R2b2) Otherwise, v sends 1 to each incident face which is adjacent to f_1.
- (R2c) If $m_3(v) = 0$, let f_1, f_2, f_3, and f_4 denote the faces of G incident to v in a cyclic order such that the degree of f_1 is the smallest one among all faces incident to v, then we do like this:
  - (R2c1) if $M_5(v) = 0$, then v sends $\frac{1}{2}$ to each incident face.
  - (R2c2) if $M_5(v) = 1$, then v sends $\frac{2}{3}$ to each of f_1, f_2, and f_4 when f_1 is a nontriangular 5-face; or v sends 1 to each of f_2 and f_4 when f_1 is a 5^*-face.
  - (R2c3) if $M_5(v) = 2$, then
    - (R2c3.1) v sends $\frac{2}{3}$ to each nontriangular 5-face and $\frac{1}{3}$ to each other incident face when $m_5(v) = 2$.

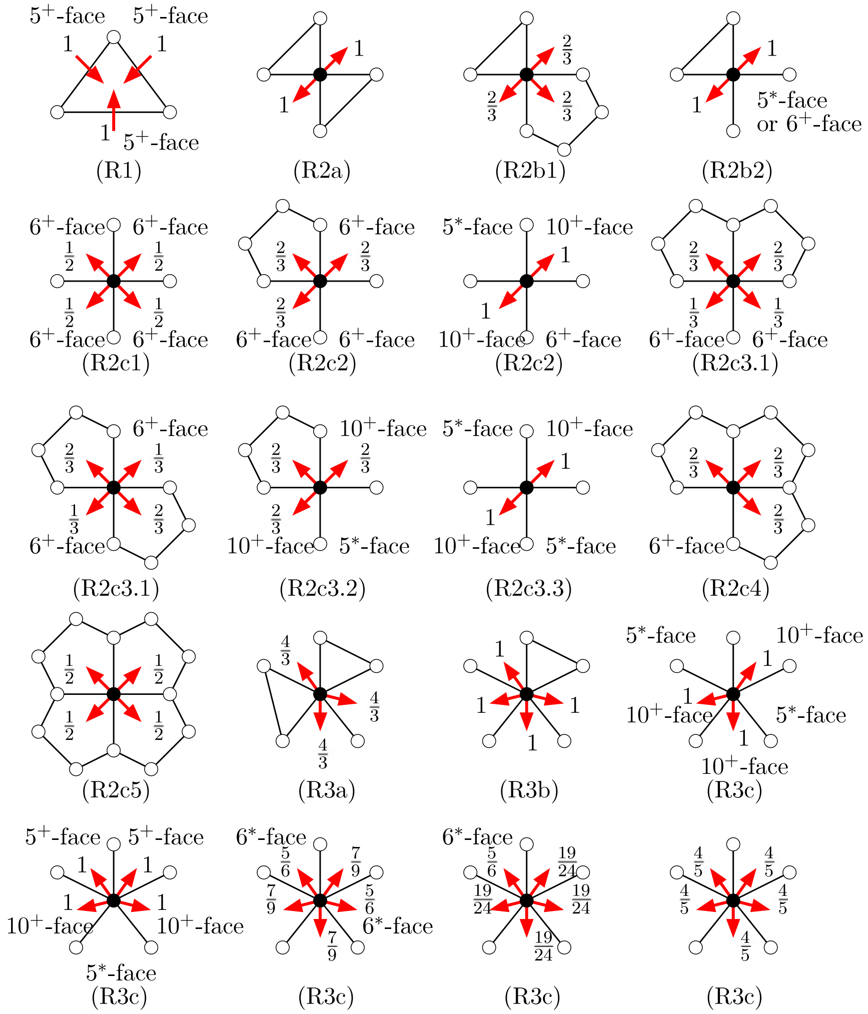

Fig. 5. Some of discharging rules (R1)–(R3).

(R2c3.2) v sends $\frac{2}{3}$ to each incident face except the unique 5^*-face when $m_5(v) = 1$ and $m_{5^*}(v) = 1$.
(R2c3.3) v sends 1 to each incident face which is not a 5^*-face when $m_{5^*}(v) = 2$.
(R2c4) if $M_5(v) = 3$, then v gives $\frac{2}{3}$ to each incident nontriangular 5-face.
(R2c5) if $M_5(v) = 4$, then v gives $\frac{1}{2}$ to each incident nontriangular 5-face.

(R3) Let v be a 5-vertex and f be a non-3-face incident to v. Then
(R3a) $\tau(v \to f) = \frac{4}{3}$ if $m_3(v) = 2$.
(R3b) $\tau(v \to f) = 1$ if $m_3(v) = 1$.
(R3c) if $m_3(v) = 0$, v sends 1 to each incident face different from a 5^*-face when $m_{5^*}(v) \geq 1$; or sends $\frac{5}{6}$ to each incident 6^*-face and sends $\frac{4-\frac{5}{6}m_{6^*}(v)}{5-m_{6^*}(v)}$ to each other incident face when $m_{5^*}(v) = 0$.

(R4) Let f be a 7^+-face. If f' is adjacent to f by a good common edge e, then
(R4a) $\tau(f \to f') = \frac{1}{3}$ if f' is a nontriangular 5-face and e is a (3, 3)-edge.
(R4b) $\tau(f \to f') = \frac{1}{6}$ if f' is a 6^*-face and e is a (3, 3)-edge or a (3, 4)-edge.

(R5) Each 10^+-face sends 1 to each adjacent 5^*-face by a good common $(3^+, 3^+)$-edge.

(R6) Each 6^+-vertex sends 1 to each incident face.

Let us check that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$.

Since $\delta(G) \geq 3$, $d(v) \geq 3$ for each vertex $v \in V(G)$. We have to handle the following cases, depending on $d(v)$.

Case 1. $d(v) = 3$.

It is easy to see that $\omega^*(v) = \omega(v) = 2 \times 3 - 6 = 0$ by (R1)–(R6).

Case 2. $d(v) = 4$.

Clearly, $\omega(v) = 2$ and v is incident to at most two 3-faces by (O7). If $m_3(v) = 2$, then we deduce that $\omega^*(v) = 2-2\times 1 = 0$ by (R2a). If $m_3(v) = 1$ (v is incident to exactly one 3-face), then depending on the opposite face of such a 3-face, v gives either $\frac{2}{3} \times 3 = 2$, or $1 \times 2 = 2$ by (R2b1) or (R2b2). Hence, $\omega^*(v) = 0$. Finally, we only need to consider the case of $m_3(v) = 0$. We divide the discussion into five subcases in the light of the value of $M_5(v)$.

Subcase 2.1. $M_5(v) = 0$.

This implies that the degree of each face incident to v is at least 6 by the absence of 4-faces. According to (R2c1), $\omega^*(v) \geq 2 - \frac{1}{2} \times 4 = 0$.

Subcase 2.2. $M_5(v) = 1$.

It is easy to observe that v sends either $\frac{2}{3} \times 3 = 2$ if $m_5(v) = 1$, or $1 \times 2 = 2$ if $m_{5^*}(v) = 1$ by (R2c2). Thus, v gives totally at most 2 to incident faces. Hence, $\omega^*(v) \geq 2 - 2 = 0$.

Subcase 2.3. $M_5(v) = 2$.

If $m_5(v) = 2$, then $\omega^*(v) \geq 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 2 = 0$ by (R2c3.1). If $m_5(v) = m_{5^*}(v) = 1$, then such a nontriangular 5-face and 5*-face cannot be adjacent to each other by Claim 5. Thus, applying (R2c3.2), $\omega^*(v) \geq 2 - \frac{2}{3} \times 3 = 0$. Otherwise, suppose $m_{5^*}(v) = 2$. Notice that v is incident to two 5*-faces which are opposite to each other by Claim 6. Thus, $\omega^*(v) \geq 2 - 1 \times 2 = 0$ by (R2c3.3).

Subcase 2.4. $M_5(v) = 3$.

We first notice that $m_{5^*}(v) \neq 3$ since there are no adjacent 5*-faces in G by Claim 6. If $1 \leq m_{5^*}(v) \leq 2$, then there exists at least one nontriangular 5-face adjacent to one 5*-face, contradicting the Claim 5. Thus, $m_{5^*}(v) = 0$, and so $m_5(v) = 3$. According to (R2c4), we have that $\omega^*(v) \geq 2 - \frac{2}{3} \times 3 = 0$.

Subcase 2.5. $M_5(v) = 4$.

One can observe that $m_{5^*}(v) = 0$ by Claims 5 and 6. It implies that v is incident to exactly four nontriangular 5-faces. Consequently, we have that $\omega^*(v) \geq 2 - \frac{1}{2} \times 4 = 0$ by (R2c5).

Case 3. $d(v) = 5$.

Obviously, $\omega(v) = 4$ and $m_3(v) \leq 2$ by (O7). It is easy to observe that v sends either $\frac{4}{3} \times 3 = 4$ by (R3a) if $m_3(v) = 2$, or $1 \times 4 = 4$ by (R3b) if $m_3(v) = 1$. Therefore, $\omega^*(v) \geq 4 - 4 = 0$ if $m_3(v) > 0$. Now we may assume that $m_3(v) = 0$. This implies that each face incident to v is a 5^+-face combining the fact that G does not contain any 4-faces. By Claim 6, we have that $m_{5^*}(v) \leq 2$. Moreover, each non-3-face adjacent to a 5*-face must be a 10^+-face by Claim 14. So by (R3c), $\omega^*(v) \geq 4 - 1 \times 4 = 0$ if $m_{5^*}(v) \geq 1$; or $\omega^*(v) \geq 4 - \frac{5}{6}m_{6^*}(v) - \frac{4-\frac{5}{6}m_{6^*}(v)}{5-m_{6^*}(v)}(5 - m_{6^*}(v)) = 0$ if $m_{5^*}(v) = 0$.

Case 4. $d(v) \geq 6$.

According to (R6), we have that $\omega^*(v) \geq (2d(v) - 6) - 1 \times d(v) = d(v) - 6 \geq 0$.

Let $f \in F(G)$. Then $b(f)$ is a cycle since G is 2-connected. We write $f = [v_1 v_2 \cdots v_{d(f)}]$ and suppose that f_i is the face of G adjacent to f with $v_i v_{i+1} \in b(f) \cap b(f_i)$ for $i = 1, 2, \ldots, d(f)$, where (and in the following discussion) all indices are taken modulo $d(f)$. We observe that $d(f) \neq 4$ and $d(f) \neq 9$ by (O6). For $i \geq 3$, let $n_i(f)$ denote the number of i-vertices incident to f. Let $m_5(f)$, $m_{5^*}(f)$, and $m_{6^*}(f)$ denote the number of nontriangular 5-faces, heavy 5-faces, and heavy 6-faces adjacent to f.

Case 5. $d(f) = 3$.

Let f be a 3-face and then $\omega(f) = -3$. Since $\delta(G) \geq 3$, f is adjacent to three faces and each adjacent face is neither a 3-face nor a 4-face by the absence of 4-cycles in G. It implies that f gets 3×1 from its adjacent faces by (R1). Thus, $\omega^*(f) \geq -3 + 1 \times 3 = 0$.

Case 6. $d(f) = 5$.

Let $f = [v_1 \cdots v_5]$ and then $\omega(f) = -1$. Clearly, f is adjacent to at most one 3-face by (O1).

- We first assume that f is a nontriangular 5-face, which means that there is no 3-face adjacent to f. Thus, f sends nothing to all its adjacent faces. Moreover, each f_i cannot be a 5*-face by Claim 5. We only have to deal with the following three possibilities, depending on the value of $n_3(f)$.

Subcase 6.1. $n_3(f) = 5$.

It means that v_i is a 3-vertex for all $i = 1, \ldots, 5$. If there exists a 6-face adjacent to f, then by Claim 8 we see that they must be adjacent to each other in an unique way as depicted by Fig. 2. It is easy to see that there is one 4^+-vertex appeared on $b(f)$, which contradicts $n_3(f) = 5$. Thus, each face adjacent to f is either a nontriangular 5-face or a 7^+-face by Claim 5 and the absence of 4-faces. Furthermore, we notice that f is adjacent to at most two nontriangular 5-faces which are not adjacent by Claim 7. So f is adjacent to at least three 7^+-faces such that each 7^+-face is adjacent to f by a good common (3, 3)-edge. Therefore, applying (R4a), we obtain that $\omega^*(f) \geq -1 + 3 \times \frac{1}{3} = 0$.

Subcase 6.2. $n_3(f) = 4$.

Let v_1 be a 4^+-vertex and v_j be a 3-vertex for all $j = 2, 3, 4, 5$. Clearly, v_1 gives at least $\frac{1}{2}$ to f by (R2), (R3) and (R6). Moreover, f_1 and f_5 cannot be any 6-face by Claim 8. If $d(f_1) = 5$ and $d(f_5) = 5$, then $d(f_j) \notin \{5, 6\}$ with $j \in \{2, 4\}$ according to Claims 7 and 9. Thus, for $j \in \{2, 4\}$, f_j is a 7^+-face by the absence of 4-faces and each f_j is adjacent to f by a good common (3, 3)-edge. By (R4a), we see that $\tau(f_2 \to f) = \frac{1}{3}$ and $\tau(f_4 \to f) = \frac{1}{3}$. So we obtain that $\omega^*(f) \geq -1 + \frac{1}{2} + \frac{1}{3} \times 2 = \frac{1}{6} > 0$.

Now we may suppose that there exists at least one face of f_1 and f_5 which is a 7^+-face, i.e., $d(f_1) \geq 7$. Then by (R2), (R3) and (R6), we see that $\tau(v_1 \to f) \geq \frac{2}{3}$. Clearly, for each $i \in \{2, 3, 4\}$, f_i is adjacent to f by a good common (3, 3)-edge. According to Claims 7, 9 and 10, we see that there exists at least one face of f_2, f_3, f_4 which is a 7^+-face. Hence, $\omega^*(f) \geq -1+\frac{1}{3}+\frac{2}{3} = 0$ by (R4a).

Subcase 6.3. $n_3(f) \leq 3$.

It means that there are at least two vertices of degree at least 4. By (R2), (R3) and (R6), we derive that $\omega^*(f) \geq -1+\frac{1}{2}\times 2 = 0$.

- Now, we may suppose that f is a 5^*-face. It implies that f is adjacent to exactly one 3-face. Without loss of generality, let $f_1 = [vv_1v_2]$ be such a 3-face that it is adjacent to f. By Claim 1, $v \neq v_i$ for all $i = 3, 4, 5$. Since $\delta(G) \geq 3$, $d(v_i) \geq 3$ with $i \in \{1, 2, \ldots, 5\}$. By Claim 14, for each $j \in \{2, 3, 4, 5\}$, we see that $d(f_j) \geq 10$ and thus both v_3v_4 and v_4v_5 are good common $(3^+, 3^+)$-edges. By (R5), $\tau(f_3 \to f) = 1$ and $\tau(f_4 \to f) = 1$. Hence, $\omega^*(f) \geq -1 - 1 + 1 \times 2 = 0$ by (R1).

Case 7. $d(f) = 6$.

Let $f = [v_1 \cdots v_6]$ and then $\omega(f) = 0$. If f is a nontriangular 6-face, then it is easy to deduce that $\omega^*(f) = \omega(f) = 0$ by (R1)–(R6). Now, we assume that f is a 6^*-face. Without loss of generality, assume $f_1 = [vv_1v_2]$ is a 3-face adjacent to f. It is obvious that $v \notin b(f)$ by Claim 1. Furthermore, f is adjacent to at most one 3-face by (O1). So f only needs to send 1 to the unique 3-face f_1. Obviously, for each $j \in \{2, \ldots, 6\}$, $d(f_j) \notin \{3, 4, 5, 6\}$ by (O1), (O6), Claim 11 and (O3). Note that $v_3v_5 \notin E(G)$ and $v_3v_6 \notin E(G)$ by (C1) and the absence of 4-cycles. This implies that each v_i has at least one outgoing neighbor which does not lie on $b(f)$. Since there is no orchid in G, f is incident to at least one 4^+-vertex. It implies that $n_3(f) \leq 5$. Next, in each case, we will show that the total charge obtained by f is at least 1 and thus $\omega^*(f) \geq -1 + 1 = 0$.

Subcase 7.1. $n_3(f) = 5$.

It means that there is exactly one 4^+-vertex incident to f. If $d(v_2) \geq 4$, then $\tau(v_2 \to f) \geq 1$ by (R2b2), (R3a), (R3b) and (R6) since $d(f_2) \neq 5$. Otherwise, by symmetry, suppose some v_i is a 4^+-vertex where $i \in \{3, 4\}$. Denote v^* be such a 4^+-vertex. First, we observe that each adjacent face different from f_1 is a 7^+-face by the discussion above. If $d(v^*) \geq 5$, then $\tau(v^* \to f) \geq \frac{5}{6}$ by (R3) and (R6). Since v_5v_6 is a good common (3, 3)-edge, f_5 sends $\frac{1}{6}$ to f by (R4b). Thus, f gets at least $\frac{5}{6} + \frac{1}{6} = 1$ from v^* and f_5. If $d(v^*) = 4$, then the opposite face of f, which is incident to f by v^*, cannot be a 3-face or a 5-face by (O3). So v^* is incident to four 6^+-faces and thus v^* gives $\frac{1}{2}$ to f by (R2c1). Consequently, f gets at least $\frac{1}{2} + \frac{1}{6} \times 3 = 1$ by (R4b).

Subcase 7.2. $0 \leq n_3(f) \leq 4$.

It implies that there are at least two 4^+-vertices incident to f. It is easy to see that every 5^+-vertex sends at least $\frac{5}{6}$ to f by (R3) and (R6). Moreover, every 4-vertex v_i sends at least $\frac{1}{2}$ to f since the opposite face to f by v_i cannot be any 3-face or 5-face by (O3). Hence, f receives at least $\frac{1}{2} \times 2 = 1$ from its incident 4^+-vertices.

In what follows, for simplicity, let $p_5(f)$, $p_{5^*}(f)$, and $p_{6^*}(f)$ denote the number of nontriangular 5-faces, 5^*-faces, and 6^*-faces receiving a charge $\frac{1}{3}$, 1, $\frac{1}{6}$ from f, respectively. Clearly, $p_5(f) \leq m_5(f)$, $p_{5^*}(f) \leq m_{5^*}(f)$ and $p_{6^*}(f) \leq m_{6^*}(f)$.

Case 8. $d(f) = 7$.

Then $\omega(f) = 1$. Let $m_3(f)$ be the number of 3-faces adjacent to f. Clearly, $m_3(f) \leq 1$ by Claim 2.

- We first assume that f is a nontriangular 7-face. Note that $d(f_i) \geq 5$ since G contains no 4-faces. By (O2), $m_{5^*}(f) = 0$. By (O4), $p_5(f) \leq 3$. We will divide the argument into four subcases according to the value of $p_5(f)$.

Subcase 8.1. $p_5(f) = 3$.

Suppose f_1, f_3, f_5 are such three 5-faces that each of them takes a charge $\frac{1}{3}$ from f. By (R4a), we see that all common edges v_1v_2, v_3v_4 and v_5v_6 are good (3, 3)-edges. This implies that $d(v_i) = 3$ with $i \in \{1, \ldots, 6\}$. By Claim 11, one can easily conclude that none of f_2, f_4, f_6, f_7 can be a 6^*-face. Thus, $p_{6^*}(f) \leq m_{6^*}(f) = 0$. Consequently, we deduce that $\omega^*(f) \geq 1 - \frac{1}{3} \times 3 = 0$ by (R4a).

Subcase 8.2. $p_5(f) = 2$.

We may suppose that f_i is a 5-face which takes $\frac{1}{3}$ from f. It means that $d(v_i) = d(v_{i+1}) = 3$ and v_iv_{i+1} is a good common edge. Thus, f_{i-1} and f_{i+1} cannot be any 6^*-face by Claim 11. It follows immediately that $p_{6^*}(f) \leq 7 - (2 + 3) = 2$ since $p_5(f) = 2$. Consequently, we have that $\omega^*(f) \geq 1 - \frac{1}{3} \times 2 - \frac{1}{6} \times 2 = 0$ by (R4).

Subcase 8.3. $p_5(f) = 1$.

Without loss of generality, let f_1 be such a nontriangular 5-face that v_1v_2 is a good common (3, 3)-edge. This implies that neither f_2 nor f_7 can be a 6^*-face. Thus, $p_{6^*}(f) \leq 7 - 3 = 4$. Hence, we have $\omega^*(f) \geq 1 - \frac{1}{3} - \frac{1}{6} \times 4 = 0$ by (R4a) and (R4b).

Subcase 8.4. $p_5(f) = 0$.

If $p_{6^*}(f) = 0$, then according to (R4), we obtain that $\omega^*(f) \geq 1 - 0 = 1$. Otherwise, let f_1 be a 6^*-face which takes a charge $\frac{1}{6}$ from f. By (R4b), f_1 must be adjacent to f by a good common (3, 3)-edge or (3, 4)-edge, i.e., $d(v_1) = 3$ and $d(v_2) \in \{3, 4\}$. It is easy to observe that f_7 cannot be any 6^*-face because of Claim 13. Thus, $p_{6^*}(f) \leq 6$ and $\omega^*(f) \geq 1 - \frac{1}{6} \times 6 = 0$ by (R4b).

- Now we may assume $m_3(f) = 1$, which implies that f is a 7^*-face and it is adjacent to exactly one 3-face. Without loss of generality, let $f_1 = [vv_1v_2]$ be such a 3-face that f sends 1 to f_1. By Claim 1, we notice that v does not lie on $b(f)$. Moreover, for each $j \in \{2, \ldots, 7\}$, we deduce that f_j is neither a 5-face nor a 6^*-face by (O5). It implies that f sends nothing to each f_j with $j \in \{2, \ldots, 7\}$. Applying (R1), we deduce that $\omega^*(f) \geq 1 - 1 = 0$.

Case 9. $d(f) = 8$.

Clearly, $\omega(f) = 2$ and f cannot be adjacent to any 3-face by Claim 3. So we only need to consider the size of $p_5(f)$ and $p_{6^*}(f)$ since they may take charge from f. It is easy to calculate that $p_5(f) \leq 6$ by the fact that there is no sunflower in G (and 2-connectivity of G). We have to consider the following possibilities by the value of $p_5(f)$.

Subcase 9.1. $p_5(f) = 6$.

It implies that f is incident to at least seven 3-vertices. Thus, the remaining two faces adjacent to f, which are not nontriangular 5-faces, cannot be any 6^*-faces by Claim 11. So $\omega^*(f) \geq 2 - 6 \times \frac{1}{3} = 0$ by (R4).

Subcase 9.2. $p_5(f) = 5$.

One can easily notice that there is at most one of f_i with $i \in \{1, \ldots, 8\}$ which is a 6^*-face because no 5-face can be adjacent to a 6^*-face by Claim 11 again. Therefore, $\omega^*(f) \geq 2 - 5 \times \frac{1}{3} - \frac{1}{6} = \frac{1}{6} > 0$.

Subcase 9.3. $0 \leq p_5(f) \leq 4$.

By (R4), we derive that

$$\begin{aligned}\omega^*(f) &\geq 2 - \frac{1}{3}p_5(f) - \frac{1}{6}p_{6^*}(f)\\ &\geq 2 - \frac{1}{3}p_5(f) - \frac{1}{6}(8 - p_5(f))\\ &= \frac{2}{3} - \frac{1}{6}p_5(f)\\ &\geq \frac{2}{3} - \frac{1}{6} \cdot 4\\ &= 0.\end{aligned}$$

Next, we will discuss several cases where $d(f) \geq 10$. Let f be such a 10^+-face that f' is adjacent to f. We call f' *special* if it takes charge 1 from f. By $F_s(f)$ denote the set of all special faces adjacent to f. Let S_i be a face adjacent to f by an edge e_i for $i = 1, 2$. If e_1 is not incident to e_2, then we call S_1 and S_2 *mutually disjoint*. According to (R1) and (R5), we see that only 3-face and 5^*-face may take charge 1 from f, respectively. It implies that each special face is either a 3-face or a 5^*-face. We first observe that:

Observation 1. *If f is adjacent to two special faces by two consecutive edges uv and vw of $b(f)$, then $\tau(v \to f) \geq 1$.*

Proof. Let f_{uv} and f_{vw} denote such two special faces adjacent to f by sharing the common edge uv and vw, respectively. It suffices to consider the following three cases.

- Assume that f_{uv} and f_{vw} are both 3-faces. By the absence of 4-cycles, we see that $d(v) \geq 4$ and thus $\tau(v \to f) \geq 1$ by (R2a), (R3a) and (R6).
- Assume that f_{uv} and f_{vw} are both 5^*-faces. By Claim 6, $d(v) \geq 4$. So by (R2c3.3), (R3c) and (R6), we derive that $\tau(v \to f) = 1$.
- Finally, w.l.o.g., we assume that f_{uv} is a 3-face and f_{vw} is a 5^*-face. By (R5), we know that the edge vw is a good common $(3^+, 3^+)$-edge, which implies that $d(v) \geq 4$. Applying (R2b2), (R3b) and (R6), we have that $\tau(v \to f) = 1$.

This completes the proof of Observation 1. □

If there exist two special faces which share at least one common vertex v that lies on $b(f)$, i.e., let f_i and f_{i+1} be such two special faces that $v_{i+1} \in V(f_i) \cap V(f_{i+1})$ and $v_{i+1} \in V(f)$, then we see that $\tau(v_{i+1} \to f) \geq 1$ by Observation 1 and f sends at most 2×1 to f_i and f_{i+1}. It means that f takes charge 1 from v_{i+1} and then sends it to f_{i+1}. Thus, we can consider that f_{i+1} takes nothing from f. So in what follows, our main focus is on the special faces adjacent to f that are mutually disjoint. For our convenience, let $F_s^*(f)$ denote the maximal subset of $F_s(f)$ such that any two faces in $F_s^*(f)$ are mutually disjoint, and let $F_s^{**}(f) = F_s(f) \backslash F_s^*(f)$. We refer to faces in $F_s^*(f)$ and $F_s^{**}(f)$ as S^*-faces and S^{**}-faces, respectively. Obviously, $|F_s^*(f)| \leq \left\lfloor \frac{d(f)}{2} \right\rfloor$. By Observation 1 and arguments above, we can assume that faces in F_s^{**} do not take charge from f.

Observation 2. *If $f_i \in F_s^*(f)$ then neither of f_{i-1} nor f_{i+1} can be a 5- or 6^*-face which takes charge from f by (R4).*

Proof. W.l.o.g., suppose that $f_i \in F_s^*(f)$ while $f_{i-1} \notin F_s^*(f)$ and $f_{i+1} \notin F_s^*(f)$. In order to prove Observation 2, it suffices to show that f_{i-1} gets nothing from f if it is a 5- or 6^*-face.

First suppose that f_i is a 3-face. If f_{i-1} takes a charge $\frac{1}{3}$ or $\frac{1}{6}$, then by (R4a) and (R4b), we see that $d(v_i) = 4$ and f_{i-1} is a 6^*-face. This contradicts (O3). Now we assume that f_i is a 5^*-face. If $d(v_i) = 3$, then f_{i-1} cannot be any nontriangular 5-face by Claim 5 and any 6^*-face by Claim 11 and thus we are done. Now suppose that $d(v_i) \geq 4$. Note that if f_{i-1} is a nontriangular 5-face, then f sends nothing to it because $v_{i-1}v_i$ is not a (3, 3)-edge. If f_{i-1} is a 6^*-face, then we argue as follows: if v_i is a 5^+-vertex, then $\tau(f \to f_{i-1}) = 0$ since $v_{i-1}v_i$ is neither a (3, 3)-edge nor a (3, 4)-edge; if v_i is a 4-vertex, then f_i is the opposite face of f_{i-1} by a 4-vertex v_i, which contradicts (O3). This completes the proof of Observation 2. □

Corollary 5. $p_5(f) + p_{6^*}(f) \leq d(f) - 2|F_s^*(f)|$, *where the equality holds only if $d(f)$ is even and $|F_s^*(f)| = \frac{d(f)}{2}$.*

Case 10. $d(f) = 10$.

Then $\omega(f) = 4$ and $|F_s^*(f)| \le 5$. We divide the argument into the following three subcases according to the value of $|F_s^*(f)|$.

Case 10.1. $|F_s^*(f)| = 5$.

Note that $p_5(f)+p_{6^*}(f) = 0$ by Corollary 5. By definition, f is adjacent to five S^*-faces that are mutually disjoint. W.l.o.g., suppose that f_1, f_3, f_5, f_7, f_9 are all these S^*-faces. If f_j is an S^*-face for some fixed $j \in \{2, 4, 6, 8, 10\}$, then $\tau(f \to f_j) = 1$, while $\tau(v_j \to f) \ge 1$ and $\tau(v_{j+1} \to f) \ge 1$ by Observation 1. Therefore, $\omega^*(f) \ge 4 - 1 \times 5 - |F_s^{**}(f)| + 2|F_s^{**}(f)| = -1 + |F_s^{**}(f)| \ge 0$. In what follows, for each $j \in \{2, 4, 6, 8, 10\}$, we suppose that f_j is not an S^*-face. Since G does not contain lotus, there exists at least one 4^+-vertex which lies on $b(f)$, say v_1. If v_1 is a 5^+-vertex, then v_1 sends at least 1 to f by (R3) and (R6). If v_1 is a 4-vertex, then we have two cases: If $d(v_{10}) = 3$, then f_{10} is not a nontriangular 5-face since f_9 is an S^*-face. So $\tau(v_1 \to f) = 1$ by (R2b2), (R2c2) and (R2c3.3); otherwise, $d(v_{10}) \ge 4$ and f receives at least $\frac{2}{3} \times 2 = \frac{4}{3}$ from v_1 and v_{10} in total by (R2b1), (R2b2), (R2c2), (R2c3.2), (R2c3.3), (R3) and (R6). Thus, in each case, we always have that $\omega^*(f) \ge 4 - 1 \times 5 + 1 = 0$.

Case 10.2. $|F_s^*(f)| = 4$.

It implies that f is adjacent to exactly four S^*-faces by four common edges which are disjoint each other. Denote S_i be such an S^*-face adjacent to f by a common edge e_i, where $i = 1, 2, 3, 4$. Note that e_i cannot be incident to e_j for each pair $(i, j) \subset \{1, \dots, 4\}$. Thus, it follows that there exist two vertices v_j, v_k in $V(f)$ which are not incident to any edge e_i with $i \in \{1, \dots, 4\}$. W.l.o.g., assume $j < k$.

First we consider the case that $k = j + 1$. Namely, v_jv_k is an edge of $b(f)$. W.l.o.g., we assume that $v_jv_k = v_{10}v_9$ such that f_1, f_3, f_5, f_7 are S^*-faces and f_9 is not. By Observation 2, we assert that none of $f_2, f_4, f_6, f_8, f_{10}$ gets charge from f if it is a 5- or 6^*-face. It follows that $p_5(f) + p_{6^*}(f) \le 1$. If $p_5(f) + p_{6^*}(f) = 0$ then we are done since $\omega^*(f) \ge 4 - 1 \times 4 = 0$. Otherwise, suppose that f_9 is a nontriangular 5-face or a 6^*-face which gets a charge $\frac{1}{3}$ or $\frac{1}{6}$ from f, respectively. It follows that neither f_8 nor f_{10} is an S^*-face. If f_j is an S^*-face for some $j = 2, 4, 6$, then similarly we have that $\omega^*(f) \ge 4 - 1 \times 4 - \frac{1}{3} - |F_s^{**}(f)| + 2|F_s^{**}(f)| = |F_s^{**}(f)| - \frac{1}{3} \ge \frac{2}{3}$. So in the following, we assume that f_j is not an S^*-face for each $j = 2, 4, 6$. By the absence of lotus in G, there exists at least one vertex in $V(f)$ whose degree is at least 4. Let v^* be such a 4^+-vertex. W.l.o.g., we have two subcases below, according to the situation of v^*.

- If $v^* \in \{v_1, \dots, v_8\}$, then by (R2b1), (R2b2), (R2c2), (R2c3.2), (R2c3.3), (R3) and (R6) it follows that $\tau(v^* \to f) \ge \frac{1}{3}$. Thus, in each case, we always have that $\omega^*(f) \ge 4 - 1 \times 4 - \frac{1}{3} + \frac{1}{3} = 0$.
- Assume that $v^* = v_9$. Namely, $d(v_9) \ge 4$. Moreover, we may further assume that f_9 is a 6^*-face and $d(v_9) = 4$ and $d(v_{10}) = 3$ for otherwise f_9 gets nothing from f by (R4a) or (R4b). Thus v_9 sends at least $\frac{1}{2}$ to f according to (R2c1), (R2c2) and Claim 11. This yields $\omega^*(f) \ge 4 - 1 \times 4 - \frac{1}{3} + \frac{1}{2} = \frac{1}{6} > 0$.

Now we suppose that $k > j + 1$. It means that $v_kv_j \notin b(f)$. In this case, it is easy to deduce that $p_5(f) + p_{6^*}(f) = 0$ by Observation 2. In other words, f only sends charges to its S^*-faces. Therefore, $\omega^*(f) \ge 4 - 1 \times 4 = 0$ by (R1) and (R5).

Case 10.3. $0 \le |F_s^*(f)| \le 3$.

If $|F_s^*(f)| = 3$, that by Observation 2 we have that $p_5(f) + p_{6^*}(f) \le 10 - (3 + 4) = 3$. So, $\omega^*(f) \ge 4 - 3 \times 1 - \frac{1}{3} \times 3 = 0$ by (R4). If $0 \le |F_s^*(f)| \le 2$, then by Observation 2, we have that $p_5(f) + p_{6^*}(f) \le 10 - 2|F_s^*(f)|$ and therefore, $\omega^*(f) \ge 4 - |F_s^*(f)| - \frac{1}{3}(10 - 2|F_s^*(f)|) = \frac{2}{3} - \frac{1}{3}|F_s^*(f)| \ge \frac{2}{3} - \frac{1}{3} \times 2 = 0$.

Case 11. $d(f) = 11$.

Clearly, $\omega(f) = 5$ and $|F_s^*(f)| \le 5$. If $|F_s^*(f)| = 5$, then $p_5(f) + p_{6^*}(f) \le 11 - (5 + 6) = 0$. So $\omega^*(f) \ge 5 - 1 \times 5 = 0$. If $0 \le |F_s^*(f)| \le 4$, then $p_5(f) + p_{6^*}(f) \le 11 - 2|F_s^*(f)|$ by Observation 2. Then $\omega^*(f) \ge 5 - |F_s^*(f)| - \frac{1}{3}(11 - 2|F_s^*(f)|) = \frac{4}{3} - \frac{1}{3}|F_s^*(f)| \ge 0$.

Case 12. $d(f) \ge 12$.

By Observation 2, we have that $p_5(f) + p_{6^*}(f) \le d(f) - 2|F_s^*(f)|$. Moreover, $|F_s^*(f)| \le \lfloor \frac{1}{2} d(f) \rfloor$. Thus, we have that

$$
\begin{aligned}
\omega^*(f) &\ge (d(f) - 6) - |F_s^*(f)| - \frac{1}{3}(d(f) - 2|F_s^*(f)|) \\
&= \frac{2}{3}d(f) - 6 - \frac{1}{3}|F_s^*(f)| \\
&\ge \frac{2}{3}d(f) - 6 - \frac{1}{3} \cdot \frac{d(f)}{2} \\
&= \frac{1}{2}d(f) - 6 \\
&\ge \frac{1}{2} \times 12 - 6 \\
&= 0.
\end{aligned}
$$

Therefore, we complete the proof of Theorem 1 if G is 2-connected.

2.2. The case G is not 2-connected

Suppose now that G is not a 2-connected plane graph and we will construct a 2-connected plane graph G^* with $\delta(G^*) \geq 3$ having neither 4-cycles nor 9-cycles and satisfying structural properties (C1) to (C5). This obviously contradicts the result just established before.

We remark that the following proof is stimulated by the technique used in [3].

Let B be an end block of G with the unique cut-vertex x. Let f be the outside face of G. Notice that $d_B(x) \geq 2$ and $d_B(v) \geq 3$ for each $v \in V(B) \setminus \{x\}$. Choosing another vertex y of B such that $y \neq x$ and y lies on the boundary of B. W.l.o.g., assume that x and y are both belonging to $b(f)$. Then we take ten copies of B, i.e., B_k with $k = 1, \ldots, 10$. In each copy B_k, the vertices corresponding to x and y are denoted by x_k and y_k, respectively. Then one can embed B_k, $k = 1, \ldots, 10$, into f in the following way: first, let $B = B_1$. Next, for each $k = 2, \ldots, 10$, consecutively embed B_k into f by identifying x_k with y_{k-1}. Finally, identify y_{10} with a vertex $u \in V(f) \setminus V(B)$. Then the first resulting graph, denoted by G_1.

Obviously, in the processing of constructing G_1, we confirm that there are no new adjacent cycles established. Furthermore, no 4-cycles and 9-cycles are formed. Thus, it is easy to deduce that G_1 satisfies the following structural properties.

(A1) Fewer end blocks than G;
(A2) The minimum degree is at least 3;
(A3) Neither 4-cycles nor 9-cycles;
(A4) A 5-cycle or a 6-cycle is adjacent to at most one 3-cycle;
(A5) A 5^*-cycle is neither adjacent to a 5^*-cycle normally, nor adjacent to an i-cycle with $i \in \{7, 8\}$;
(A6) A 6^*-cycle is not adjacent to a 6-cycle;
(A7) A nontriangular 7-cycle is not adjacent to two 5-cycles which are normally adjacent;
(A8) A 7^*-cycle is neither adjacent to a 5-cycle nor a 6^*-cycle.

Furthermore, we confirm that G_1 also satisfies the following two structural properties:

(P1) G_1 has neither orchid, nor sunflower, nor lotus;
(P2) A 6^*-cycle is not incident to an i-cycle C with $i \in \{3, 5\}$, where C is opposite to such a 6^*-cycle by a 4-vertex.

(P1) For some $k \in \{2, \ldots, 10\}$, notice that we just identify some vertex x_k with y_{k-1}. It implies that any new cycle, which is not completely belong to some B_k, must be an 11^+-cycles, i.e., $C^* = x_1 \cdots x_{10} u \cdots x_1$. Thus, any orchid, sunflower, or lotus cannot be established.

(P2) Assume to the contrary that G_1 contains a 6^*-cycle, denoted by C_6^*, which is incident to a 3-cycle C_3 or a 5-cycle C_5 by a 4-vertex v^*. Clearly, v^* must be equal to u or some vertex x_k with $k \in \{2, \ldots, 10\}$. However, $d_{G_1}(u) = d_{B_{10}}(u) + d_{G \setminus B_1}(u) \geq 2+3 = 5$ or $d_{G_1}(x_k) = d_{B_{k-1}}(x_k) + d_{B_k}(x_k) \geq 3+2 = 5$ for all $k \in \{2, \ldots, 10\}$. We always get a contradiction to $d_{G_1}(v^*) = 4$.

Now, if G_1 is 2-connected, then we well done. Otherwise, we may repeat the process described above and finally obtain a desired G^*.

Thus, we complete the proof of Theorem 1. □

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References

[1] O.V. Borodin, Structural properties of plane graphs without adjacent triangles and an application to 3-colorings, J. Graph Theory 21 (1996) 183–186.
[2] O.V. Borodin, A.N. Glebov, A. Raspaud, M.R. Salavatipour, Planar graphs without cycles of length from 4 to 7 are 3-colorable, J. Combin. Theory, Ser. B 93 (2005) 303–311.
[3] M. Chen, H. Lu, Y. Wang, A note on 3-choosability of planar graphs, Inform. Process. Lett. 105 (5) (2008) 206–211.
[4] P. Erdös, A.L. Rubin, H. Taylor, Choosability in graphs, Congr. Numer. 26 (1979) 125–157.
[5] T.R. Jensen, B. Toft, Graph Coloring Problems, Wiley-Interscience, New York, 1995.
[6] L. Shen, Y. Wang, A sufficient condition for a planar graph to be 3-choosable, Inform. Process. Lett. 104 (2007) 146–151.
[7] L. Shen, Y. Wang, Q. Wu, Planar graphs without cycles of length 4, 7, 8, or 9 are 3-choosable, Discrete Appl. Math. 159 (4) (2011) 232–239.
[8] C. Thomassen, Every planar graph is 5-choosable, J. Combin. Theory Ser. B 62 (1994) 180–181.
[9] M. Voigt, List colourings of planar graphs, Discrete Math. 120 (1993) 215–219.
[10] M. Voigt, A non-3-choosable planar graph without cycles of length 4 and 5, Discrete Math. 307 (7–8) (2007) 1013–1015.
[11] Y. Wang, H. Lu, M. Chen, Planar graphs without cycles of length 4, 5, 8 or 9 are 3-choosable, Discrete Math. 310 (2010) 147–158.
[12] L. Zhang, B. Wu, A note on 3-choosability of planar graphs without certain cycles, Discrete Math. 297 (2005) 206–209.
[13] L. Zhang, B. Wu, Three-coloring planar graphs without certain small cycles, Graph Theory Notes N. Y. 46 (2004) 27–30.