q-Analogs of symmetric function operators

Michael Zabrocki[*]

Mathematics and Statistics-61, York University, 4700 Keele Street, Toronto, Ont. Canada M3J 1P3

Abstract

For any homomorphism V on the space of symmetric functions, we introduce an operation that creates a q-analog of V. By giving several examples we demonstrate that this quantization occurs naturally within the theory of symmetric functions. In particular, we show that the Hall–Littlewood symmetric functions are formed by taking this q-analog of the Schur symmetric functions and the Macdonald symmetric functions appear by taking the q-analog of the Hall–Littlewood symmetric functions in the parameter t. This relation is then used to derive recurrences on the Macdonald q, t-Kostka coefficients.

Résumé

Pour un homomorphisme V sur l'espace des fonctions symétriques, nous présentons une opération qui crée un q-analogue de V. En donnant plusieurs exemples nous démontrons que cette quantization se produit naturellement dans la théorie de fonctions symétriques. En particulier, nous prouvons que les fonctions symétriques de Hall–Littlewood sont constituées en prenant ce q-analogue des fonctions symétriques de Schur et les fonctions symétriques de Macdonald apparaissent en prenant le q-analogue des fonctions symétriques de Hall–Littlewood dans le paramètre t. Cette relation est alors employée pour dériver des récurrence sur les coefficients Macdonald q, t-Kostka. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Hall–Littlewood and Macdonald symmetric functions are two examples of families of symmetric functions that depend on a parameter q such that setting this

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[*] Corresponding author.

E-mail address: zabrocki@mathstat.yorku.ca (M. Zabrocki).

URL: http://www.math.yorku.ca/∼zabrocki

parameter q equal to 0 yields one class of symmetric functions which is not a product of generators and setting the parameter q equal to 1 yields a multiplicative basis. There are other classes of symmetric functions with the same property, and in this article we will show that practically any of these families are instances of the same q-twisting of the symmetric function found by setting $q=0$.

This remarkable fact has lead to a completely elementary proof of the polynomiality of the q, t-Kostka coefficients [4] and in this article we use the very same observation to derive a combinatorial recurrence on these coefficients as well as algebraic formulas for operators that add a column to the partition indexing a Macdonald symmetric function.

The first section of this article will introduce some necessary notation and the definition of this q-analog. In the second section we give several examples where it arises. Some examples will be nothing more than showing that $\tilde{V}^q$ for some V is a formula that is well known in the literature. Other examples present some completely new equations, the most important of which will concern the relation of the Hall–Littlewood symmetric functions to the Macdonald symmetric functions. This section will show that this single q-analog appears in the creation of several different classes of Schur positive symmetric functions.

In the third section we derive some formulas related to an operator that adds a column to the Hall–Littlewood symmetric functions. The q-analog of this operator adds a column to the Macdonald symmetric functions. In the fourth section, these equations are used to give a formula for the action of this operator on the Schur basis giving a combinatorial rule for computing the Macdonald symmetric functions (i.e. a 'Morris-like' recurrence for the q, t-Kostka coefficients).

2. Notation

A partition of n is a sequence of non-negative integers $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots)$ such that $\sum_i \lambda_i = n$. The length of a partition is the largest index i such that λ_i is non-zero, and it will be denoted here by $\ell(\lambda)$. A partition will be drawn as a sequence of rows of boxes aligned at the left edge with λ_i cells in the ith row. We will use the French convention and draw these diagrams with the largest row on the bottom and the smallest row on the top. The conjugate partition λ' is the sequence whose ith entry is the number of cells in the ith column of the diagram for λ.

The partition will be sometimes be identified with its diagram in the sort of language that is used. For instance, the operations of adding rows or columns to partitions indexing bases for the symmetric functions are important here. The notation (m, μ) is used to represent the sequence μ with a part of size m prepended, which will be a partition as long as $\mu_1 \leqslant m$. The notation $1^m | \mu$ will be used to represent the partition $(\mu_1 + 1, \mu_2 + 1, \ldots, \mu_m + 1)$ (as long as $\ell(\mu) \leqslant m$).

Let Λ be the space of symmetric functions with the standard bases for this space, h_λ homogeneous, e_λ elementary, m_λ monomial, f_λ forgotten, p_λ power, and s_λ the Schur symmetric functions defined as they are in [M]. The involution ω that sends p_k to $(-1)^{k-1} p_k$ relates these bases by $\omega(h_\lambda) = e_\lambda$, $\omega(m_\lambda) = f_\lambda$ and $\omega(s_\lambda) = s_{\lambda'}$. The standard

inner product on this space determines the dual bases

$$\langle p_\lambda, p_\mu/z_\mu \rangle = \langle h_\lambda, m_\mu \rangle = \langle e_\lambda, f_\mu \rangle = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}, \tag{1}$$

where $z_\lambda = \prod_{i>1} i^{n_i(\lambda)} n_i(\lambda)!$ with $n_i(\lambda)$ equal to the number of parts of size i in λ, and we have set $\delta_{\lambda\lambda} = 1$ and $\delta_{\lambda\mu} = 0$ if $\lambda \neq \mu$.

For any element f of Λ, let $f^\perp$ be the operation that is dual to multiplication by f with respect to the standard inner product. By definition we have that for any dual bases $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$, the action of $f^\perp$ on another symmetric function g is given by the formula

$$f^\perp g = \sum_\lambda \langle g, f a_\lambda \rangle b_\lambda. \tag{2}$$

'Plethystic' notation is a device for expressing the substitution of the monomials of one expression in a symmetric function. Assume that E is a formal series in a set of variables $x_1, x_2, \ldots$ with possible special parameters q and t. For $k \geqslant 1$, set $p_k[E]$ to be E with x_i replaced by x_i^k and q and t replaced by q^k and t^k, respectively, that is

$$p_k[E(x_1, x_2, \ldots, q, t)] = E(x_1^k, x_2^k, \ldots; q^k, t^k). \tag{3}$$

For an arbitrary symmetric function P, $P[E]$ will represent the formal series found by expanding P in terms of the power symmetric functions and then substituting $p_k[E]$ for p_k. More precisely, if the power sum expansion of the symmetric function P is given by $P = \sum_\lambda c_\lambda p_\lambda$ then $P[E]$ is given by the formula

$$P[E] = \sum_\lambda c_\lambda p_{\lambda_1}[E] p_{\lambda_2}[E] \cdots p_{\lambda_{\ell(\lambda)}}[E]. \tag{4}$$

The symmetric functions in the infinite set of variables $x_1, x_2, x_3, \ldots$ will be denoted by Λ^X. Λ and Λ^X are isomorphic and in this exposition we will identify the two spaces when it is convenient. In plethystic notation, the isomorphism that identifies the two spaces is given by $f \mapsto f[X]$ where $X = x_1 + x_2 + x_3 + \cdots$ since under this map p_k is sent to $x_1^k + x_2^k + x_3^k + \cdots$. Also use the notation $X_n = x_1 + x_2 + \cdots + x_n$ to represent when a symmetric function being evaluated a finite set of variables. For sets of variables using other letters we use a similar convention.

The symbol $\Omega = \sum_{n \geqslant 0} h_n$ will represent a special generating function, and we will use the plethystic notation for symmetric functions with this expression as well with the following identities.

$$\Omega[X + Y] = \Omega[X]\Omega[Y], \tag{5}$$

$$\Omega[X] = \prod_i \frac{1}{1 - x_i}, \tag{6}$$

$$\Omega[XY] = \sum_\lambda s_\lambda[X] s_\lambda[Y]. \tag{7}$$

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Operators that have the property that they add a row or a column to the partition indexing a symmetric function will be known as 'creation operators'. The creation operators that will be used repeatedly are those that add a row to the Schur symmetric functions (due to Bernstein, see [20, p. 69], [15, p. 96]) and the Hall–Littlewood symmetric functions (due to Jing, see [2,5] or [15, p. 238]). In the third section, the operator introduced in [18] that adds a column to the partition indexing a Hall–Littlewood symmetric function will be developed further.

Define the following involution on the space $\mathrm{Hom}(\Lambda, \Lambda)$ that is a useful tool for deriving identities within the theory of symmetric functions. Let V be an element of $\mathrm{Hom}(\Lambda, \Lambda)$ and $P \in \Lambda$. We define the flip of V by the formula

$$\bar{V}P[X] = V^Y P[X - Y]|_{Y=X}. \tag{8}$$

It seems to arise naturally when one considers the sorts of operators that will concern us here (see [18] and [19]).

The degree of a symmetric function $P \in \Lambda$ is the highest power of z in $P[zX]$ and will be denoted by $\deg(P)$. If $P[zX] = z^{\deg(P)}P[X]$ then we will say that P is of homogeneous degree.

Use this involution here to define a q-twisting of a symmetric function operator. Let V once again be an element of $\mathrm{Hom}(\Lambda, \Lambda)$ and let F^q be defined by $F^q P[X] = P[X(1-q)]$. Our q-analog is defined when it acts of the symmetric function $P \in \Lambda$ by the formula

$$\tilde{V}^q P[X] = V^Y P[qX + (1-q)Y]|_{Y=X} = \overline{\bar{V}F^q}P[X]. \tag{9}$$

It is easily seen that this q-analog has the following fundamental property.

Remark 1. Let V be an element of $\mathrm{Hom}(\Lambda, \Lambda)$ and create the q-twisting of this operator from formula (9), $\tilde{V}^q$, and act this new operator on a symmetric function $P[X]$ to create an expression such as

$$\tilde{V}^q P[X] \tag{10}$$

This q-analog has the property that when $q=0$, the expression becomes

$$VP[X] \tag{11}$$

and if $q=1$, then it reduces to the product

$$V(1)P[X] \tag{12}$$

This paper is concerned with generalizations of the standard bases, the Hall–Littlewood and Macdonald symmetric functions, which depend on additional parameters q and t. There are two important scalar products on the symmetric functions related to these bases. They are defined by their values on the power symmetric

basis.

$$\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} 1 - t^{\lambda_i}, \tag{13}$$

$$\langle p_\lambda, p_\mu \rangle_{qt} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} (1 - q^{\lambda_i})(1 - t^{\lambda_i}). \tag{14}$$

The Macdonald symmetric functions $H_\mu[X; q, t]$ are defined by the following three conditions:

1. $\langle H_\lambda[X; q, t], H_\mu[X; q, t] \rangle_{qt} = 0$ if $\lambda \neq \mu$.
2. $F^t H_\mu[X; q, t] = \sum_{\lambda \leqslant \mu} c_{\lambda\mu} m_\lambda[X]$ for suitable coefficients $c_{\lambda\mu}$ and the sum is over all partitions λ that are smaller than μ in the standard dominance order.
3. $\langle H_\mu[X], h_n[X] \rangle = t^{n(\mu)}$ where $n(\mu) = \sum_i (i - 1)\mu_i$.

The expansion of the $H_\mu[X; q, t]$ basis in the Schur basis for the symmetric functions defines the coefficients $K_{\lambda\mu}(q, t)$, that is $H_\mu[X; q, t] = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(q, t) s_\lambda[X]$.

The Hall–Littlewood basis is defined similarly with respect to the $\langle , \rangle_t$ scalar product; simply stated $H_\mu[X; t] = H_\mu[X; 0, t]$.

The symmetric functions $H_\mu[X; q, t]$ and $H_\mu[X; t]$ are the two families of symmetric functions that will interest us the most here. A symmetric function with the property that when expressed in terms of the Schur basis their coefficients are polynomials in q and t with non-negative coefficients will be called *Schur positive*. The Macdonald and Hall–Littlewood functions are just two examples of families with this property.

3. Examples

3.1. Schur symmetric functions I

In [20] an operator attributed to Bernstein that adds a row to the Schur function is given by $S_m = \sum_{i \geqslant 0} (-1)^i h_{m+i} e_i^\perp$. The formula has a very convenient form when expressed in terms of plethystic notation. Let $P[X]$ be a symmetric function in the X variables. Define a generating function of operators given by

$$S(z)P[X] = P\left[X - \frac{1}{z}\right] \Omega[zX]. \tag{15}$$

Now for any $m \in \mathbb{Z}$, set $S_m P[X] = S(z)P[X]|_{z^m}$. If $m \geqslant \mu_1$, then it easily follows that $S_m s_\mu[X] = s_{(m,\mu)}[X]$. S_m is a creation operator for the Schur basis since we have the formula

$$S_{\mu_1} S_{\mu_2} \cdots S_{\mu_{\ell(\mu)}} 1 = s_\mu[X]. \tag{16}$$

Now if we set $H(z) = \widetilde{S(z)}^q$ and $H_m^q = H(z)|_{z^m}$, then this is a q-analog of the operator S_m and we may calculate that

$$H(z)P[X] = S(z)P[qX + (1-q)Y]|_{Y=X}$$

$$= P\left[qX + (1-q)\left(X - \frac{1}{z}\right)\right]\Omega[zX]$$

$$= P\left[X - \frac{1-q}{z}\right]\Omega[zX]. \tag{17}$$

Remarkably, this is the formula for the Hall–Littlewood creation operator of Jing [5] in the notation used by Garsia [2]. These operators have the property that

Theorem 2 (Jing [5]). *Let $H_m^q = \widetilde{S_m}^q$. Then*

$$H_{\mu_1}^q H_{\mu_1}^q \cdots H_{\mu_{\ell(\mu)}}^q 1 = H_\mu[X;q]. \tag{18}$$

The fact that $H_\mu[X;0] = s_\mu[X]$ and $H_\mu[X;1] = h_\mu[X]$ follows from Remark 1.

3.2. Schur symmetric functions II

For any sequence of integers $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, let x^γ represent the monomial $x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}$. We say that γ is a dominant weight if $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_n$.

The symmetric group S_n acts on any polynomial in the X_n variables by permuting their indices. For any $\sigma \in S_n$, set ε_σ to be the sign of the permutation. Also set $\delta = (n-1, n-2, \ldots, 1, 0)$. Then define for any polynomial f in the x_i variables a symmetrization operator $\pi_n(f) = J(f)/J(1)$ where

$$J(f) = \sum_{\sigma \in S_n} \varepsilon_\sigma \sigma(x^\delta f). \tag{19}$$

When λ is a partition, π_n sends x^λ to the Schur function $s_\lambda[X_n]$. When λ is any dominant weight, set $s_\lambda(X_n) = \pi_n(x^\lambda)$ to be the resulting Laurent polynomial.

Let η be a sequence of positive integers whose sum is n (a composition) and define the set of ordered pairs $\mathrm{Roots}_\eta = \{(i,j) : 1 \leqslant i \leqslant \eta_1 + \eta_2 + \cdots + \eta_r < j \leqslant n \text{ for some } r\}$. Consider the following formal power series given by the formula

$$H_{\mu\eta}(X_n;q) = \pi_n\left(x^\mu \prod_{(i,j) \in \mathrm{Roots}_\eta} (1 - qx_i/x_j)^{-1}\right). \tag{20}$$

This formal series has an expansion in terms of the Schur functions indexed by all dominant integral weights. Define $K_{\lambda\mu\eta}(q)$ as the coefficient of $s_\lambda(X_n)$ in $H_{\mu\eta}(X_n;q)$, so that

$$H_{\mu\eta}(X_n;q) = \sum_\lambda K_{\lambda\mu\eta}(q)s_\lambda(X_n), \tag{21}$$

where the sum is over all dominant integral weights λ. The $K_{\lambda\mu\eta}(q)$ are known as the generalized or parabolic Kostka polynomials. For a more complete exposition of these polynomials we refer the reader to [16], or [6].

Now consider a composition of the Bernstein Schur function operators $S_v = S_{v_1} S_{v_2} \cdots S_{v_{\ell(v)}}$ and define $H_v^q = \widetilde{S}_v{}^q$. By a calculation similar to (17), one may show that

$$H_v^q P[X] = P[X - (1-q)Z^*]\Omega[ZX] \prod_{1 \leqslant j < i \leqslant \ell(v)} 1 - z_i/z_j \Bigg|_{z_1^{v_1} z_2^{v_2} \cdots z_{\ell(v)}^{v_{\ell(v)}}}, \qquad (22)$$

where $Z^* = \sum_{i=1}^{\ell(v)} 1/z_i$.

In work with Shimozono [17], we demonstrated that H_v^q is an operator with interesting properties related to the generalized Kostka coefficients. In particular, they can be used as generating functions for the generalized Kostka coefficients. They are also operators which act on symmetric functions and can be used to build a family of symmetric functions.

Theorem 3 (Shimozono and Zabrocki [17]). *Let η be a composition of k and $\mu \in \mathbb{Z}^k$, set $\mu^{(i)} = (\mu_{\eta_1 + \cdots + \eta_{i-1}}, \ldots, \mu_{\eta_1 + \cdots + \eta_i})$. For any $v \in \mathbb{Z}^\ell$ set $H_v = \widetilde{S}_v{}^q$ where $S_v = S_{v_1} \cdots S_{v_{\ell(v)}}$, then we have*

$$H_{\mu^{(1)}}^q H_{\mu^{(2)}}^q \cdots H_{\mu^{(\ell)}}^q = \sum_\lambda K_{\lambda\mu\eta}(q) H_\lambda^q, \qquad (23)$$

where the sum is over all dominant weights λ. In particular, when this operator is applied to the symmetric function 1, *we arrive at a class of symmetric functions and may set*

$$H_{\mu\eta}[X; q] := H_{\mu^{(1)}}^q H_{\mu^{(2)}}^q \cdots H_{\mu^{(k)}}^q 1 = \sum_\lambda K_{\lambda\mu\eta}(q) s_\lambda[X], \qquad (24)$$

where the sum is over all partitions λ.

In addition, these operators also seem to be fundamentally related to the new class of symmetric functions referred to as 'atoms' $A_\lambda^{(k)}[X; q]$ (see [9,10,11]). In particular, when the partition indexing the operator is a rectangle, it was conjectured that $H_{(\ell^{k+1-\ell})}^q$ is a creation operator for this class of symmetric functions.

By Remark 1, the coefficients $\langle H_{\mu\eta}[X; q], s_\lambda[X] \rangle = K_{\lambda\mu\eta}(q)$ have the property that when $q = 1$, they are the Littlewood–Richardson coefficients $c_{\mu^{(1)}\mu^{(2)}\ldots\mu^{(k)}}^\lambda$ and when $q = 0$, the $K_{\lambda\mu\eta}(0) = 1$ if $\mu = \lambda$ and 0 otherwise. It is conjectured that the coefficients $K_{\lambda\mu\eta}(q)$ are polynomials in q with non-negative integer coefficients.

3.3. Homogeneous creation operator

Multiplication by h_k is an operator that adds a row to the homogeneous symmetric functions. The q-analog of this operator is once again h_k and so this is not particularly

interesting. However, in [19] we gave a formula for an operator that adds a column to the homogeneous symmetric functions and in [18] we gave a combinatorial description of the action of this operator on the Schur function basis. Let H_{1^m} be a family of operators with the property that

$$H_{1^{\lambda_1}} H_{1^{\lambda_2}} \cdots H_{1^{\lambda_{\ell(\lambda)}}} 1 = h_{\lambda'}[X]. \tag{25}$$

The q-twisting of this operator is another example where this q-analog appears naturally to produce a family of Schur positive symmetric functions. It develops that $\widetilde{H_{1^m}}^q$ is an operator that adds a column to the symmetric functions $(q;q)_\lambda h_\lambda[X/1-q]$, where $(q;q)_k = (1-q)(1-q^2)\cdots(1-q^k)$ and $(q;q)_\lambda = (q;q)_{\lambda_1}(q;q)_{\lambda_2}\cdots(q;q)_{\lambda_{\ell(\lambda)}}$. This family has the property that when $q=1$ the symmetric functions become $h_{1^{|\lambda|}}$ and when $q=0$ they become h_λ.

Theorem 4. *Let H_{1^m} be an operator with the property that $H_{1^m} h_\lambda[X] = h_{1^m | \lambda}[X]$ for $\ell = \ell(\lambda) \leqslant m$. Then we have*

$$\widetilde{H_{1^{\lambda_1}}}^q \widetilde{H_{1^{\lambda_2}}}^q \cdots \widetilde{H_{1^{\lambda_\ell}}}^q 1 = (q;q)_{\lambda'} h_{\lambda'}\left[\frac{X}{1-q}\right]. \tag{26}$$

Proof.

$$\widetilde{H_{1^m}}^q \left((q;q)_\lambda h_\lambda\left[\frac{X}{1-q}\right] \right) = (q;q)_\lambda H_{1^m}^Y \left(h_\lambda\left[\frac{qX + (1-q)Y}{1-q}\right] \right)\Bigg|_{Y=X}. \tag{27}$$

Since the summation formula for a homogeneous symmetric function in two sets of variables is given as $h_m[X+Y] = \sum_{i=0}^m h_i[X] h_{m-i}[Y]$, then for a partition λ, $h_\lambda[X+Y] = \prod_{n=1}^{\ell(\lambda)} \sum_{i=0}^{\lambda_n} h_i[X] h_{\lambda_n - i}[Y]$. It follows that (27) reduces to:

$$= (q;q)_\lambda \prod_{n=1}^m \sum_{i=0}^{\lambda_n} h_i\left[\frac{qX}{1-q}\right] h_{\lambda_n - i + 1}[X]$$

$$= (q;q)_\lambda \prod_{n=1}^m \left(\sum_{i=0}^{\lambda_n+1} h_i\left[\frac{qX}{1-q}\right] h_{\lambda_n - i + 1}[X] - h_{\lambda_n+1}\left[\frac{qX}{1-q}\right] \right)$$

$$= (q;q)_\lambda \prod_{n=1}^m h_{\lambda_n+1}\left[\frac{X}{1-q}\right] - q^{\lambda_n+1} h_{\lambda_n+1}\left[\frac{X}{1-q}\right]$$

$$= (q;q)_\lambda \prod_{n=1}^m (1 - q^{\lambda_n+1}) h_{\lambda_n+1}\left[\frac{X}{1-q}\right]$$

$$= (q;q)_{1^m|\lambda} h_{1^m|\lambda}\left[\frac{X}{1-q}\right]. \tag{28}$$

There are elementary proofs that the functions $(q;q)_\lambda h_\lambda[X/1-q]$ are Schur positive. Once again we have a case of a Schur positive q-analog arising from a Schur positive family of symmetric functions $h_\lambda[X]$. Remark 1 implies that the limit as q goes to 1 of these symmetric functions is $h_{1^n}[X]$ when λ is a partition of n and when $q=0$ we have that they reduce to $h_{\lambda'}[X]$.

3.4. Macdonald's operators

Macdonald introduced operators D_n^r (see [15, p. 315]) such that the Macdonald polynomials $P_\lambda(X;q,t)$ are characterized as eigenfunctions of this family of operators. What we will show is that the Macdonald operators are the q-twisting of the same operators with q set equal to 0.

Let T_{x_i} be an operator on polynomials with the property $T_{x_i}P[X_n]=P[X-x_i]$. Consider the operator

$$D_n^r(t)=\sum_{I\subseteq\{1,\dots,n\}} A_I(X_n;t)\prod_{i\in I} T_{x_i}, \tag{29}$$

where $A_I(X_n;t)=t^{\binom{r}{2}}\prod_{i\in I, j\notin I}(tx_i-x_j)/(x_i-x_j)$ and the sum is over all subsets I of size r.

Now define $D_n^r(q,t)=\widetilde{D_n^r(t)}^q$, then calculate that

$$D_n^r(q,t)P[X_n]=\sum_{\substack{I\subseteq\{1,\dots,n\}\\|I|=r}} A_I(Y_n;t)\prod_{i\in I} T_{y_i}P[qX_n+(1-q)Y_n]\big|_{Y_n=X_n}$$

$$=\sum_{\substack{I\subseteq\{1,\dots,n\}\\|I|=r}} A_I(Y_n;t)P[X_n-(1-q)X_I]. \tag{30}$$

If we set $T_{q,x_i}P[X_n]=P[X_n-(1-q)x_i]=\widetilde{T_{x_i}}^q$, then

$$D_n^r(q,t)=\sum_{\substack{I\subseteq\{1,\dots,n\}\\|I|=r}} A_I(X_n;t)\prod_{i\in I} T_{q,x_i} \tag{31}$$

and these are the operators D_n^r as they are defined in [15]. As was presented in this reference, set $D_n(u;q,t)=\sum_{r=0}^n u^r D_n^r(q,t)$.

Consider again the Bernstein operators, S_m, as they were defined in equation (15) and also consider $\tilde{S}_m=\omega S_m\omega=(-1)^m S(-z)|_{z^m}$. From these operators create a q,t analog by applying this parameter deformation twice. Set $\tilde{D}_m=\widetilde{\widetilde{S}_m}^{q^{1/t}}$ and $\tilde{D}_m^*=\widetilde{\widetilde{S}_m}^{t^{1/q}}$.

A simple calculation yields that

$$\tilde{D}_m P[X] = P\left[X + \frac{(1-q)(1-1/t)}{z}\right] \Omega[-zX]|_{z^m} \tag{32}$$

and

$$\tilde{D}_m^* P[X] = P\left[X - \frac{(1-1/q)(1-t)}{z}\right] \Omega[zX]|_{z^m}. \tag{33}$$

These families of operators were studied in [3] and more extensively in [1] to show the polynomiality of the q,t-Catalan numbers. In particular, when $m=0$ it is known that these operators are related to the operators D_n^1 and that they have the family $H_\mu[X;q,t]$ as eigenfunctions. Theorem 1.2 of [3] was the following result (translated on the $H_\mu[X;q,t]$ basis).

Theorem 5 (Garsia et al. [3]). *For μ a partition of n, we have*

$$\tilde{D}_0 H_\mu[X;q,t] = \left(1 - (1-1/t)\prod_{i\geqslant 1} t^{-i}(1-q^{\mu_i})\right) H_\mu[X;q,t], \tag{34}$$

$$\tilde{D}_0^* H_\mu[X;q,t] = \left(1 - (1-t)\prod_{i\geqslant 1} t^{i}(1-q^{-\mu_i})\right) H_\mu[X;q,t]. \tag{35}$$

3.5. Hall–Littlewood creation operator

Consider now an operator that adds a column to the Hall–Littlewood symmetric functions $H_\lambda[X;t]$. One such operator was introduced in [19] where the combinatorial action on the Schur function basis was discussed and some explicit formulas were presented. We will discuss some of these operators in more detail in the following section. For now we present the following theorem.

Theorem 6. *Let λ be a partition such that $\ell = \ell(\lambda) \leqslant m$. Any operator $H_{1^m}^t$ with the property $H_{1^m}^t H_\mu[X;t] = H_{1^m|\lambda}[X;t]$ satisfies the equation*

$$\widetilde{H_{1^{\lambda_1}}^t}{}^q \widetilde{H_{1^{\lambda_2}}^t}{}^q \cdots \widetilde{H_{1^{\lambda_\ell}}^t}{}^q = H_{\lambda'}[X;q,t]. \tag{36}$$

The most elementary proof of this theorem can be seen in [4]. It is selfcontained and uses nothing more than the identities in the original paper of Macdonald. We present here a short proof that follows by demonstrating that if the theorem is true for one such operator $H_{1^m}^t$, then it is true for all such operators.

Some of the first proofs of the polynomiality of the q,t-Kostka coefficients used operators of this type to show that the q,t-Kostka polynomials satisfied recurrences that did not have denominators. Almost any of the operators given by Kirillov and Noumi [7,8] and Lapointe and Vinet [12,13] are of this type. We need only one example and the following lemma.

Lemma 7. *Let λ be a partition such that $\ell(\lambda) \leqslant m$. Assume that there exists some operator $H_{1^m}^t$ such that*

$$H_{1^m}^t H_\lambda[X; t] = H_{1^m | \lambda}[X; t] \tag{37}$$

and

$$\widetilde{H_{1^m}^t}^q H_\lambda[X; q, t] = H_{1^m | \lambda}[X; q, t]. \tag{38}$$

Then for any operator $H_{1^m}^{\prime\,t}$ that satisfies Eq. (37) *will also satisfy Eq.* (38).

Proof. It follows from the definition of the Macdonald symmetric functions and the property that $H_\mu[X; t, q] = \omega H_{\mu'}[X; q, t]$ that we have the triangularity relation $H_\mu[X(1 - q); q, t] = \sum_{\lambda \geqslant \mu} a_{\lambda\mu}(q, t) s_\lambda[X]$. Since $H_\mu[X; t] = \sum_{\lambda \geqslant \mu} K_{\lambda\mu}(t) s_\lambda[X]$, then there exist coefficients $b_{\lambda\mu}(q, t)$ such that $H_\mu[(1 - q)X; q, t] = \sum_{\lambda \geqslant \mu} b_{\lambda\mu}(q, t) H_\lambda[X; t]$.

Consider the expansion of $H_\mu[X + Y; q, t] = \sum_{\nu \subseteq \mu} H_{\mu/\nu}[X; q, t] H_\nu[Y; q, t]$ which may be seen as a transformation of formula (7.9) on p. 345 of [15].

Since our operator $H_{1^m}^t$ adds a column of size m to $H_\nu[X; t]$ and $\widetilde{H_{1^m}^t}^q$ adds a column to $H_\mu[X; q, t]$, then $H_{1^m | \mu}[X; q, t]$ is given by the formula

$$\widetilde{H_{1^m}^t}^q H_\mu[X; q, t] = \sum_{\lambda \subseteq \mu} H_{\mu/\lambda}[qX; q, t] H_{1^m}^t H_\lambda[(1 - q)X; q, t]$$

$$= \sum_{\lambda \subseteq \mu} H_{\mu/\lambda}[qX; q, t] \sum_{\nu \geqslant \lambda} b_{\lambda\nu}(q, t) \, H_{1^m}^t H_\nu[X; t]$$

$$= \sum_{\lambda \subseteq \mu} H_{\mu/\lambda}[qX; q, t] \sum_{\nu \geqslant \lambda} b_{\lambda\nu}(q, t) \, H_{1^m | \nu}[X; t]. \tag{39}$$

The right-hand side of this expression is independent of the operator that is used to derive it, therefore absolutely any operator $H_{1^m}^{\prime\,t}$ that has the property that $H_{1^m}^{\prime\,t} H_\mu[X; t] = H_{1^m | \mu}[X; t]$, also has the property that $\widetilde{H_{1^m}^{\prime\,t}}^q$ adds a column to the Macdonald symmetric functions $H_\mu[X; q, t]$. $\quad\square$

To prove the theorem it is necessary to produce at least one operator that satisfies the property of the lemma. Fortunately, this is relatively easy since practically any in the development in [7,8,12,13] such that one can set $q = 0$ without needing to take a limit satisfies the properties of Lemma 7 (once transformed to the $H_\mu[X; q, t]$ basis).

Consider Expression 3 of [13]. Let $D_I^r(u; q, t)$ be the Macdonald operator of Eq. (31) that acts only on the variables x_i for i in the set I. A formula for an operator that adds a column of size m onto the $J_\mu[X; q, t] := H_\mu[X(1 - t); q, t]$ basis is given by

$$B_m^{(3)}(q, t) = \sum_{|I| = m} \sum_{r=0}^{m} t^{-r} x_I \prod_{\substack{i \in I \\ j \notin I}} \frac{x_i - x_j/t}{x_i - x_j} D_I^r(-t; q, t), \tag{40}$$

where x_I here represents $\prod_{i \in I} x_i$. It is easy to demonstrate by acting on an arbitrary symmetric function that

$$B_m^{(3)}(q,t) = \widetilde{B_m^{(3)}(0,t)}^q. \tag{41}$$

Let F_t be an operator that sends the symmetric function $H_\mu[X;q,t]$ to the symmetric function $J_\mu[X;q,t]$. More precisely, for an arbitrary symmetric function $P[X]$ set $F_t P[X] = P[X(1-t)]$ and denote the inverse of this operator F_t^{-1}. Since $B_m^{(3)}(q,t)$ adds a column of size m to the $J_\mu[X;q,t]$ basis, $F_t^{-1} B_m^{(3)}(q,t) F_t$ is an operator that adds a column to the $H_\mu[X;q,t]$ basis.

When $q=0$ in the operator $B_m^{(3)}(q,t)$ it becomes an operator that adds a column to the $H_\mu[X(1-t);t]$ basis. That is, we have

$$B_m^{(3)}(0,t) H_\mu[X(1-t);t] = H_{1^m|\mu}[X(1-t);t]. \tag{42}$$

Since $F_t H_\mu[X;t] = H_\mu[X(1-t);t]$, this implies $F_t^{-1} B_m^{(3)}(0,t) F_t$ is an operator that adds a column to the $H_\mu[X;t]$ basis.

To demonstrate the theorem, it remains to show that the q-twist of $F_t^{-1} B_m^{(3)}(0,t) F_t$ is exactly the operator $F_t^{-1} B_m^{(3)}(q,t) F_t$. This follows from the fact that conjugation by F_t commutes with the q-twisting for any symmetric function operator.

Lemma 8. *For $V \in \mathrm{Hom}(\Lambda, \Lambda)$ we have*

$$F_t^{-1} \tilde{V}^q F_t = (\widetilde{F_t^{-1} V F^{tq}}). \tag{43}$$

Proof. This follows by acting both the left and the right-hand side of this equation on an arbitrary symmetric function.

This leads us to several other formulas for operators with similar properties. Consider the following corollary.

Corollary 9. *Define $h_\lambda(q) = \prod_{s \in \lambda} 1 - q^{a_\lambda(s) + l_\lambda(s) + 1}$. Let $H_{1^m}^q$ be an operator with the property $H_{1^m}^q H_\lambda[X;q] = H_{1^m|\lambda}[X;q]$ for $\ell = \ell(\lambda) \leqslant m$. Then*

$$\widetilde{H_{1^{\lambda_1}}^q}^t \widetilde{H_{1^{\lambda_2}}^q}^t \cdots \widetilde{H_{1^{\lambda_\ell}}^q}^t 1 = \omega H_\lambda[X;q,t] \tag{44}$$

and

$$\widetilde{H_{1^{\lambda_1}}^q}^q \widetilde{H_{1^{\lambda_2}}^q}^q \cdots \widetilde{H_{1^{\lambda_\ell}}^q}^q = h_{\lambda'}(q) s_{\lambda'}\left[\frac{X}{1-q}\right]. \tag{45}$$

Proof. This follows from Theorem 6 and the following two identities about Macdonald's symmetric functions.

$$H_\mu[X;t,q] = \omega H_{\mu'}[X;q,t], \tag{46}$$

$$H_\mu[X;q,q] = h_\mu(q) s_\mu\left[\frac{X}{1-q}\right]. \tag{47}$$

In the next two sections we will develop this last example in more detail. The observation that the Macdonald polynomials are built up from the q-twisting of the operators that build the Hall–Littlewood symmetric functions allows us to derive several interesting formulas for these operators.

Remark 10. Unfortunately, the analog of Section 3.2 does not seem to extend to these operators as a way of generalizing the Maconald symmetric functions. Consider the operator $H_{1^\lambda}^t := H_{1^{\lambda_1}}^t H_{1^{\lambda_2}}^t \ldots H_{1^{\lambda_{\ell(\lambda)}}}^t$. We would hope that a composition of $\widetilde{H_{1^\lambda}^t}{}^q$ are Schur positive if reasonable conditions are placed on λ. By calculating examples we begin to be encouraged by such a conjecture, however for large enough examples it seems to break down (for example if $\lambda^{(1)} = (4)$, $\lambda^{(2)} = (2,2)$, $\lambda^{(3)} = (1,1)$, then $\widetilde{H_{1^{\lambda^{(1)}}}^t}{}^q \widetilde{H_{1^{\lambda^{(2)}}}^t}{}^q \widetilde{H_{1^{\lambda^{(3)}}}^t}{}^q 1$ is not Schur positive).

4. Ribbons and Hall–Littlewood symmetric functions

In [18] we gave a combinatorial formula for the action of an operator that adds a column to the Hall–Littlewood symmetric functions. We will recall some of the definitions and theorems from that work and use them to derive some useful formulas.

The definition of a ribbon is a skew partition that contains no 2×2 subdiagrams. For a non-empty partition λ, define $\lambda^{\mathrm{rc}} = (\lambda_2 - 1, \lambda_3 - 1, \ldots, \lambda_{\ell(\lambda)} - 1)$ (the rc indicates that λ has the first row and first column removed). If R is a ribbon of size m (denoted by $R \models m$) then R will be equal to $\lambda/\lambda^{\mathrm{rc}}$ for some partition λ with $\lambda_1 + \ell(\lambda) - 1 = m$.

Set $D(R)$ equal to the descent set of R, that is the set $\{i \mid i + 1\text{st cell lies below the }i\text{th cell in }R\}$ when the cells are labeled with the integers 1 to n from left to right and top to bottom. Therefore every ribbon can be identified with a subset of $\{1, \ldots, m-1\}$.

There is a natural statistic associated with a ribbon. Define the major index of a ribbon to be $\mathrm{maj}(R) = \sum_{i \in D(R)} i$. Its complementary statistic will be $\mathrm{comaj}(R) = \binom{|R|}{2} - \mathrm{maj}(R)$.

From formula (15), S_m is an operator that adds a row to the partition indexing a Schur symmetric function. By conjugating S_m by ω, one obtains an operator that adds a column. Define $\tilde{S}_m = \omega S_m \omega$. In plethystic notation, this operator is given as

$$\tilde{S}_m P[X] = (-1)^m P\left[X + \frac{1}{z}\right] \Omega[-zX]|_{z^m}. \tag{48}$$

Now for each ribbon of size m, define an operator that raises the degree of a symmetric function by m. For $R = \lambda/\lambda^{\mathrm{rc}}$ set

$$S^R = s_{\lambda^{\mathrm{rc}}}^\perp \tilde{S}_{\lambda_1'} \tilde{S}_{\lambda_2'} \cdots \tilde{S}_{\lambda_{\lambda_1}'}, \tag{49}$$

where λ_i' is the length of the ith column in λ. This is a combinatorial operator in the sense that all calculations can be computed on the Schur basis using the Littlewood-Richardson rule and the commutation relations $\tilde{S}_a \tilde{S}_b = -\tilde{S}_{b-1} \tilde{S}_{a+1}$ and $\tilde{S}_a \tilde{S}_{a+1} = 0$ so that

the operator S^R can be thought of as an operator that acts on s_λ by adding the ribbon R to the left of λ.

The main theorem in [18] was the following result.

Theorem 11 (Theorem 1.1 of [18]). *The operator $H_{1^m}^q = \sum_{R \models m} q^{\mathrm{comaj}(R)} S^R$ has the property that $H_{1^m}^q H_\mu[X;q] = H_{1^m|\mu}[X;q]$ for $\ell(\mu) \leqslant m$.*

Some elegant relations develop with the flip operation and ribbon operators. Note that it follows directly from the definition that if R is a ribbon of size m and R^+ is a ribbon of size $m+1$ with $D(R) = D(R^+)$, then $S^{R^+} = S^R \tilde{S}_1$. It develops that there is also a recursive method for adding a cell below the ribbon. If R_+ is a ribbon of size $m+1$ such that $D(R_+) = D(R) \cup \{m\}$, then we have the following surprising formula.

Theorem 12 (Theorem 2.2 of [18]). *If $R \models m$ and $R_+ \models m+1$ such that $D(R_+) = D(R) \cup \{m\}$, then $S^{R_+} = \overline{\overline{S^R}} S_1$.*

This theorem can be used to produce the following plethystic formula for a ribbon operator.

Proposition 13. *Let $R \models m$, then*

$$S^R P[X] = (-1)^{m-|D(R)|} P[X + Z^*] \Omega[-(z_1 + Z_{D(R)^c+1})X]$$

$$\prod_{1 \leqslant i < j \leqslant m} (1 - z_j/z_i)|_{z_1 z_2 \cdots z_m}, \tag{50}$$

where we have set $Z^* = \sum_{i=1}^m 1/z_i$ and $Z_{D(R)^c+1} = \sum_{i \in [1, m-1] - D(R)} z_{i+1}$.

Proof. By induction using Theorem 13 and direct calculation. □

It follows from Theorems 11 and 12 that $H_{1^m}^t$ may be defined recursively. Set $H_{1^1}^t = \tilde{S}_1$ and

$$H_{1^{m+1}}^t = t^m H_{1^m}^t \tilde{S}_1 + \overline{\overline{H_{1^m}^t}} S_1. \tag{51}$$

Either from this recursive definition or from the previous proposition, one may demonstrate the following plethystic formula for the $H_{1^m}^t$ operator.

Proposition 14. *The operator $H_{1^m}^t$ of Theorem 11 has the following form in plethystic notation.*

$$H_{1^m}^t P[X] = -P[X + Z^*] \Omega[-z_1 X] \prod_{i=2}^m (1 - t^{i-1} \Omega[-z_i X])$$

$$\prod_{1 \leqslant i < j \leqslant m} (1 - z_j/z_i) \Bigg|_{z_1 z_2 \cdots z_m}. \tag{52}$$

Note that because the coefficient of z_1 in the expression

$$P[X + Z^*] \prod_{i=2}^{m} (1 - t^{i-1}\Omega[-z_i X]) \prod_{1 \leqslant i < j \leqslant m} (1 - z_j/z_i) \tag{53}$$

is zero, we also have the following equivalent expression.

Corollary 15. *The operator* $H_{1^m}^t$ *of Theorem* 11 *has the following form in plethystic notation.*

$$H_{1^m}^t P[X] = P[X + Z^*] \prod_{i=1}^{m} (1 - t^{i-1}\Omega[-z_i X]) \prod_{1 \leqslant i < j \leqslant m} (1 - z_j/z_i) \Bigg|_{z_1 z_2 \cdots z_m} . \tag{54}$$

In the next section Theorem 6 will be used to develop methods for computing Macdonald polynomials and the q, t-Kostka coefficients from these formulas. One may use some of the properties of the ribbon operators to derive several other formulas for operators $H_{1^m}^t$ and hence for $\widetilde{H_{1^m}^{t}}{}^q$, but this particular formula seems like a natural extension to the ribbon operator formula for the Hall–Littlewood symmetric functions.

5. Generalized ribbons and Macdonald symmetric functions

Consider the following generalization of the plethystic formulas presented in the previous section. Since we know from Theorem 6 that the operator $\widetilde{H_{1^m}^{t}}{}^q$ is an operator that adds a column to the Macdonald symmetric functions, the q-analog of Eq. (54) yields the following theorem.

Theorem 16. *The following operator adds a column to the Macdonald symmetric functions* $H_\mu[X; q, t]$ *if* $\ell(\mu) \leqslant m$.

$$H_{1^m}^{qt} P[X] = P[X + (1-q)Z^*] \prod_{i=1}^{m} (1 - t^{i-1}\Omega[-z_i X])$$

$$\prod_{1 \leqslant i < j \leqslant m} (1 - z_j/z_i) \Bigg|_{z_1 z_2 \cdots z_m} \tag{55}$$

Proof. This follows from Theorem 6 and Corollary 15. Calculate (55) by using Eqs. (54) and (9) to show

$$\widetilde{H_{1^m}^{t}}{}^q P[X] = H_{1^m}^{tY} P[qX + (1-q)Y]|_{Y=X}$$

$$= P[qX + (1-q)Y + (1-q)Z^*] \prod_{i=1}^{m} (1 - t^{i-1}\Omega[-z_i Y])$$

$$\left. \prod_{1 \leqslant i < j \leqslant m} (1 - z_j/z_i) \right|_{z_1 z_2 \cdots z_m} \Bigg|_{Y=X} . \tag{56}$$

We will develop this operator further and show that the combinatorial definition of a ribbon operator can be generalized and used to give a formula analogous to Theorem 11.

Let $V \in \mathrm{Hom}(\Lambda, \Lambda)$ be an operator that does not involve the parameter q and let $P \in \Lambda$ also not include the parameter q. By setting $q=0$ in the expression $\tilde{V}^q P[X]$,

$$\tilde{V}^q P[X]\big|_{q=0} = V^Y P[qX + (1-q)Y]\big|_{Y=X}\big|_{q=0} = VP[X]. \tag{57}$$

We remark that the highest power of q that appears in this expression is the degree of the symmetric function P. By acting $\tilde{V}^q$ on a Schur function, it can be seen that

$$\tilde{V}^q s_\lambda[X] = \sum_{\mu \subseteq \lambda} q^{|\mu|} V^Y (s_\mu[X - Y] s_{\lambda/\mu}[Y]) \Bigg|_{Y=X} . \tag{58}$$

The coefficient of $q^{|\lambda|}$ in this expression will be the term

$$\tilde{V}^q s_\lambda[X]\big|_{q^{|\lambda|}} = V^Y s_\lambda[X - Y]\big|_{Y=X} = \bar{V} s_\lambda[X]. \tag{59}$$

The coefficients of q^k may be interpreted then as a discrete interpolation between V and $\bar{V}$. Define notation for the coefficient of q^k of this expression so that

$$\tilde{V}^q P[X]\big|_{q^k} = \bar{V}^{(k)} P[X] = \sum_{\lambda \vdash k} V^Y (s_\lambda[X - Y](s_\lambda^\perp P)[Y]) \Bigg|_{Y=X} . \tag{60}$$

By linearity, this notation may be extended to any operator V that may now depend on the parameter q. This yields the following proposition.

Proposition 17. *For $V \in \mathrm{Hom}(\Lambda, \Lambda)$,*

$$\tilde{V}^q = \sum_{k \geqslant 0} q^k \bar{V}^{(k)}, \tag{61}$$

where $\bar{V}^{(k)}$ is defined in Eq. (60).

$\overline{S^{R(k)}}$ can be developed in detail thereby giving a combinatorial method for calculating the coefficient of q^i in a Macdonald polynomial.

Define a notion of a generalized ribbon operator that starts with a ribbon $R \models m$ with $R = \lambda/\lambda^{\mathrm{rc}}$ and associate with this a sequence $v = (v_1, v_2, \ldots, v_m)$ with $v_i \geqslant 0$. Generalize

the notion of a ribbon by setting the 'thickness' of the ith cell of the ribbon to be $v_i + 1$ so that when the sequence consists of m zeros this gives the standard ribbon.

Let $\ell = \ell(\lambda)$ and say that $D(R) = \{i_1 > i_2 > \cdots > i_{\ell-1}\}$ and $\{1,\ldots,m\} - D(R) = \{j_1 < j_2 < \cdots < j_{\lambda_1}\}$. Let $\alpha = \lambda^{\mathrm{rc}} - (v_{i_1+1}, v_{i_2+1}, \ldots, v_{i_{\ell-1}+1})$ (as vectors) and $\beta' = \lambda' + (v_1, v_{j_1+1}, v_{j_2+1}, \ldots, v_{j_{\lambda_1-1}+1})$ (neither α nor β' are necessarily partitions). Then set $S^{(R,v)} = (-1)^{|\lambda^{\mathrm{rc}}| - |\alpha|} s_\alpha^\perp \tilde{S}_{\beta'_1} \tilde{S}_{\beta'_2} \cdots \tilde{S}_{\beta'_{\lambda_1}}$. Call $S^{(R,v)}$ a generalized ribbon operator.

The formulation of these operators leads to a simple construction with a picture: draw the original ribbon and place v_i cells either to the left of the ith cell if $i - 1$ is a descent of the ribbon or above the cell if it is not. α is the sequence representing the space underneath the diagram and β' is the sequence representing the heights of the columns of the diagram. The sign represents the number of cells that are 'underneath' the ribbon. We present a couple of examples to give a better picture of these truly combinatorial constructions.

Example 18. Consider the ribbon $R = $ ⬜ of size 4 with $D(R) = \{3, 2\}$. If $v = (0, 0, 0, 0)$, then $S^{(R,v)} = S^R$. If v is one of $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ then $S^{(R,v)}$ is equivalent to the following ribbon operators (respectively):

$$= s_{11}^\perp \tilde{S}_4 \tilde{S}_3 \qquad = s_{11}^\perp \tilde{S}_3 \tilde{S}_4 \qquad - = -s_1^\perp \tilde{S}_3 \tilde{S}_3$$

$$- = -s_{01}^\perp \tilde{S}_3 \tilde{S}_3 \tag{62}$$

The second and fourth generalized ribbons are 0. The second because it contains the operation of adding a column of size 3 on a column of size 4, and the fourth because a row of size 0 is added on a row of size 1 in the skew part of the operator.

Example 19. Let $R = $. Now let $v = (1, 1, 0, 1, 0, 2, 0, 1, 0)$ This is represented by the following picture where a dot is placed in each of the cells representing the original ribbon and there are v_i cells either to the left of the ith cell if $i - 1 \in D(R)$ or above if $i - 1 \notin D(R)$ (and the v_1 cells always go above the first cell in the ribbon).

$$= s_{4321}^\perp \tilde{S}_6 \tilde{S}_6 \tilde{S}_5 \tilde{S}_5 \tilde{S}_3. \tag{63}$$

Representing these operators with a diagram of this sort works fine if $i - 1$ is a descent and v_i is so large that it creates a negative index in α. Interpret this to mean that skewing by a Schur function with a negative index kills the term and the result is 0.

Note also that some 'straightening' using the relation $S_m S_n = -S_{n-1} S_{m+1}$ may be necessary.

Example 20. R is as above, but $v = (1, 1, 0, 1, 0, 4, 0, 1, 5)$. Then R can be represented by the image

$$-\boxed{} = -s^{\perp}_{-1,3,2,1}\tilde{S}_6\tilde{S}_6\tilde{S}_5\tilde{S}_7\tilde{S}_3 = s^{\perp}_{-1,3,2,1}\tilde{S}_6\tilde{S}_6\tilde{S}_6\tilde{S}_6\tilde{S}_3$$

$$= s^{\perp}_{2,1,1,1}\tilde{S}_6\tilde{S}_6\tilde{S}_6\tilde{S}_6\tilde{S}_3 = \boxed{}$$ (64)

The final image comes from first straightening the columns of the generalized ribbon and then straightening Schur function that one skews by with appropriate sign changes.

Generalized ribbon operators are related to the original notion of a ribbon operator by the following easily statable theorem.

Theorem 21. *Let R be a ribbon of size m and $k \geqslant 0$ an integer.*

$$\overline{S^{R(k)}} = \sum_{v} S^{(R,v)} e_v^{\perp},$$ (65)

where the sum is over all sequences v having length m and whose sum is k and the condition that $v_i \geqslant 0$ and e_v is the elementary symmetric function indexed by the sequence v. Let

$$H^{qt}_{1^m} = \sum_{R \models m} \sum_{v} q^{|v|} t^{\mathrm{comaj}(R)} S^{(R,v)} e_v^{\perp},$$ (66)

where here the sum is over all sequences v having length m and non-negative entries. Then $H^{qt}_{1^m} H_\mu[X; q, t] = H_{1^m | \mu}[X; q, t]$ for $\ell(\mu) \leqslant m$.

This theorem is a combinatorial rule for computing Macdonald symmetric functions. Before we present the proof, we give an example of how this theorem works.

Example 22. Computation of a Macdonald symmetric function with generalized ribbons.
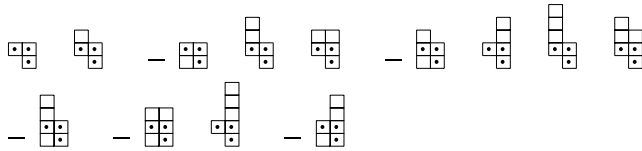
We will use formula (66) to compute $H_{222}[X; q, t]$. This is a long and involved example, but it demonstrates the power of this recurrence since with a reasonable amount of work one can calculate a Macdonald polynomial of size 6 or higher by hand.
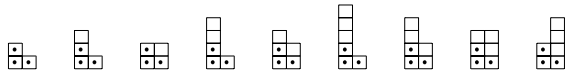
Start with the formula for $H_{111}[X; q, t] = \boxed{} + (t + t^2)\boxed{} + t^3 \boxed{}$ (this may be calculated by acting $H^{qt}_{1^3}$ on 1).
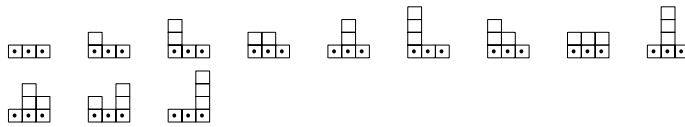
The sum in Eq. (66) over v is finite because only terms such that $|v|$ is less than or equal to the degree of the symmetric function that is being acted on are needed. In this computation quite a few operators are necessary.

We list all of the relevant operators (those which are non-zero) and place a dot in the cells that consist of a the core of the operator so that it is easy to read the sequence v from the picture. The sign associated to each picture of the operator is -1 to the power of the number of cells under the ribbon.

$$(67)$$

$$(68)$$

$$(69)$$

$$(70)$$

To complete this computation, calculate $e_\lambda^\perp$ on the symmetric function $H_{111}[X;t]$ for $|\lambda| \leqslant 3$. This is given by the following list

$$e_1^\perp H_{111}[X;t] = (1 + t + t^2)(ts_2 + s_{1,1}), \tag{71}$$

$$e_2^\perp H_{111}[X;t] = (1 + t + t^2)s_1, \tag{72}$$

$$e_{11}^\perp H_{111}[X;t] = (1 + 2t + 2t^2 + t^3)s_1, \tag{73}$$

$$e_3^\perp H_{111}[X;t] = 1, \tag{74}$$

$$e_{21}^\perp H_{111}[X;t] = 1 + t + t^2, \tag{75}$$

$$e_{111}^\perp H_{111}[X;t] = 1 + 2t + 2t^2 + t^3. \tag{76}$$

The computation proceeds as follows. The coefficient of q^0 is just the Hall–Littlewood symmetric function $H_{222}[X;t]$, calculated by acting $+\ t$ $+\ t^2$ $+\ t^3$ on $H_{111}[X;t]$. So the coefficient of q^0 is

$$s_{222} + (t + t^2)s_{321} + t^3 s_{33} + t^3 s_{411} + (t^2 + t^3 + t^4)s_{42} + (t^4 + t^5)s_{51} + t^6 s_6. \tag{77}$$

The coefficient of q^1 is the operator $\square\!\!\!\square + t\,\square\!\!\!\square - t\,\square\!\!\!\square + t^2\,\square\!\!\!\square + t^2\,\square\!\!\!\square + t^3\,\square\!\!\!\square$ when it acts on the symmetric function $(1 + t + t^2)(ts_2 + s_{1,1})$.

$$(1 + t + t^2)(t^4 s_{51} + (t^2 + t^3)s_{411} + t^3 s_{42} + (t + t^2)s_{321}$$
$$+ t^2 s_{33} + t s_{3111} + s_{2211}). \tag{78}$$

The coefficient of q^2 comes from two components,

$$\square\!\!\!\square + t\,\square\!\!\!\square + t\,\square\!\!\!\square + t^2\,\square\!\!\!\square + t^3\,\square\!\!\!\square + t^3\,\square\!\!\!\square$$

when it acts on the symmetric function $e_2^\perp H_{111}[X; q, t]$, and

$$t\,\square\!\!\!\square - t\,\square\!\!\!\square + t^2\,\square\!\!\!\square + t^3\,\square\!\!\!\square$$ when it acts on the symmetric function $e_{11}^\perp H_{111}[X; q, t]$. The first part is

$$(t^2 + t + 1)(-t^3 s_{42} + t^3 s_{411} + (t + t^2)s_{3111} - t s_{222} + s_{2211}t + s_{21111}) \tag{79}$$

and the second is

$$(1 + 2t + 2t^2 + t^3)(t^3 s_{42} + t^2 s_{321} + t s_{222}). \tag{80}$$

The sum of these two quantities is

$$(t^2 + t + 1)(t^4 s_{42} + t^3 s_{411} + (t^2 + t^3)s_{321} + (t + t^2)s_{3111}$$
$$+ t^2 s_{222} + t s_{2211} + s_{21111}). \tag{81}$$

The coefficient of q^3 comes from three different operators acting each on a different constant. The first operator is

$$\square\!\!\!\square + t\,\square\!\!\!\square + t\,\square\!\!\!\square + t^2\,\square\!\!\!\square + t^2\,\square\!\!\!\square + t^3\,\square\!\!\!\square + t^3\,\square\!\!\!\square + t^3\,\square\!\!\!\square$$

when it acts on 1, the second is

$$t\,\square\!\!\!\square - t\,\square\!\!\!\square - t\,\square\!\!\!\square + t^2\,\square\!\!\!\square + t^2\,\square\!\!\!\square + t^3\,\square\!\!\!\square$$
$$+ t^3\,\square\!\!\!\square + t^3\,\square\!\!\!\square$$

when it acts on $1 + t + t^2$, and the third is $-t\,\square\!\!\!\square + t^3\,\square\!\!\!\square$ when it acts on the symmetric function $2t + 2t^2 + t^3 + 1$. These three parts area

$$t^3 s_{33} - t^3 s_{321} + t^3 s_{3111} - (t + t^2)s_{222} + (t + t^2)s_{21111} + s_{111111}, \tag{82}$$

$$(1 + t + t^2)(-2t^3 s_{33} + t^3 s_{321} + (t^2 + 2t)s_{222} + t^2 s_{2211}), \tag{83}$$

$$(2t + 2t^2 + t^3 + 1)(-t s_{222} + t^3 s_{33}). \tag{84}$$

The sum of these three quantities is

$$t^6 s_{33} + (t^4 + t^5)s_{321} + t^3 s_{3111} + t^3 s_{222}$$

$$+ (t^2 + t + 1)t^2 s_{2211} + (t + t^2)s_{21111} + s_{111111}, \tag{85}$$

which is the coefficient of q^3 in $H_{222}[X; q, t]$.

Clearly, an enormous amount of simplification occurs when arriving at a final expression for $H_\mu[X; q, t]$. An eventual goal of a combinatorial recurrence on the q, t-Kostka coefficients will be to arrive at a combinatorial interpretation for them in terms of standard tableaux. Even if this recurrence turns out to be too complicated, these techniques (in particular, Theorems 6 and 11) can certainly be used to derive many other recurrences for the coefficients.

Before presenting the proof, we will need a few lemmas that come from the derivation of the ribbon operator. We state them without proof and refer the reader to [18].

Lemma 23. *For any operator* V, $\overline{\overline{V} S_m} = \sum_{j \geqslant 0} (-1)^{m-j} h_j V \tilde{S}_{m-j}$.

Lemma 24. $\overline{\overline{s_\lambda^\perp S_{-m}}} = (s_{(m, \lambda)})^\perp$.

Proof of Theorem 21. If $a \geqslant 0$ and v is a list, then we denote v with a prepended (resp. appended) by (a, v) (resp. (v, a)).

Let R^+ be a ribbon of size $m + 1$ that does not have m as a descent. Also let R be the ribbon of size m such that $D(R) = D(R^+)$. By the definition of $S^{(R,v)}$, notice that $S^{(R^+, (v,a))} = S^{(R,v)} \tilde{S}_{1+a}$.

Now let R_+ be a ribbon of size $m + 1$ such that m is a descent. Let $R = \lambda/\lambda^{\mathrm{rc}}$ be the ribbon of size m such that $D(R) \cup \{m\} = D(R_+)$ then remark that $R_+ = (\lambda_1, \lambda)/(\lambda_1 - 1, \lambda^{\mathrm{rc}})$. If $S^{(R,v)} = (-1)^{|v|} s_\alpha^\perp \tilde{S}_{\beta'}$, then $S^{(R_+, (v,a))} = (-1)^{|v|+a} s_{(\lambda_1-1-a, \alpha)}^\perp \tilde{S}_{(\lambda_1, \beta)'}$. It follows from Lemmas 23 and 24 and the commutation relation $\tilde{S}_m \tilde{S}_n = -\tilde{S}_{n-1} \tilde{S}_{m+1}$ that

$$S^{(R_+, (v,a))} = (-1)^{|\lambda^{\mathrm{rc}}|-|\alpha|+a} \overline{s_\alpha^\perp S_{1+a-\lambda_1}} \tilde{S}_{(\lambda_1, \beta)'}$$

$$= (-1)^{|\lambda^{\mathrm{rc}}|-|\alpha|+a} \sum_{j \geqslant 0} (-1)^{1+a-\lambda_1-j} h_j s_\alpha^\perp \tilde{S}_{1+a-\lambda_1-j} \tilde{S}_{(\lambda_1, \beta)'}$$

$$= (-1)^{|\lambda^{\mathrm{rc}}|-|\alpha|+a} \sum_{j \geqslant 0} (-1)^{1+a-j} h_j s_\alpha^\perp \tilde{S}_{\beta'} \tilde{S}_{1+a-j}$$

$$= (-1)^a \overline{S^{(R,v)}} S_{1+a}. \tag{86}$$

Use these two relations to give an inductive derivation of the following plethystic form of the operator $S^{(R,v)}$. By carrying out nearly the exact same calculation (and

using identical notation for $Z_{D(R)^c+1}$) as given in Proposition 13, derive that

$$S^{(R,v)}P[X] = (-1)^{m-|D(R)|+|v|}P[X + Z^*]\Omega[-(z_1 + Z_{D(R)^c+1})X]$$

$$\prod_{1 \leqslant i < j \leqslant m} (1 - z_j/z_i)\big|_{z_1^{1+v_1} z_2^{1+v_2} \cdots z_m^{1+v_m}}. \tag{87}$$

Now consider a formula for $\widetilde{S^{Rq}}$. Using the same calculation for Theorem 16 and the equation given in Proposition 13, demonstrate that

$$\widetilde{S^{Rq}} = (-1)^{m-|D(R)|}P[X + (1-q)Z^*]\Omega[-(z_1 + Z_{D(R)^c+1})X]$$

$$\prod_{1 \leqslant i < j \leqslant m} (1 - z_j/z_i)\big|_{z_1 z_2 \cdots z_m}. \tag{88}$$

The coefficient of q^k in this formula is

$$\overline{S^{R(k)}} = (-1)^{m-|D(R)|} \sum_{\lambda \vdash k} f_\lambda$$

$$[-Z^*]e_\lambda^\perp P[X + Z^*]\Omega[-(z_1 + Z_{D(R)^c+1})X] \prod_{1 \leqslant i < j \leqslant m} (1 - z_j/z_i)\big|_{z_1 z_2 \cdots z_m}$$

$$= (-1)^{m-|D(R)|+k} \sum_{\lambda \vdash k} m_\lambda[Z^*]e_\lambda^\perp P[X + Z^*]\Omega[-(z_1 + Z_{D(R)^c+1})X]$$

$$\prod_{1 \leqslant i < j \leqslant m} (1 - z_j/z_i)\big|_{z_1 z_2 \cdots z_m}. \tag{89}$$

By expanding $m_\lambda[Z^*]$ as $\sum_{v \sim \lambda} z^{-v}$ we see clearly that this is equivalent to Eq. (65). The formula stated for the operator $H_{1^m}^{qt}$ follows from this derivation, Theorem 6 and Proposition 17.

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