# Generalized Invertibility of Hankel and Toeplitz Matrices*

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ABSTRACT

A result for the computation of the Moore-Penrose inverse of Hankel matrices over the field of the complex numbers is given by Heinig and Rost. In this note some methods of characterizing and computing generalized inverses of Hankel and Toeplitz matrices over fields and over rings with the extended Rao condition are presented. The results were obtained by combining classical theory on the special structure of those classes of matrices with recent theory on generalized invertibility.

## 1. INTRODUCTION

Let $R$ be an associative ring with unity, and let Mat($R$) be the category of finite matrices over $R$. An involution $*$ on Mat($R$) is a unary operation such that if $A$ is $m \times n$, $A^*$ is $n \times m$ and

(1) for all $A$ in Mat($R$), $A^{**} = A$;
(2) for all $B, C$ in Mat($R$) for which $B + C$ is defined, $(B + C)^* = B^* + C^*$;
(3) for all $F, G$ in Mat($R$) for which $F \cdot G$ is defined, $(F \cdot G)^* = G^* \cdot F^*$.

A matrix $S$ is called symmetric with respect to $*$ if $S^* = S$.

A $m \times n$ matrix is said to be von Neumann regular if the equation $AXA = A$ has at least one solution $A^{(1)}$. A common solution of the equations

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$AXA = A$ and $XAX = X$ defines a (1, 2)-inverse of $A$, denoted by $A^{(1,2)}$. The (unique) solution, if it exists, of the set of equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, $(XA)^* = XA$ is called the *Moore-Penrose inverse* $A^+$ of $A$. The *group inverse* $A^\#$ of $A$ is the unique (if it exists) (1, 2)-inverse $X$ of $A$ such that $AX = XA$.

If $A$ is an $m \times n$ matrix from Mat($R$), then $A[\theta/\beta] = [(i_1, \dots, i_p)/(j_1, \dots, j_q)]$ is the $p \times q$ submatrix of $A$ formed by all the components of $A$ belonging to rows $i_1 < \cdots < i_p$ and columns $j_1 < \cdots < j_q$ ($p \leqslant m$, $q \leqslant n$). If $\theta = \beta$, we abbreviate $A[\theta/\beta]$ by $A[\theta]$. We denote by $H = (h_{i+j})$, $i = 0, \dots, m-1$, $j = 0, \dots, n-1$, an $m \times n$ Hankel matrix on Mat($R$) (usually it will be supposed that $m \leqslant n$). By $T = (t_{i-j})$, $i = 0, \dots, m-1$, $j = 0, \dots, n-1$, we mean a Toeplitz matrix over $R$. The symbols $I$ and $J$ ($I_n$ and $J_n$) will denote, respectively, the identity matrix and the counteridentity matrix (of order $n$).

Let

$$D_{\lambda-1} = \begin{cases} \det H[(1, \dots, \lambda)] & \text{if} \quad \lambda = 1, \dots, m, \\ 1 & \text{if} \quad \lambda = 0. \end{cases}$$

The *r-characteristic* of $H$ is defined by Iohvidov [9] as the maximal natural number $\lambda$ $(0 \leqslant \lambda \leqslant m)$ such that $D_{\lambda-1}$ is different from zero. We will refer to it as $r(H)$.

Let $b_{\mu\nu} = \det H[(1, \dots, r, r+\mu+1)/(1, \dots, r, r+\nu+1)]$ be the bordered minor

$$b_{\mu\nu} = \left|\begin{array}{c:c} & h_{r-\nu} \\ D_{r-1} & \vdots \\ \hdashline h_{r+\mu} \;\cdots & h_{2r+\mu+\nu} \end{array}\right|,$$

and let $B$ be the matrix $B = (b_{\mu\nu})$, $0 \leqslant \mu \leqslant m - r - 1$, $0 \leqslant \nu \leqslant n - r - 1$.

LEMMA 1.1 (Frobenius' Theorem). *Let $H$ be an $n \times n$ Hankel matrix of rank $\rho$ and $r(H) = r$. Then the minor $D_{\rho-1}$ of order $\rho$ consisting of the first $r$ lines (rows and columns) of the matrix $H$ and its last $\rho - r$ lines is different from zero.*

*Proof.* See [6, 9]. ∎

LEMMA 1.2 [6]. *$B[(1, \dots, m-r)]$ is a Hankel matrix such that $b_{\mu+\nu} = 0$ for $\mu + \nu \leqslant m - r - 1$.*

REMARK 1. The matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

of rank 2 and $r(H) = 1$ is an example which illustrates that Frobenius' Theorem is not true for any nonsquare Hankel matrix $H$. However, we can find several submatrices of $H$ where the Theorem is verified. In the following section we will prove that there exists a well-defined submatrix of $H$ to which Frobenius' Theorem can always be extended.

## 2. THE EXTENDED FROBENIUS' THEOREM

Let $F$ be a field, and let $H \in \mathrm{Mat}(F)$ be of rank $\rho$ with $r(H) = r$.

LEMMA 2.1. *The rank of the* $(m-r) \times (n-r)$ *matrix* $B$ *is equal to* $\rho - r$.

*Proof.* From the definition of $B$ it follows that to any minor of order $l$,

$$B_{l-1}^{\sim} = \det B[(i_1, \dots, i_l)/(j_1, \dots, j_l)],$$

$$1 \leqslant i_k \leqslant m - r, \; 1 \leqslant j_k \leqslant n - r, \quad k = 1, \dots, l,$$

corresponds a minor $D_{r+l-1}^{\sim}$ of $H$ (of order $r + l$) such that

$$D_{r+l-1}^{\sim} = \det H[(1, \dots, r, r+i_1, \dots, r+i_l)/ (1, \dots, r, r+j_1, \dots, r+j_l)].$$

Since $H$ is of rank $\rho$ and $D_{r-1} \neq 0$, it follows that:

(1) $D_{r+l-1}^{\sim} = 0$ for $l > \rho - r$. Consequently, by Sylvester's determinant identity [6], $B_{l-1}^{\sim}$ must also vanish for $l > \rho - r$.

(2) There are nonzero minors

$$D_{\rho-1}^{\sim} = \det H[(1, \dots, r, i_1, \dots, i_{\rho-r})/(1, \dots, r, j_1, \dots, j_{\rho-r})]$$

of order $\rho$ (bordering $D_{r-1}$) to which corresponds a nonzero minor of $B$ with order $\rho - r$.

Then there exist nonzero minors of $B$ with maximal order $\rho - r$. ■

REMARK 2. If $\rho = r$, then obviously $B$ is the null matrix.

If $\rho \neq r$, then it is clear that there must exist a minimal natural $t$, $0 \leqslant t \leqslant m - r - 1$, and a minimal natural $p$, $1 \leqslant p \leqslant n - r - 1$, such that $b_{tp} \neq 0$ (i.e., if $\mu < t$ then $b_{\mu\nu} = 0$, and if $\nu < p$ then $b_{t\nu} = 0$).

LEMMA 2.2. *If* $s := r + p + 1$, *then* $m \leqslant s \leqslant n$ *and* $B[(1, \dots, m - r)/(1, \dots, s - r)]$ *is a Hankel matrix of rank* $m - r - t$.

*Proof.* The case $r = \rho$ being trivial, let $r < \rho$.

From Lemma 1.2 and the foregoing definition of $t$ and $p$ it follows that $b_{\mu\nu} = 0$ for $\mu = 1, \dots, t - 1$, $b_{\mu\nu} = 0$ for $\mu + \nu < t + m - r - 1$, and $b_{t\mu} = 0$ for $\mu \leqslant p - 1$. Consequently, a nonzero minor of $B$ with maximal order $\rho - r$ has to belong to the last $m - r - t$ rows of $B$. Next we prove that

$$\operatorname{rank} B = \operatorname{rank} B[(1, \dots, m - r)/(1, \dots, s - r)] = m - r - t.$$

From the definition of $b_{\mu\nu}$ it follows that

$$b_{\mu\nu} = h_{2r+\mu+\nu} D_{r-1} + \left| \begin{array}{ccc:c} & D_{r-1} & & \begin{matrix} h_{r+\nu} \\ \vdots \\ h_{2r+\nu-1} \end{matrix} \\ \hdashline h_{r+\mu} & \cdots & h_{2r+\mu-1} & 0 \end{array} \right|. \tag{E}$$

The definition of $b_{tp}$ such that $b_{tp} \neq 0$ implies that the rank of $H_{r+t, r+p+1} := H[(1, \dots, r + t)/(1, \dots, r + p + 1)]$ (and also of $H_{r+t+1, r+p} := H[(1, \dots, r + t + 1)/(1, \dots, r + p)])$ is equal to $r$, but the rank of $H[(1, \dots, r + t + 1)/(1, \dots, r + p + 1)]$ is bigger than $r$. The nonsingularity of $H[(1, \dots, r)]$, together with simple reasoning on the linear dependence of the columns of the matrix $H_{r+t, r+p+1}$, leads us to the formula

$$h_q = \sum_{j=0}^{r-1} \alpha_j h_{q-j-1} \qquad (q = r, r + 1, \dots, 2r + p + t - 1)$$

for some scalars $\alpha_j$, $j = 0, \dots, r - 1$ (see [9, pp. 55, 59]).

Therefore,

$$b_{\mu\nu} = h_{2r+\mu+\nu}D_{r-1} + \sum_{j=0}^{r-1} \alpha_j \left(b_{\mu,\,\nu-j-1} - h_{2r+\mu+\nu-j-1}D_{r-1}\right)$$

for $\mu + \nu \leqslant t + p$. This means that $b_{\mu\nu}$ only depends on $\mu + \nu$ for $\mu + \nu \leqslant t + p$. Moreover, since $h_{i+j} = h_{ij}$, $0 \leqslant i \leqslant m - 1$, $0 \leqslant j \leqslant n - 1$, a simple computation on the equality (E) leads to the conclusion that $b_{t+1,\,p} = b_{t+2,\,p-1}, \ldots, b_{m-r-1,\,p-1} = b_{m-r-2,\,p}$, and hence $b_{\mu\nu} = b_{\mu+\nu}$ for $t + p < \mu + \nu < m - r - 1 + p$.

Thus, if $s := r + p + 1$, we can conclude that $B[(1, \ldots, m - r)/(1, \ldots, s - r)]$ is a Hankel matrix of the form

$$\begin{bmatrix} & & & & & & 0 \\ & \bigcirc & & & & & \vdots \\ & & & & & & 0 \\ & & & & & & b_{tp} \\ & & & & & & \vdots \\ & & & & & & b_{m-r-2,\,p} \\ 0 & \cdots & 0 & b_{tp} & \cdots & b_{m-r-1,\,p} & b_{m-r-1,\,p} \end{bmatrix}$$

which clearly is of rank $m - r - t$.

The inequality $m - r - 1 \leqslant p \leqslant n - r - 1$, i.e., $m \leqslant s \leqslant n$ is a consequence of Lemma 1.2 together with the definition of $p$. ■

THEOREM 2.1 (Extended Frobenius' Theorem). *Let $H$ be an $m \times n$ Hankel matrix on* Mat$(F)$ *with rank $\rho$ and $r(H) = r$. Let $H_{m,\,s}$ be the $m \times s$ submatrix $H[(1, \ldots, m)/(1, \ldots, s)]$ $(m \leqslant s \leqslant n)$.*

*If $k := m - r - t$, then the minor $D_{r+k-1}^{\sim}$ of order $r + k$, consisting of the first $r$ lines of $H_{m,\,s}$ and its last $k$ lines, is different from zero.*

*Proof.* By Lemma 2.2, a nonzero minor of $B$ with maximal order $\rho - r$ consists of elements of the last $m - r - t =: k$ rows of $B$ (i.e. $\rho = r + k$, $0 \leqslant k \leqslant \rho - r$).

Since $b_{tp} \neq 0$, the minor of $B$ (of order $k$) given by

$$B_{k-1}^{\sim} = \det B\left[(t + 1, \ldots, m - r)/(p + 2 - k, \ldots, p + 1)\right]$$

is nonnull. As was mentioned in the proof of Lemma 2.1, to $B_{k-1}^{\sim}$ corresponds a minor $D^{\sim}_{r+k-1}$ of $H$ (of order $\rho$), bordering $D_{r-1}$, such that

$$D^{\sim}_{r+k-1} = \det H\big[(1,\dots,r,m-k+1,\dots,m)/ \\ (1,\dots,r,s-k+1,\dots,s)\big],$$

which, by the Sylvester's determinant identity, is also nonnull. Therefore,

$$\operatorname{rank} H\big[(1,\dots,r,m-k+1,\dots,m)/(1,\dots,r,s-k+1,\dots,s)\big] \\ = r+k = \operatorname{rank} H_{m,s} = \operatorname{rank} H. \quad \blacksquare$$

## 3. GENERALIZED INVERSES OF HANKEL MATRICES OVER FIELDS

*3.1*

It follows from Theorem 2.1 that for any Hankel matrix over a field $F$ there are matrices $X$, $Y$ such that

$$H := \begin{bmatrix} I_r & O \\ X & Y \\ O & I_k \end{bmatrix} \begin{bmatrix} H_{r,n} \\ H_{k,n} \end{bmatrix}$$

is a full rank factorization of $H$, with

$$H_{r,n} = H\big[(1,\dots,r)/(1,\dots,n)\big] \quad \text{and}$$

$$H_{k,n} = H\big[(m-k+1,\dots,m)/(1,\dots,n)\big].$$

If $K_{n,r}$ denotes the right inverse of $H_{r,n}$, then

$$[X \quad Y] = \big[H[(r+1,\dots,m-k)/(1,\dots,n)]K_{n,r} \quad Y\big].$$

From the structure of $B = (b_{\mu\nu})$ it follows that $Y = O$. Therefore, considering Theorem 3 in [12], the Moore-Penrose inverse $H^{*}$ exists with respect to any involution $*$ on Mat$(F)$ iff

$$I_r + X^*X \quad \text{and} \quad \begin{bmatrix} H_{r,n} \\ H_{k,n} \end{bmatrix} \begin{bmatrix} H_{r,n} \\ H_{k,n} \end{bmatrix}^*$$

are invertible. In that case,

$$H^{\dagger} = \begin{bmatrix} H_{r,n} \\ H_{k,n} \end{bmatrix}^* \left\{ \begin{bmatrix} H_{r,n} \\ H_{k,n} \end{bmatrix} \begin{bmatrix} H_{r,n} \\ H_{k,n} \end{bmatrix}^* \right\}^{-1} \begin{bmatrix} (I_r + X^*X)^{-1} & O \\ O & I_k \end{bmatrix} \begin{bmatrix} I_r & O \\ X & O \\ O & I_k \end{bmatrix}.$$

If $H$ is a square Hankel matrix, then its group inverse can be characterized using Theorem 1 in [12]. Therefore, the group inverse $H^{\#}$ exists iff

$$\begin{bmatrix} H_{r,n} \\ H_{k,n} \end{bmatrix} \begin{bmatrix} I_r & O \\ X & O \\ O & I_k \end{bmatrix}$$

is invertible. In this case,

$$H^{\#} = \begin{bmatrix} I_r & O \\ X & O \\ O & I_k \end{bmatrix} \left\{ \begin{bmatrix} H_{r,n} \\ H_{k,n} \end{bmatrix} \begin{bmatrix} I_r & O \\ X & O \\ O & I_k \end{bmatrix} \right\}^{-2} \begin{bmatrix} H_{r,n} \\ H_{k,n} \end{bmatrix},$$

and $H^{\#} = H^{\dagger}$ with respect to the involution $T : (a_{ij}) \to (a_{ji})$ on Mat$(F)$.

*3.2*

Let $F$ be an algebraically closed field, and let $H$ on Mat$(F)$ be of nonmaximal rank. Then $H$ has a kernel $L$ (see [5]). If, in particular, $m = n$, then $L^T H = H L^T = O$ implies that $H$ has also a cokernel $L^T$. Hence, respectively from Theorems 1 and 2 in [14], it follows that

(1) if $H^*H + LL^*$ is invertible then $H^{\dagger} = (H^*H + LL^*)^{-1}H^*$;
(2) if $L^T L$ and $H^2 + L(L^T L)^{-1}L^T$ are invertible then

$$H^{\#} = H\left[H^2 + L(L^T L)^{-1}L^T\right]^{-1} = \left[H^2 + L(L^T L)^{-1}L^T\right]^{-1} H.$$

REMARK 3. Theorem 5.7 in [5] is a different version of the extended Frobenius' Theorem. Its proof deals with the factorization

$$H = P\begin{bmatrix} H[(1,\dots,r)] & O \\ O & W \end{bmatrix}Q,$$

with $P$, $Q$ invertible matrices. We remark that this factorization, together with Theorems 1 and 2 in [7], gives another possibility of characterizing Moore-Penrose and group inverses of Hankel matrices over the complexes.

3.3

Let $F$ be an algebraically closed field, and let $H$ be a square Hankel matrix over $F$. M. Fiedler proved [3] that if rank $H = \rho$, then $H = H_P + H_D$ with $H_P, H_D$ Hankel matrices such that rank $H_P = r$, rank $H_D = \rho - r$ $(= k)$. Also from results in [3], it follows that there exist von Neumann regular inverses $H_P^{(1)}$, $H_D^{(1)}$ such that $H_P^{(1)} + H_D^{(1)}$ is a von Neumann regular inverse $H^{(1)}$ of $H$, and consequently $H^{(1)}HH^{(1)}$ is a $(1,2)$-inverse of $H$. Since $H$ is symmetric with respect to the involution $T$ and since there exist idempotent matrices $HH^{(1)}$ and $H^{(1)}H$ such that $H = HH^{(1)}H$, it follows that $H$ is "$H^{(1,2)}$-reduced to $H$" (see [8]).

Then we can consider the following proposition as a corollary of Theorem 1 in [8]:

PROPOSITION 3.1. *$H^{*}$ exists with respect to $T$ iff $H^2H^{(1)} + I - HH^{(1)}$ and $H^{(1)}H^2 + I - H^{(1)}H$ are invertible. In that case,*

$$H^{*} = H^{\#} = HH^{(1)}(H^2H^{(1)} + I - HH^{(1)})^{-2}H.$$

## 4. HANKEL MATRICES OVER RINGS WITH THE EXTENDED RAO CONDITION

We recall (see [15]) that a *ring with the extended Rao condition* is a ring with unity and involution $a \to \bar{a}$ such that

$$\text{if } a_1 = \sum_{i=1}^{n} a_i\bar{a}_i \text{ then } a_i = 0 \qquad \text{whenever} \quad i \neq 1.$$

In this section we will show that, over such rings, Hankel matrices which have a Moore-Penrose inverse have a simple structure and also that their Moore-Penrose inverse is easily determined.

PROPOSITION 4.1. *Let $R$ be an associative ring with unity and with an involution $a \to \bar{a}$ that satisfies the extended Rao condition. Let $H$ be an $m \times n$ matrix from the category* $\mathrm{Mat}(R)$ *with the involution $(a_{ij}) \to (a_{ij})^* = (\bar{a}_{ji})$. If $H$ is a noninvertible Hankel matrix, then it has a Moore-Penrose inverse $H^+$ if and only if*

$$H = \begin{bmatrix} H_r & O \\ O & O \end{bmatrix} + \begin{bmatrix} O & O \\ O & H_k \end{bmatrix}, \qquad r + k < m,$$

*where $H_r$ and $H_k$, respectively, are an $r \times r$ invertible upper triangular and an $k \times k$ invertible lower triangular Hankel matrix, or the zero matrix. In this case*

$$H^+ = \begin{bmatrix} H_r & O \\ O & O \end{bmatrix}^+ + \begin{bmatrix} O & O \\ O & H_k \end{bmatrix}^+ .$$

*Moreover, $H^+$ is a Hankel matrix iff only the main auxiliary diagonals in $H_r$ and $H_k$ (if not absent) are nonzero.*

*Proof.* It is known (see [15]) that $H$ has a Moore-Penrose inverse with respect to $* : (a_{ij}) \to (\bar{a}_{ji})$ iff

$$H = P_\beta \begin{bmatrix} M & O \\ O & O \end{bmatrix} P_\sigma$$

with $P_\beta$, $P_\sigma$ permutation matrices and $M$ an invertible matrix. This means that $H$ must have at least one zero row (and one zero column), which implies that $H$ must be of the form

$$\begin{bmatrix} * & \cdots * & 0 & \cdots & & 0 \\ \vdots & \iddots & \iddots & & & \vdots \\ * & \iddots & & & & 0 \\ 0 & & & & \iddots & * \\ \vdots & & & \iddots & \iddots & \vdots \\ 0 & \cdots & & 0 & * \cdots & * \end{bmatrix},$$

i.e.,

$$H = \begin{bmatrix} H_r & O & O \\ O & O & O \\ O & O & H_k \end{bmatrix} \qquad \text{with} \quad r + k < m$$

and with $H_r$, $H_k$ as stated above.

Since $(A+B)^{*} \neq A^{*} + B^{*}$ in general, we prove that

$$H^{*} = \begin{bmatrix} H_r & O \\ O & O \end{bmatrix}^{*} + \begin{bmatrix} O & O \\ O & H_k \end{bmatrix}^{*}$$

by checking the Moore-Penrose equations.

The rest of the proposition follows from the fact that $H_r^{-1}$ and $H_k^{-1}$, respectively, are a lower triangular and a upper triangular matrix. ■

## 5. GENERALIZED INVERTIBILITY OF TOEPLITZ MATRICES

Let $T$ be an $m \times n$ Toeplitz matrix on Mat($R$). As is well known, there exist $m \times n$ Hankel matrices $H$, $H'$ and counteridentity matrices $J_m$, $J_n$ such that $T = J_m H = H' J_n$. Hence $T^{*}$ exists iff $H^{*}$ exists with respect to any involution $*$ on Mat($R$) with $R$ either a field or a ring with the extended Rao condition. This follows from the fact that over such $R$, by the Skölem-Noether theorem, $J^{*} = J\ (=J^{-1})$ for any involution $*$ on Mat($R$). Moreover, $T^{*} = H^{*} J_m = J_n H'^{*}$.

If $T$ is a square Toeplitz matrix, then it is not true that $T^{\#}$ exists iff $H^{\#}$ exists. This can be illustrated with examples:

(1) Let $R = Z_2$, and let

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{on Mat}(R).$$

There is $T$ on Mat($R$) such that

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad H = JT.$$

However, $H^{\#}$ exists and $T^{\#}$ does not.

(2) Let $R$ be the field of complex numbers. Let

$$T = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \qquad \text{on Mat}(R).$$

There is

$$H = \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix} \quad \text{on } \mathrm{Mat}(R)$$

such that $T = JH$. However, if $i^2 = -1$, then $T^\#$ exists but $H^\#$ does not.

REMARK 4. If $H^\#$ exists, then from Theorem 1 in [7] it follows that $T^\#$ exists iff $HH^\#T + I - HH^\#$ is invertible. In that case

$$T^\# = T(HH^\#T + I - HH^\#)^{-2}.$$

From the same theorem an analogous conclusion can be obtained for $H^\#$ if $T^\#$ exists.

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