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# Approximation on Banach spaces of functions on the sphere ☆

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**Abstract**

Many approximation results were proved on $L_p(S^{d-1})$, $1 \leqslant p \leqslant \infty$ where $S^{d-1}$ is the unit sphere in $R^d$. We will show here that most of these results extend to Banach spaces on the sphere for which operation by a $d \times d$ orthogonal matrix is a continuous isometry.


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*Keywords:* Spherical homogeneous Banach spaces; Cesàro summability; Smoothness and rate of approximation on the sphere

## 1. Introduction

For functions on $T$ (or $R$ or $R^d$) many approximation theorems are extendable to Banach spaces of functions for which translation is a continuous isometry, that is, satisfying

$$\|f(x+u)\|_B = \|f(x)\|_B \text{ (translation is an isometry)} \tag{I}$$

and

$$\|f(x+u) - f(x)\|_B = o(1) \quad u \to 0 \text{ (translation is continuous).} \tag{II}$$

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Such results were described in [13,7,8] and several other papers. For functions on the unit sphere of $R^d$, $S^{d-1}$ given by

$$S^{d-1} = \{x \in R^d : |x|^2 = x_1^2 + \cdots + x_d^2 = 1\},$$

the elements of $SO(d)$

$$SO(d) = \{\rho : \rho \quad \text{is a} \quad d \times d \quad \text{real orthogonal matrix}, \quad \det \rho = 1\}$$

replace the translations in (I) and (II) (as $x \in S^{d-1}$ does not imply $x + a$ is in $S^{d-1}$).

In this paper we deal with Banach spaces of functions on $S^{d-1}$ for which all $\rho \in SO(d)$ are isometries, that is

$$\|f(\rho\, \cdot)\|_B = \|f(\cdot)\|_B \equiv \|f(I\, \cdot)\|_B \quad \text{for all } \rho \in SO(d). \tag{1.1}$$

Furthermore, the operation by $\rho$ is assumed to be continuous, that is,

$$\|f(\rho\, \cdot) - f(\cdot)\|_B \to 0 \quad \text{as} \quad |\rho - I| \to 0, \tag{1.2}$$

where

$$|\rho - I|^2 = \max_{x \in S^{d-1}} (\rho x - x, \rho x - x) = \max_{x \in S^{d-1}} 2\,(1 - (\rho x \cdot x))\,. \tag{1.3}$$

(In relation to earlier results given in [10] we note that $\max_{x \in S^{d-1}} (\rho x \cdot x) \geqslant \cos t$ is equivalent to $|\rho - I| \leqslant 2|\sin \frac{t}{2}|$.) Using (1.1), one may write (1.2) in the form

$$\|f(\rho\, \cdot) - f(\tau\, \cdot)\|_B \to 0 \quad \text{as } |\rho - \tau| \to 0. \tag{1.2$'$}$$

Clearly, for $L_p(S^{d-1})$ (1.1) is satisfied for $1 \leqslant p \leqslant \infty$ and (1.2) for $1 \leqslant p < \infty$. The subspace of $L_\infty(S^{d-1})$ for which (1.2) is satisfied is $C(S^{d-1})$. We note that for $L_p(S^{d-1})$, $1 \leqslant p < \infty$

$$\|f\|_p = \left\{ \int_{S^{d-1}} |f(x)|^p \, dx \right\}^{1/p} = \left\{ \omega_d \int_{SO(d)} |f(\rho v)|^p \, d\rho \right\}^{1/p}, \tag{1.4}$$

where $v$ is any point in $S^{d-1}$, $d\rho$ represents the Haar measure on $SO(d)$ normalized to satisfy $\int_{SO(d)} d\rho = 1$ and $\omega_d = m(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ (see [14, p. 9]). For any fixed vector $v \in S^{d-1}$ functions on $S^{d-1}$ could be construed as functions on $\tau \in SO(d)$, $f(\tau v)$ and we require that the norm on $B$ can be represented as a norm of functions on the elements of $SO(d)$ which satisfy

$$\|f(\cdot\, v_1)\|_B = \|f(\cdot\, v_2)\|_B \quad \text{for } v_i \in S^{d-1} \tag{1.5}$$

and

$$\|f(\cdot\, \rho v) - f(\cdot\, v)\|_B \to 0 \quad \text{as } |\rho - I| \to 0. \tag{1.6}$$

We note that both (1.1) and (1.5) can be considered as analogues of (I) while both (1.2) and (1.6) can be considered as analogues of (II). Moreover, for the spaces of functions discussed below the function can be considered as $f(\rho x)$ with fixed $\rho$ $\rho \in SO(d)$ and variable $x \in S^{d-1}$ ($f(x) = f(Ix)$) or as $f(\tau v)$ with fixed $v \in S^{d-1}$ and variable $\tau \in SO(d)$.

We also assume

$$C^m(S^{d-1}) \subset B \subset L_1(S^{d-1}) \quad \text{where } \|f\|_B \geqslant \|f\|_{L_1} \text{ and some fixed } m. \tag{1.7}$$

We follow the classical concept of homogeneous Banach spaces, HBS (see for instance [13, p. 14]) and define the spherical homogeneous Banach spaces which we denote by SHBS to be Banach spaces of functions on $x \in S^{d-1}$ ($f(\rho x)$ with fixed $\rho$) and on $\tau \in SO(d)$ ($f(\tau v)$ with fixed $v$) that satisfy (1.1), (1.2), (1.5), (1.6) and (1.7). In Section 7 we give several examples of SHBS spaces.

In some papers (see [4,9]) the importance of the boundedness of the Cesàro summability for many approximation processes was discussed. For $L_p(S^{d-1})$ the boundedness was proved in the classical paper of Bonami and Clerc [3]. In Section 2 we define the Cesàro summability on $B$, which is SHBS, and state its boundedness, which is proved in Section 3. The immediate corollaries of the results in Sections 2 and 3 are described in Section 4. Extension of theorems on averages on the rim of the cap of the sphere and their combinations are given in Section 5. Some applications of the results of Section 5 and further results are given in Section 6. The Jackson inequality using the recent moduli of smoothness [10] is not treated here for SHBS as at the present point in time the proof is too long and involved (see [11] for $L_p(S^{d-1})$). I intend to get back to this problem when I succeed in simplifying the proof sufficiently. (Of course I may lose and someone else may publish such a result first.) In Section 7, we will discuss some spaces of functions that are SHBS.

## 2. The Cesàro summability

Following [4,9], the boundedness of the Cesàro summability for functions in a given Banach space is useful for the proof of approximation theorems on that space.

The Laplace–Beltrami operator $\widetilde{\Delta}$, given by

$$\widetilde{\Delta} f(x) = \Delta f\left(\frac{x}{|x|}\right), \quad x \in S^{d-1} \quad \text{where } \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \tag{2.1}$$

is the tangential component of the Laplacian $\Delta$. The eigenspace of $\widetilde{\Delta}$, $H_k$ given by

$$\widetilde{\Delta}\varphi = -k(k+d-2)\varphi \quad \text{for } \varphi \in H_k \tag{2.2}$$

has the dimension $\dim H_k \equiv d_k = \frac{d+2k-2}{k}\binom{d+k-3}{k-1}$ (see [14, p. 140]). For $B$ satisfying (1.7) i.e. satisfying $C^m(S^{d-1}) \subset B \subset L_1(S^{d-1})$ for some integer $m$ we have

$$B \supset H_k \quad \text{and} \quad B^* \supset H_k \quad \text{for all } k, \tag{2.3}$$

where $B^*$ is the dual to $B$. We define the projection $P_k f$ by

$$P_k f = \sum_{\ell=1}^{d_k} \langle f, Y_{k,\ell}\rangle \, Y_{k,\ell}, \tag{2.4}$$

where $\{Y_{k,\ell}\}_{\ell=1}^{d_k}$ is an orthonormal basis of $H_k$ (in $L_2(S^{d-1})$).

It is clear that (2.4) is defined on $B$ satisfying (1.7) and maps $B$ onto $H_k$. The Cesàro summability of order $\ell$ given by

$$C_n^\ell f = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \cdots \left(1 - \frac{k}{n+\ell}\right) P_k f \tag{2.5}$$

is defined for $f \in B$ satisfying (1.7) and maps $B$ onto span $\left\{\cup_{k=0}^n H_k\right\}$. The boundedness and convergence results for the Cesàro summability on $B$ are given in the following theorem.

**Theorem 2.1.** *For a function $f \in B$ where $B$ is a SHBS space i.e. satisfying* (1.1), (1.2), (1.5), (1.6) *and* (1.7), *and for the Cesàro summability $C_n^\ell f$ given by* (2.5) *we have*

$$\|C_n^\ell f\|_B \leqslant C\|f\|_B \quad \text{for } \ell > \frac{d-2}{2}, \tag{2.6}$$

$$\|C_n^\ell f\|_B \leqslant \|f\|_B \quad \text{for } \ell > d-1, \tag{2.7}$$

*and*

$$\|C_n^\ell f - f\|_B = o(1) \quad \text{as } n \to \infty \text{ for } \ell > \frac{d-2}{2}. \tag{2.8}$$

For $L_p(S^{d-1})$ (2.6) and (2.8) were proved in [3].

Before proceeding with the proof of Theorem 2.1, we discuss the operator $C_n^\ell f$. We note that as $B \subset L_1(S^{d-1})$

$$C_n^\ell f(y) = \int_{S^{d-1}} f(x) K_n^\ell(x \cdot y)\, dx, \tag{2.9}$$

where

$$K_n^\ell(x \cdot y) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \cdots \left(1 - \frac{k}{n+\ell}\right) \sum_{m=1}^{d_k} Y_{k,m}(x) Y_{k,m}(y) \tag{2.10}$$

with $Y_{k,m}(x)$ an orthonormal basis of $H_k$. The kernel $K_n^\ell(x \cdot y)$ satisfies

$$\int_{S^{d-1}} |K_n^\ell(x \cdot y)|\, dx \leqslant C \quad \text{for } \ell > \frac{d-2}{2} \tag{2.11}$$

and

$$K_n^\ell(x \cdot y) \geqslant 0 \quad \text{for } \ell > d-1. \tag{2.12}$$

For any $\ell$ orthogonality implies

$$\int_{S^{d-1}} K_n^\ell(x \cdot y)\, dx = 1. \tag{2.13}$$

## 3. Discussion of vector valued integrals and proof of Theorem 2.1

As mentioned in the Introduction, we may view elements of $B$ as $f(Ix) = f(x) \in B$ with variable $x \in S^{d-1}$ or as $f(\tau v)$ with variable $\tau \in SO(d)$ and a fixed $v \in S^{d-1}$. For a SHBS space we assumed that (1.5) was satisfied. From (1.2) given in the equivalent form

$$\|f(\rho\, \cdot) - f(\cdot)\|_B \leqslant \varepsilon \quad \text{for } |\rho - I| < \delta = \delta(\varepsilon) \tag{3.1}$$

we deduce in the following lemma its analogous form when the variable is $\tau \in SO(d)$.

**Lemma 3.1.** *For a SHBS space $B$ and $f \in B$ we have*

$$\|f(\cdot\, v_1) - f(\cdot\, v)\|_B < \varepsilon \quad \text{for } |v_1 - v| < \delta = \delta(\varepsilon), \tag{3.2}$$

*where $|v_1 - v|$ is the Euclidean distance.*

**Proof.** For $v_1$ and $v$ in $S^{d-1}$ satisfying $|v - v_1| < \delta$ there exists a transformation $\sigma \in SO(d)$ such that $\sigma v_1 = v$ and $|\sigma - I| = |v - v_1|$. (The matrix which rotates $v_1$ to $v$ and keeps elements in the Euclidean subspace perpendicular to a plane containing the vectors $v_1$ and $v$ will do.) We now note that

$$f(\tau v_1) - f(\tau v) = f(\tau\sigma v) - f(\tau v),$$

and (1.6) implies (3.2). □

With $\tau e = y$ ($e = (0, \dots, 0, 1)$) we may write (2.9) as

$$\begin{aligned} C_n^\ell f(\tau e) &= \int_{S^{d-1}} f(x) K_n^\ell(x \cdot \tau e)\, dx \\ &= \int_{S^{d-1}} f(x) K_n^\ell(\tau^{-1}x \cdot e)\, dx \\ &= \int_{S^{d-1}} f(\tau z) K_n^\ell(z \cdot e)\, dz. \end{aligned}$$

We can now define the integral as a vector valued Riemann-type integral (considering $f(\tau z)$ as vector in $B$ i.e. a function on $\tau \in SO(d)$ for any $z$ integrated on the variable $z \in S^{d-1}$). This procedure is legitimate as $K_n^\ell(z \cdot e)$ is continuous in $z$ and so is $f(\tau z)$ using Lemma 3.1. (For each $z \in S^{d-1}$ $f(\tau z)$ is a function on $SO(d)$ that is a vector or element of $B$.) We cover $S^{d-1}$ by non-overlapping sets $E_i$ satisfying $|z - z_i| \leqslant \eta$ for some collection of points $z_i$ and estimate $\int_{S^{d-1}} f(\tau z) K_n^\ell(z \cdot e)\, dz$ by $\sum_{i=1}^N \mu(E_i) f(\tau z_i) K_n^\ell(z_i \cdot e)$ which converges to the integral. Using the facts $B \supset L_1$ and $C^m(S^{d-1})$ is dense in $L_1(S^{d-1})$, the vector value integral is the same as $C_n^\ell f(\tau e)$ in $L_1$, and hence in $B$. This procedure is routine and follows more or less the textbook treatment (see [13, p. 257]).

We are now ready for the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Using the definition of the integral

$$C_n^\ell f(\tau e) = \int_{S^{d-1}} f(\tau z) K_n^\ell(z \cdot e)\, dz$$

as a vector valued Riemann-type integral, we have

$$\begin{aligned} \|C_n^\ell f(\tau e)\|_B &\leqslant \int_{S^{d-1}} \|f(\tau z)\|_B |K_n^\ell(z \cdot e)|\, dz \\ &\leqslant \|f(\tau v)\|_B \int_{S^{d-1}} |K_n^\ell(z \cdot e)|\, dz, \end{aligned}$$

which, applying (2.11), yields (2.6). Now using (2.12) and (2.13), we have (2.7) as well. We now prove (2.8). The identity (2.13) implies

$$\begin{aligned} \|C_n^\ell f(\tau e) - f(\tau e)\|_B &\leqslant \int_{S^{d-1}} \|f(\tau z) - f(\tau e)\|_B |K_n^\ell(z \cdot e)|\, dz \\ &= \left\{ \int_{|z-e|<\delta} + \int_{|z-e|\geqslant\delta} \right\} \|f(\tau z) - f(\tau e)\|_B |K_n^\ell(z \cdot e)|\, dz \\ &\equiv I_1 + I_2. \end{aligned}$$

From Lemma 3.1 and (2.11) we derive

$$I_1 \leqslant \varepsilon \int_{S^{d-1}} |K_n^\ell(z, e)|\, dz \leqslant \varepsilon M.$$

We also have

$$I_2 \leqslant 2\|f\|_B \int_{|z-e|\geqslant\delta} |K_n^\ell(z\cdot e)|\, dz,$$

and as

$$\int_{|z-e|\geqslant\delta} |K_n^\ell(z\cdot e)|\, dz < \varepsilon$$

for sufficiently large $n$ and $\ell > \frac{d-2}{2}$, we complete the proof of Theorem 2.1. □

## 4. Applications of Theorem 2.1

We itemize some applications of Theorem 2.1.

(A) Combinations of spherical polynomials are dense in any SHBS space. We obtain (A) using (2.8) and recalling that any element of span $\cup_{k=0}^n H_k$ is a combination of spherical polynomials. Moreover, if $P_k f = 0$ for all $k$, then $C_n^\ell f = 0$ for all $n$ and $f = 0$ by (2.8).

(B) The Riesz means $R_{n,\alpha,\ell} f$ given by

$$R_{n,\alpha,\ell} f = \sum_{k<n} \left(1 - \left(\frac{k(k+d-2)}{n(n+d-2)}\right)^\alpha\right)^\ell P_k f \tag{4.1}$$

are bounded for $B \in$ SHBS when $\ell > \frac{d-2}{2}$, that is

$$\|R_{n,\alpha,\ell} f\|_B \leqslant C(d,\ell)\|f\|_B \quad \text{for } \alpha > 0 \text{ and } \ell > \frac{d-2}{2}, \tag{4.2}$$

and also

$$\|R_{n,\alpha,\ell} f - f\|_B = o(1) \quad \text{as } n\to\infty \text{ for } \alpha > 0 \text{ and } \ell > \frac{d-2}{2}. \tag{4.3}$$

We follow [9, Theorem 2.1] for (4.2) and [9, Corollary 2.2] for (4.3).

(C) The Bernstein inequality

$$\|(-\widetilde{\Delta})^\alpha \varphi\|_B \leqslant C n^{2\alpha} \|\varphi\|_B \tag{4.4}$$

for $\alpha > 0$, $B \in$ SHBS and $\varphi \in$ span $\left(\cup_{k=0}^n H_k\right)$ is satisfied.

For integer $\alpha$ (4.4) follows from (B) and [4, Theorem 2.2] using (4.2) and (4.3). For other $\alpha$ we define

$$(-\widetilde{\Delta})^\alpha f \sim \sum_{k=1}^\infty (k(k+d-2))^\alpha P_k f \tag{4.5}$$

whenever the right-hand side is an expansion of a function in $B$, in which case we say $f \in \mathcal{D}\left((-\widetilde{\Delta})^\alpha\right)$. (For $\varphi \in$ span $\cup_{k=0}^n H_k$ we always have $\varphi \in \mathcal{D}\left((-\widetilde{\Delta})^\alpha\right)$.) The result (4.4) now follows from [9, Theorem 3.2] using (4.2) and (4.3) again.

(D) The de la Vallée Poussin type operator given by

$$\eta_\lambda f \equiv \sum_{k=0}^{\infty} \eta\left(\frac{k}{\lambda}\right) P_k f = \int_{S^{d-1}} G_\lambda(x\cdot y) f(y)\, dy \tag{4.6}$$

is bounded for $B \in$ SHBS where $\eta(x) = 1$ for $x \leqslant 1$, $\eta(x) = 0$ for $x \geqslant 2$, and $\eta(x) \in C^\infty$. That is,

$$\|\eta_\lambda f\|_B \leqslant C(\eta)\|f\|_B. \tag{4.7}$$

Inequality (4.7) follows from Theorem 2.1 in a routine manner (see for instance [6]).

As $\eta_\lambda\varphi = \varphi$ for $\varphi \in \operatorname{span} \cup_{k\leqslant\lambda} H_k$, we have for $B \in$ SHBS

$$\begin{aligned}\|f - \eta_\lambda f\|_B &\leqslant (C(\eta)+1)\, E_\lambda(f)_B \\ &\equiv (C(\eta)+1) \inf\left(\|f-\varphi\|_B : \varphi \in \operatorname{span} \bigcup_{k\leqslant\lambda} H_k\right),\end{aligned} \tag{4.8}$$

and obviously $\|f - \eta_\lambda f\|_B = o(1)$ as $\lambda \to \infty$ for such $B$.

(E) We may define the $K$-functional

$$K\left(f, (-\widetilde{\Delta})^\alpha, t^{2\alpha}\right)_B = \inf_{g\in\mathcal{D}\left((-\widetilde{\Delta})^\alpha\right)} \left(\|f-g\|_B + t^{2\alpha}\|(-\widetilde{\Delta})^\alpha g\|_B\right) \tag{4.9}$$

and obtain the realization result for any positive $\alpha$

$$K\left(f, (-\widetilde{\Delta})^\alpha, t^{2\alpha}\right)_B \approx \|f - \eta_{a/t} f\|_B + t^{2\alpha}\|(-\widetilde{\Delta})^\alpha \eta_{a/t} f\|_B, \quad a > 0 \tag{4.10}$$

when we examine [9, Theorem 6.2]. We note that the constants of the equivalence (4.10) depend on $a$. Equivalence (4.10) implies the Jackson inequality

$$E_n(f)_B \leqslant CK\left(f, (-\widetilde{\Delta})^\alpha, 1/n^{2\alpha}\right)_B. \tag{4.11}$$

(F) We also have for the $K$-functional of (4.9), $B \in$ SHBS, and $0 < \alpha < \beta$, the Marchaud inequality

$$K\left(f, (-\widetilde{\Delta})^\alpha, t^{2\alpha}\right)_B \leqslant Ct^{2\alpha} \int_t^1 \frac{K\left(f, (-\widetilde{\Delta})^\beta, u^{2\beta}\right)_B}{u^{2\alpha+1}}\, du \tag{4.12}$$

following [9, Theorem 6.5].

There are other results which are valid for all $B \in$ SHBS, but the above is an indication of the usefulness of Theorem 2.1.

## 5. Multipliers and applications to averaging on a sphere

For $f \in B$ and $B \in$ SHBS we deal with a multiplier operator $T_\mu$ given by

$$T_\mu f \sim \sum_{k=0}^{\infty} \mu_k P_k f, \tag{5.1}$$

that is an operator that satisfies

$$T_\mu \varphi = \mu_k \varphi \quad \text{for all } \varphi \in H_k.$$

The basic result using (2.6) is the following theorem.

**Theorem 5.1.** *For* $f \in B$, $B \in SHBS$ *and* $T_\mu$ *given by* (5.1) *the conditions* $\lim_{k\to\infty} \mu_k = 0$ *and*

$$\sum_{k=0}^{\infty} |\Delta^{\ell+1}\mu_k| \binom{k+\ell}{\ell} \leqslant M \quad \text{with } \ell > \frac{d-2}{2}, \tag{5.2}$$

*where* $\Delta\mu_k = \mu_{k+1} - \mu_k$ *and* $\Delta^m \mu_k = \Delta(\Delta^{m-1}\mu_k)$, *imply*

$$\|T_\mu f\|_B \leqslant CM\|f\|_B \tag{5.3}$$

*with M of* (5.2) *and C of* (2.6).

**Proof.** We show that

$$\sum_{k=0}^{\infty} \binom{k+\ell}{\ell} (\Delta^{\ell+1}\mu_k) C_k^\ell f \sim \sum_{k=0}^{\infty} \mu_k P_k f. \tag{5.4}$$

This follows essentially from

$$P_k f = \overleftarrow{\Delta}^{\ell+1} \binom{k+\ell}{\ell} C_k^\ell f, \quad \overleftarrow{\Delta} a_k = a_k - a_{k-1}, \quad \overleftarrow{\Delta}^{\ell+1} a_k = \overleftarrow{\Delta}(\overleftarrow{\Delta}^\ell a_k)$$

with $C_k^\ell f = 0$ for negative $k$ and the Abel transformation repeated $\ell + 1$ times. Equivalently, one can compare the projections

$$\begin{aligned} P_n &\left\{ \sum_{k=0}^{\infty} \binom{k+\ell}{\ell} (\Delta^{\ell+1}\mu_k) C_k^\ell f \right\} \\ &= \sum_{k=n}^{\infty} \binom{k+\ell}{\ell} \Delta^{\ell+1}\mu_k \left(1 - \frac{n}{k+1}\right) \cdots \left(1 - \frac{n}{k+\ell}\right) P_n f \\ &= \mu_n P_n f \\ &= P_n \left\{ \sum_{k=0}^{\infty} \mu_k P_k f \right\}. \end{aligned}$$

For this we need $\lim_{k\to\infty} k^j \Delta^j \mu_k = 0$ for $j = 1, \ldots, \ell$, which are self-evident in the applications we use below (Theorem 5.3), and in fact they follow from (5.2) and $\lim \mu_k = 0$. We now use $\|C_k^\ell f\|_B \leqslant C\|f\|_B$ for $\ell > \frac{d-2}{2}$ and (5.2) and as

$$T_\mu f = \sum_{k=0}^{\infty} \binom{k+\ell}{\ell} \left(\Delta^{\ell+1}\mu_k\right) C_k^\ell f, \tag{5.5}$$

we have

$$\|T_\mu f\|_B \leqslant C \sum_{k=0}^{\infty} \binom{k+\ell}{\ell} |\Delta^{\ell+1}\mu_k| \, \|f\|_B. \qquad \square$$

Theorem 5.1 has several applications and in many investigations the estimate

$$\sum_{k=0}^{\infty} \binom{k+\ell}{\ell} |\Delta^{\ell+1} \mu_k| \leqslant M \quad \text{and the limit } \lim_{k\to\infty} \mu_k = 0$$

for various $\mu_k$ were crucial in the proof of approximation results, in particular for $L_1(S^{d-1})$ or $C(S^{d-1})$.

The average on the rim of the cap of $S^{d-1}$, $S_\theta f$ given by

$$S_\theta f(y) = \frac{1}{m_\theta} \int_{x\cdot y=\cos\theta} f(x)\, d\gamma, \quad S_\theta 1 = 1 \tag{5.6}$$

(where $d\gamma$ is the measure on $\{z : z \cdot y = \cos\,\theta\}$ induced by the Lebesgue measure) is the crucial concept used in most of the investigations in approximation theory on $L_p(S^{d-1})$, $1 \leqslant p \leqslant \infty$. We now show that these results carry over to any SHBS space $B$.

**Theorem 5.2.** *For the SHBS space B on $S^{d-1}$ and $f \in B$ $S_\theta f$, given by* (5.6), $S_\theta : B \to B$ *and satisfies*

$$\|S_\theta f\|_B \leqslant \|f\|_B. \tag{5.7}$$

**Proof.** For a given $\theta$ we may follow earlier considerations ( in Section 3) and write (5.6) as

$$S_\theta f(\tau e) = \frac{1}{m_\theta} \int_{z\cdot e=\cos\,\theta} f(\tau z)\, d\gamma, \quad S_\theta 1 = 1. \tag{5.8}$$

Using Lemma 3.1, $f(\tau z)$ as a function in $\tau \in SO(d)$ is continuous on $z \in S^{d-1}$, and hence on $S^{d-1} \cap \{z : z \cdot e = \cos\,\theta\}$. Moreover, the weight in (5.8) is continuous on $\{z : z \cdot e = \cos\theta\}$ (as it is a constant). Therefore, we may view (5.8) as a Riemann vector-valued integral on $\{z : z \cdot e = \cos\,\theta\} \cap S^{d-1}$, and as such we have

$$\begin{aligned} \|S_\theta f(\tau e)\|_B &\leqslant \frac{1}{m_\theta} \int_{z\cdot e=\cos\theta} \|f(\tau z)\|_B\, d\gamma \\ &\leqslant \|f(\tau z)\|_B = \|f(z)\|_B. \end{aligned}$$

We now recall that $B \subset L_1(S^{d-1})$ and that $C^m(S^{d-1})$ is dense in $L_1(S^{d-1})$ (in the $L_1$ norm). Thus the definition of (5.6) and the vector-valued Riemann integral coincide in $L_1$ and hence in $B$. □

**Theorem 5.3.** *For $B \in SHBS$, $f \in B$ and $S_\theta f$ given by* (5.6) *we have*

$$\|\widetilde{\Delta} S_\theta^m f\|_B \leqslant C \max\left(\frac{1}{\theta^2}, \frac{1}{(\pi-\theta)^2}\right) \|f\|_B \quad \textit{for } m > \frac{2([\frac{d}{2}]+3)}{d-2}, \tag{5.9}$$

$$\left\| f + \frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} S_{j\theta} f \right\|_B \approx K\left(f, (-\widetilde{\Delta})^\ell, \theta^{2\ell}\right)_B \quad \textit{for } 0<\theta<\frac{\pi}{2\ell}, \tag{5.10}$$

*and in particular*

$$\|f - S_\theta f\|_B \approx K(f, -\widetilde{\Delta}, \theta^2)_B \quad \textit{for } 0 < \theta < \frac{\pi}{2}. \tag{5.11}$$

**Proof.** We set $Q_n^{(\lambda)}(u)$ to be the normalized ultraspherical polynomial given by

$$\frac{1}{(1-u^2)^{\lambda-\frac{1}{2}}}\frac{d}{du}(1-u^2)^{\lambda+\frac{1}{2}}\frac{d}{du}Q_n^{(\lambda)}(u)=-n(n+2\lambda)Q_n^{(\lambda)}(u) \quad \text{and} \quad Q_n^{(\lambda)}(1)=1.$$

It was shown in [1, Proof of Theorem 3.1] that

$$\theta^2\sum_{k=1}^{\infty}\left|\Delta^{\ell+1}\left\{k(k+d-2)\left(Q_k^{(\lambda)}(\cos\theta)\right)^m\right\}\right|\binom{k+\ell}{\ell}\leqslant C_1$$

for $d\geqslant 3$, $\lambda=\frac{d-2}{2}$ and $m>2\,\frac{\ell+3}{d-2}$. (The limits $\{k^j\Delta^j k(k+d-2)Q_k^{(\lambda)}(\cos\theta)^m\}\to 0$ for $0\leqslant j\leqslant \ell$ are self evident.) Now using Theorem 5.1 for $\ell>\frac{d-2}{2}$, we obtain (5.9) for $0<\theta\leqslant\frac{\pi}{2}$. For $\frac{\pi}{2}\leqslant\theta\leqslant\pi$ we obtain (5.9) using considerations of [1]. The equivalences (5.10) and hence (5.11) follow from [5]. We first recall (E) of Section 4 (see (4.10) there). The equivalence (5.10) constitutes the analogues of (5.3), (5.4) and (5.5) of [5] for $B\in$ SHBS. The proof of (5.3), (5.4) and (5.5) of [5] utilizes (5.2) of Theorem 5.1 here, and using the theorem, we can transfer the proof from $L_p(S^{d-1})$ to any $B\in$ SHBS. □

## 6. Further applications

The Jackson inequality is given by the following theorem.

**Theorem 6.1.** *For $B\in SHBS$, $f\in B$, $S_\theta f$ given by* (5.6) *and $E_n(f)_B$ given in* (4.8) *we have*

$$E_n(f)_B\leqslant C\left\|f+\frac{2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}S_{j/n}f\right\|_B \tag{6.1}$$

*for $n\geqslant 2\ell/\pi$.*

**Proof.** This is just a combination of (5.10) and (4.11) and the interesting case is $n\geqslant\ell$. □

As a special case of (6.1) we have

$$E_n(f)_B\leqslant C\|f-S_{1/n}f\|_B. \tag{6.2}$$

For SHBS spaces which are lattice compatible we can prove a Bernstein inequality different than (4.4) (see for the $L_p$ analogous result [11, Theorem 8.4]).

**Definition 6.2.** We say that $B\in$ SHBS is lattice compatible if for $g\in B$ and $f\in L_1(S^{d-1})$ $|f|\leqslant|g|$ implies $f\in B$ and $\|f\|_B\leqslant\|g\|_B$. In particular, $|f|\in B$ implies $f\in B$ and $\||f|\|_B=\|f\|_B$.

**Theorem 6.3.** *For $B\in SHBS$ which is lattice compatible we have*

$$\left\|\max_{\xi\perp x}\left|\left(\frac{\partial}{\partial\xi}\right)^r\varphi_n(x)\right|\right\|_B\leqslant Cn^r\|\varphi_n\|_B \quad \textit{for } \varphi_n\in \operatorname{span}\bigcup_{0\leqslant k\leqslant n}H_k. \tag{6.3}$$

The derivatives $\left(\frac{\partial}{\partial\xi}\right)^r g(x)$ are defined by

$$\frac{\partial}{\partial\xi}\, g(x) = \frac{d}{dt}\, g(e^{tM}x)\Big|_{t=0}, \quad \left(\frac{\partial}{\partial\xi}\right)^r g(x) = \left(\frac{d}{dt}\right)^r g(e^{tM}x)\Big|_{t=0}, \tag{6.4}$$

where $M$ is the skew-symmetric matrix satisfying $e^{tM}x = x\cos t + \xi \sin t$, $e^{tM}\xi = \xi\cos t - x\sin t$ and $e^{tM}u = u$ for $u \perp \operatorname{span}(x,\xi)$. In the coordinates $(x,\xi,u_3,\dots,u_d)$ $M$ consists of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ at the upper left corner and zeros elsewhere. We note that as $\max_{\xi\perp x} \frac{\partial}{\partial\xi}\, g(x)$ is the tangential gradient of $g$ at $x$, one can consider $\max_{\xi\perp x} \left(\frac{\partial}{\partial\xi}\right)^r g(x)$ as a generalization of the tangential gradient.

**Lemma 6.4.** *Suppose $f \in B$, $B \in SHBS$ where $B$ is lattice compatible and suppose $G(t) \in C^r[-1,1]$ satisfying*

$$\int_{-1}^{1} |G^{(r-\ell)}(t)|(1-t^2)^{(d+r-2\ell-3)/2}\,dt \leqslant M \quad \text{for } 2\ell \leqslant r,$$
$$\int_{-1}^{1} G^{(k)}(t)(1-t^2)^{(d-3)/2}\,dt \leqslant M \quad \text{for } 0<k<\frac{r}{2}. \tag{6.5}$$

*Then $F$ given by*

$$F(x) = \int_{S^{d-1}} f(y)G(x\cdot y)\,dy \tag{6.6}$$

*satisfies $\sup_{\xi\perp x}\left|\left(\frac{\partial}{\partial\xi}\right)^r F(x)\right| \in B$ and*

$$\left\| \sup_{\xi\perp x}\left|\left(\frac{\partial}{\partial\xi}\right)^r F(x)\right| \right\|_B \leqslant CAM\|f\|_B, \tag{6.7}$$

*where*

$$A = m(S^{d-1}) \Big/ \int_{-1}^{1}(1-t^2)^{(d-3)/2}\,dt.$$

We observe that for $L_p(S^{d-1})$ and $r=1$ Lemma 6.4 is Lemma 9.1 of [11]. For $r>1$ we had to assume (6.5) rather than [11, (9.3)] in Lemma 9.1 of [11]. Theorem 8.4 of [11] is generalized by Theorem 6.3 here.

**Proof of Lemma 6.4.** As $f \in L_1(S^{d-1})$ and $G(t) \in C^r[-1,1]$,

$$\left|\left(\frac{\partial}{\partial\xi}\right)^r F(x)\right| = \left|\int_{S^{d-1}} f(y)\left(\frac{\partial}{\partial\xi}\right)^r G(x\cdot y)\,dy\right|$$

is defined for all $x$. (The derivative is taken on the variable $x$.)

We now note that

$$\frac{d}{dt}\,(e^{Mt}x\cdot y)\Big|_{t=0} = (Mx\cdot y) = (\xi\cdot y),$$

and as $y = (x \cdot y)x + \left(1 - (x \cdot y)^2\right)^{1/2} z$ where $|z| = 1$ and $(z \cdot x) = 0$, we have for $\xi \perp x$

$$(\xi \cdot y) = \left(1 - (x \cdot y)^2\right)^{1/2} (\xi \cdot z) \quad \text{or} \quad |(\xi \cdot y)| \leqslant \left(1 - (x \cdot y)^2\right)^{1/2}.$$

Furthermore,

$$\left|\left(\frac{d}{dt}\right)^{\ell} (e^{tM} x \cdot y)\right| = |(M^{\ell} e^{tM} x \cdot y)| \leqslant 1 \quad \text{for all } t \text{ and } \ell = 0, 1, \ldots .$$

Therefore, for $\xi \perp x$

$$\begin{aligned}\left(\frac{d}{dt}\right)^{r} G(e^{tM} x \cdot y)\Big|_{t=0} &= G^{(r)}(x \cdot y)\left(1 - (x \cdot y)^2\right)^{r/2} \Phi_0(M, x, y)\\ &\quad + G^{(r-1)}(x \cdot y)\left(1 - (x \cdot y)^2\right)^{(r-2)/2} \Phi_1(M, x, y) + \cdots\\ &\quad + G^{(r-[\frac{r}{2}])}(x \cdot y)\left(1 - (x \cdot y)^2\right)^{(r-2[\frac{r}{2}])/2} \Phi_{[\frac{r}{2}]}(M, x, y)\\ &\quad + G^{(r-[\frac{r}{2}]-1)}(x \cdot y)\Phi_{[\frac{r}{2}]+1}(M, x, y) + \cdots\\ &\quad + G'(x \cdot y)\Phi_{r-1}(M, x, y), \end{aligned} \tag{6.8}$$

where $|\Phi_{\ell}(M, x, y)| \leqslant C_{\Phi}(r)$ and $C_{\Phi}(r)$ is independent of $M, x,$ and $y$. (For $r = 1$ we have $\frac{d}{dt} G(e^{tM} x \cdot y)\big|_{t=0} = G'(x \cdot y)\left(1 - (x \cdot y)^2\right)^{1/2} \Phi_0(M, x, y)$, $C_{\Phi}(1) = 1$ and $\Phi_0(M, x, y) = (\xi \cdot z)$.)

Using (6.8), we write for $(\xi \cdot x) = 0$ (or $(Mx \cdot x) = 0$)

$$\begin{aligned}&\sup_{\xi \perp x}\left|\left(\frac{\partial}{\partial \xi}\right)^{r} F(x)\right|\\ &\quad = \sup_{\substack{M\\(Mx\cdot x)=0}}\left|\left(\frac{d}{dt}\right)^{r} F(e^{Mt}x)\right|_{t=0}\\ &\quad \leqslant \int_{S^{d-1}} |f(y)| \left|\left(\frac{d}{dt}\right)^{r} G(e^{Mt} x \cdot y)\right|_{t=0} dy\\ &\quad \leqslant C_1\left[\max_{0 \leqslant \ell < [\frac{r}{2}]} \int_{S^{d-1}} |f(y)|\, |G^{(r-\ell)}(x \cdot y)|\left(1 - (x \cdot y)^2\right)^{(r-2\ell)/2} dy\right.\\ &\qquad \left. + \max_{[\frac{r}{2}] < \ell < r} \int_{S^{d-1}} |f(y)|\, |G^{(r-\ell)}(x \cdot y)|\, dy\right].\end{aligned}$$

Hence, following the argument in Theorem 3.2, we have for $\xi \perp \tau e$

$$\begin{aligned}&\sup_{\xi \perp \tau e}\left|\left(\frac{\partial}{\partial \xi}\right)^{r} F(\tau e)\right|\\ &\quad \leqslant C_1\left[\max_{0 \leqslant \ell < [\frac{r}{2}]} \int_{S^{d-1}} |f(\tau z)|\, |G^{(r-\ell)}(z \cdot e)|\left(1 - (z \cdot e)^2\right)^{(r-2\ell)/2} dz\right.\\ &\qquad \left. + \max_{[\frac{r}{2}] < \ell < r} \int_{S^{d-1}} |f(\tau z)|\, |G^{(r-\ell)}(z \cdot e)|\, dz\right].\end{aligned}$$

As $\sup_{\xi\perp\tau e}\left(\frac{\partial}{\partial\xi}\right)^r F(\tau e)$ and the expression majorizing it can be described as Riemann vector-valued integrals of $f(\tau z)$, and the latter is independent of $\xi$ provided that $\xi\perp\tau e$, we may follow Theorem 3.2 and deduce (6.5). □

**Proof of Theorem 6.3.** We use Lemma 6.4 with $G_n(t)$, the combination of ultraspherical polynomials $Q_k^{(\lambda)}(t)$ with $k<2n$ ($\lambda=\frac{d-2}{2}$), as given in (4.6). This is a de la Vallée Poussin-type kernel and we could have used other de la Vallée Poussin-type kernels (see for instance [11, p. 31]).

To apply Lemma 6.4 we need to show that for $0\leqslant\ell<\left[\frac{r}{2}\right]$

$$\int_{S^{d-1}}|G_n^{(r-\ell)}(z\cdot e)|\left(1-(z\cdot e)^2\right)^{(r-2\ell)/2}dz$$
$$=A\int_{-1}^{1}G_n^{(r-\ell)}(t)(1-t^2)^{(r-2\ell+d-3)/2}\,dt\leqslant Cn^r, \tag{6.9}$$

and that for $\left[\frac{r}{2}\right]\leqslant\ell<r$

$$\int_{S^{d-1}}|G_n^{(r-\ell)}(z\cdot e)|\,dz=A\int_{-1}^{1}|G_n^{(r-\ell)}(t)|(1-t^2)^{(d-3)/2}\,dt$$
$$\leqslant Cn^{2(r-\ell)}\leqslant Cn^r. \tag{6.10}$$

We recall that $G_n(u)$ is a polynomial of degree $2n$ (using $\eta_\lambda$ of (4.6)), and following (4.7) we have

$$\int_{S^{d-1}}|G_n(x\cdot y)|\,dy=A\int_{-1}^{1}|G_n(u)|(1-u^2)^{(d-3)/2}\,du$$
$$\equiv A\|wG_n\|_{L_1[-1,1]}\leqslant C(\eta),$$

where $C(\eta)$ is independent of $n$. (It does depend on the operator $\eta_\lambda$ defined in $D$ of Section 4.) We now use a combination of weighted Bernstein and Markov inequalities to prove (6.9) and (6.10). To show (6.9) we write (with $\varphi(u)=(1-u^2)^{1/2}$ and $w(u)=\varphi(u)^{d-3}$)

$$\begin{aligned}\|wG_n^{(r-\ell)}\varphi^{r-2\ell}\|_{L_1[-1,1]} &\leqslant C_1n^{r-2\ell}\|wG_n^{(\ell)}\|_{L_1[-1,1]}\\ &\leqslant C_2n^{r-2\ell}\|wG_n^{(\ell)}\|_{L_1\left[-1+\frac{c}{n^2},1-\frac{c}{n^2}\right]}\\ &\leqslant C_3n^{r-2\ell}n^{\ell}\|w\varphi^{\ell}G_n^{(\ell)}\|_{L_1[-1,1]}\\ &\leqslant C_4n^r\|wG_n\|_{L_1[-1,1]},\end{aligned}$$

using for the first inequality Theorem 8.4.7 of [12] $r-2\ell$ times (with different weights), and for the second inequality Theorem 8.4.8 of [12]. The third inequality is obvious, and for the fourth inequality we apply again Theorem 8.4.7 of [12] $\ell$ times. For the proof of (6.10) we follow the proof of the last few steps in the proof of (6.9). □

## 7. Examples of SHBS spaces

The models for SHBS spaces are the function spaces $L_p(S^{d-1})$ with the norm

$$\|f\|_B=\left\{\int_{S^{d-1}}|f(\tau x)|^p\,dx\right\}^{1/p}=\left\{\omega_d\int_{SO(d)}|f(\tau v)|^p\,d\tau\right\}^{1/p}, \tag{7.1}$$

where $dx$ is the induced Lebesgue measure on $S^{d-1}$ and $d\tau$ is the Haar measure normalized such that $\int_{SO(d)}d\tau=1$. Clearly (1.1), (1.2), (1.5), (1.6) and (1.7) (with $m=0$) are satisfied.

As the theorems we prove in this paper were known for $L_p(S^{d-1})$, we need other examples to establish the usefulness of the present results.

Orlicz spaces on the sphere.

The closest spaces to $L_p(S^{d-1})$ are Orlicz spaces. For $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$, $\varphi$ is increasing and left continuous and $\Phi(s) = \int_0^s \varphi(u)\, du$ is the Young function. The Orlicz class is the class of functions for which

$$M^{\Phi}(f) = \int_{S^{d-1}} \Phi(|f(\tau x)|)\, dx = \omega_d \int_{SO(d)} \Phi(|f(\tau v)|)\, d\tau \tag{7.2}$$

is finite.

The Luxemburg norm on the class is given (as usual) by

$$\rho^{\Phi}(f) = \inf\{k^{-1} : M^{\Phi}(k|f|) \leqslant 1\}.$$

With the norm $\rho^{\Phi}(f)$ we have a rearrangement invariant Banach space [2, p. 269].

We also assume that the $\Delta_2$ condition, that is

$$\Phi(2s) \leqslant C\Phi(s) < \infty \quad \text{for } s_0 \leqslant s < \infty \tag{7.3}$$

is satisfied to insure that the totality of functions for which $M^{\Phi}(f)$ is finite is a linear space (see [2, Proposition 8.5]). There is another description of the norm (the Orlicz norm) which is equivalent. Clearly (1.1) and (1.2) are satisfied, and as we did not allow $\varphi(s) = \infty$ the continuous functions are dense and we have (1.5) and (1.6) as well. Condition (1.7) is evident. We note that with $\varphi(x) = 0\ 0 \leqslant u \leqslant 1$ and $\varphi(u) = 1 + \log u$ we have the Zygmund space $L \log^+ L$.

We would like to point out that the SHBS spaces are not necessarily rearrangement invariant. For instance, the space of functions for which the norm

$$\|f\|_{p,r} = \|f\|_{L_p(S^{d-1})} + \|\widetilde{\Delta}^r f\|_{L_p(S^{d-1})} \tag{7.4}$$

is finite satisfies the conditions with $1 \leqslant p < \infty$. The norms in (7.4) can be replaced by Orlicz norms.

Also the norm

$$\|f\|_{p,r,\alpha} = \sup_t t^{-\alpha} K_r(f, t^{2r})_p, \quad \alpha < 2r \tag{7.5}$$

with

$$K_r(f, t^{2r})_p \equiv \inf \left( \|f - g\|_{L_p(S^{d-1})} + t^{2r} \|\widetilde{\Delta}^r g\|_{L_p(S^{d-1})} \right)$$

and $1 \leqslant p < \infty$ satisfies conditions (1.1), (1.2), (1.5) and (1.6). We note that in the $K$-functional above $\widetilde{\Delta}^r$ can be replaced by $\widetilde{\Delta}^{\beta}$ $(\beta > \alpha)$ and other multiplier operators.

Examples of SHBS spaces for which the norm is lattice compatible are the Orlicz spaces.

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## References

[1] E. Belinsky, F. Dai, Z. Ditzian, Multivariate approximating averages, J. Approx. Theory 125 (2003) 85–105.
[2] C. Bennett, R. Sharpley, Interpolation of Operators, Academic Press, New York, 1988.
[3] A. Bonami, J.L. Clerc, Sommes de Cesàro et multiplicateur de developpement en harmoniques sphériques, Trans. Amer. Math. Soc. 183 (1973) 223–262.
[4] W. Chen, Z. Ditzian, Best approximation and $K$-functionals, Acta Math. Hungar. 75 (1997) 165–208.
[5] F. Dai, Z. Ditzian, Combinations of multivariate averages, J. Approx. Theory 131 (2004) 268–283.
[6] F. Dai, Z. Ditzian, Littlewood Paley theory and sharp Marchaud inequality, Acta Sci. Math. Szeged 71 (2005) 65–90.
[7] Z. Ditzian, Some remarks on inequality of Landau and Kolmogorov, Aequationes Math. 12 (1975) 151–245.
[8] Z. Ditzian, Some remarks on approximation theorems on various Banach spaces, J. Math. Anal. Appl. 77 (1980) 567–576.
[9] Z. Ditzian, Fractional derivatives and best approximation, Acta Math. Hungar. 81 (4) (1998) 323–348.
[10] Z. Ditzian, A modulus of smoothness on the unit sphere, J. D'Analyse Math. 79 (1999) 189–200.
[11] Z. Ditzian, Jackson-type inequality on the sphere, Acta Math. Hungar. 102 (1–2) (2004) 1–35.
[12] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer, Berlin, 1987.
[13] Y. Katznelson, An Introduction to Harmonic Analysis, Wiley, New York, 1968.
[14] E.M. Stein, G. Weiss, An Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, NJ, 1971.