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Sets of three pairwise orthogonal Steiner triple systems

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Abstract

Two Steiner triple systems (STS) are orthogonal if their sets of triples are disjoint, and two disjoint pairs of points defining intersecting triples in one system fail to do so in the other. In 1994, it was shown (Canad. J. Math. 46(2) (1994) 239–252) that there exist a pair of orthogonal Steiner triple systems of order v for all $v \equiv 1, 3 \pmod 6$, with $v \geqslant 7$, $v \neq 9$. In this paper we show that there exist *three* pairwise orthogonal Steiner triple systems of order v for all $v \equiv 1 \pmod 6$, with $v \geqslant 19$ and for all $v \equiv 3 \pmod 6$, with $v \geqslant 27$ with only 24 possible exceptions.
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1. Introduction

A *Steiner triple system* (STS) of order v is a pair $(V, \mathscr{B})$, where V is a v-set of elements and $\mathscr{B}$ is a collection of 3-subsets (triples) of V such that every pair of elements in V is contained in a unique triple of $\mathscr{B}$. The necessary numerical condition is well-known [3,14] to be $v \equiv 1, 3 \pmod 6$.

Two STS, $(V, \mathscr{B}_1)$ and $(V, \mathscr{B}_2)$, are *orthogonal* if

(i) $\mathscr{B}_1 \cap \mathscr{B}_2 = \emptyset$, and
(ii) for u, v, x, y distinct, $\{u, v, a\}, \{x, y, a\} \in \mathscr{B}_1$ and $\{u, v, w\}, \{x, y, z\} \in \mathscr{B}_2$ implies $w \neq z$.

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Orthogonal Steiner triple systems were first introduced by O'Shaughnessey [11] in 1968 as a means to finding Room squares. After much work, the spectrum problem for orthogonal STS was completely solved [2] in 1994. The interested reader will find a nice history of the topic in that paper. The paper [8] gives a description of the algorithms used to find many small orders. In [4] further direct constructions (both in finite fields and of a hill-climbing nature) were given to find sets of three or more pairwise orthogonal STS(v) for $v \leqslant 500$. Earlier papers dealing with sets of more than two pairwise orthogonal Steiner triple systems include [5,9,15].

If there exists a set of three pairwise orthogonal Steiner triple systems on v points we say that there exists a 3OSTS(v) or that there exists 3OSTS(v). In this paper we will proceed down a well followed path. We will first construct 3OSTS(v) for many "small" values of v, then we will use Wilson fundamental construction-type recursion to get all of the large orders. We will leave only a relatively small number of outstanding cases unsolved. We begin by noting the values of $v < 500$ for which 3OSTS(v) are known to exist.

Theorem 1.1 (see Dinitz and Dukes [4]). *There exists* 3OSTS(v) *for all* $v \equiv 1, 3 \pmod 6$, *with* $25 \leqslant v \leqslant 85$. *Also*, 3OSTS($v$) *exist for all* $v \in \{19, 91, 97, 103, 109, 115, 121, 127, 133, 139, 145, 151, 157, 163, 169, 175, 181, 193, 199, 211, 223, 229, 241, 271, 277, 283, 289, 307, 313, 331, 337, 343, 349, 361, 367, 373, 379, 397, 409, 421, 433, 439, 457, 463, 487, 499\}$.

For completeness we give the following results about nonexistence of 3OSTS.

Theorem 1.2. *There does not exist a* 3OSTS(v) *for* $v = 3, 7, 9$ (see [10]), 13 *or* 15 (both [6]).

The main ingredient in the recursive construction used to find the spectrum of orthogonal Steiner triple systems was a special type of group divisible design called an *orthogonal group divisible design*, or OGDD. In this paper we extend these (basically from two to three underlying block sets) to what we term 3OGDD. Here are the definitions.

A *group divisible design* (or GDD) is a triple $(X, \mathscr{G}, \mathscr{A})$ which satisfies three properties:

1. $\mathscr{G}$ is a partition of the point set X into subsets called *groups*;
2. $\mathscr{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one point in common; and
3. every pair of points from distinct groups occurs in a unique block.

If all the blocks of a GDD have the same size k it is called k-GDD. A *transversal design*, TD(k, n) is a GDD with k groups each of size n and $|b| = k$ for all blocks $b \in \mathscr{A}$. In other words, a transversal design is a group divisible design where all the groups have the same size and each block intersects each group in exactly one point. In many places in this paper we will be using the existence of TD(k, n). In most cases these transversal designs will exist by the standard finite field construction which states that when $n = q$ is a prime power, there is a TD(k, n) for all $k \leqslant n + 1$.

For the remainder of this paper we will assume that in every GDD, all blocks have size three (except of course in the transversal design cases). Let $(X, \mathcal{G}, \mathcal{A}_1)$, $(X, \mathcal{G}, \mathcal{A}_2)$ and $(X, \mathcal{G}, \mathcal{A}_3)$ be three 3-GDDs on the same pointset X and with the same groups $\mathcal{G}$. These three 3-GDDs are *pairwise orthogonal*, and are termed a 3OGDD, if the following orthogonality conditions are satisfied:

1. if $\{x, y, z\} \in \mathcal{A}_i$ and $\{x, y, w\} \in \mathcal{A}_j$, with $i \neq j$, then z and w are in different groups;
2. if $\{\{a, b, c\}, \{a, d, e\}\} \subset \mathcal{A}_i$ and if $\{\{x, b, c\}, \{y, d, e\}\} \subset \mathcal{A}_j$, with $i \neq j$, then $x \neq y$.

Adopting the standard notation, we say that a 3OGDD has type $(g_1)^{u_1} \cdots (g_s)^{u_s}$ if the 3OGDD has u_i groups of size g_i for each $1 \leqslant i \leqslant s$, and no other groups. Note that a 3OSTS(v) is therefore the same as a 3OGDD of type 1^v.

As noted earlier, our main recursive construction will be *Wilson's fundamental construction*, here applied to 3OGDD, in conjunction with a *filling in the groups* construction. We state both of these theorems here and refer the reader to [13] or [2] for proofs of these results in the case of two orthogonal triple systems (or two OGDDs). The proofs for the 3OSTS and 3OGDDs are analogous. Note that we give several versions of the *filling in the groups* construction.

Theorem 1.3 (Wilson's fundamental construction). *Let* $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ *be a* GDD (*the* master GDD) *with groups* $G_1, G_2, \ldots G_t$. *Suppose there exists a* (*weight*) *function* $w : \mathcal{V} \to \mathbf{Z}^+ \cup \{0\}$ (*a* weight function) *which has the property that for each block* $B = \{x_1, x_2, \ldots, x_k\} \in \mathcal{B}$ *there exists a* 3OGDD *of type* $(w(x_1), w(x_2), \ldots, w(x_k))$. *Then there exists a* 3OGDD *of type*

$$\left(\sum_{x \in G_1} w(x), \sum_{x \in G_2} w(x), \ldots, \sum_{x \in G_t} w(x) \right).$$

Theorem 1.4 (Filling in groups). (a) *If there is a* 3OGDD *of type* $u^g v^h$ *and there exist a* 3OSTS$(u + 1)$ *and a* 3OSTS$(v + 1)$, *then there exists a* 3OSTS$(gu + hv + 1)$.

(b) *If there is a* 3OGDD *of type* $u^g v^h$ *and there exist a* 3OSTS(u) *and a* 3OSTS(v), *then there exists a* 3OSTS$(gu + hv)$.

(c) *If there is a* 3OGDD *of type* $u^g v^1$ *and there exist a* 3OGDD *of type* $1^u 3^1$ *and a* 3OSTS$(v + 3)$, *then there exists a* 3OSTS$(gu + v + 3)$.

In the next section we will give direct constructions for many 3OSTS and 3OGDD. In order to check for orthogonality we note that every STS defines a *third element function* $\Theta : \binom{V}{2} \to V$ given by $\Theta(\{u, v\}) = w$ if and only if $\{u, v, w\}$ is a triple. It follows from the definition that two STS $(V, \mathcal{B}_1)$ and $(V, \mathcal{B}_2)$ with third element functions Θ_1 and Θ_2, respectively, are orthogonal if and only if for each $c \in V$, the list

$(\Theta_2(\{u, v\})|\Theta_1(\{u, v\}) = c)$ consists of distinct elements none of which equal c. This verification is called the *orthogonality certificate*. A similar type of certificate exists for 3OGDD.

In Section 2 we discuss the direct constructions for the small 3OSTS and 3OGDDs that were necessary to find directly via computer algorithms. In Section 3 we study the spectrum of 3OSTS(v) with $v \equiv 1 \pmod 6$ and in Section 4 we will do the same for $v \equiv 3 \pmod 6$. We note here that most of the real work in this paper was in finding the small designs (both 3OSTS and 3OGDD) in Section 2. Once these were in place, the recursions kicked in and the results in the subsequent sections followed in a relatively straightforward manner. We must add that we were extremely pleased that in the $v \equiv 1 \pmod 6$ case that we were able to determine the spectrum completely with no exceptional cases.

2. Direct constructions

In this section, certain sets of three pairwise orthogonal STS and 3-GDDs are presented with a brief description of the various algebraic and computational methods used. The designs here will be used as ingredients in the recursive constructions that follow. We will give a complete verification of orthogonality for only the smallest design found by a certain method. The authors may be contacted for all other orthogonality certificates.

2.1. 3OSTS

All of the direct constructions in this section use the hill-climbing method of search (see [7,8]). However, all of these searches are far too difficult by just that method alone and in every case it is necessary to use some possible automorphism of the purported design. This is the art of this search.

For many orders, cyclic automorphisms may be exploited. We extend a method used in Section 3.2 of [4] to some larger values here. A hill-climb is first used to construct a Steiner triple system S in a additive cyclic group of order v subject to the condition that S is orthogonal to its (elementwise) negative, $-S$. Such STS are called *opposite orthogonal*. An easy check of opposite orthogonality is that the elements among the $(v-1)/6$ zero-sum base triples are all distinct [12]. A second hill-climb then attempts to complete another cyclic Steiner triple system T, the mate, which is orthogonal to both S and to $-S$. It should be noted that when successful, this actually produces a subgraph isomorphic to K_4 minus an edge in the graph of orthogonality between all STS(n), namely $S, -S, T, -T$. Many opposite orthogonal bases are often needed to be searched to find a mate. This method has an advantage over a three-stage hill-climb because of the relatively quick construction of opposite orthogonal STS.

Lemma 2.1. *There exist* 3OSTS *of orders* $v = 187, 205, 253, 265, 295$ *and* 319.

Proof. Below we give the $(v-1)/6$ zero-sum triples which form the base blocks for S, followed by $(v-1)/6$ base triples for the mate T. Each full system is generated in the additive group $\mathbf{Z}_v$. Orthogonality of S with $-S$ can be seen by inspecting the first set of base blocks for repetitions. The orthogonality certificate of T with S and $-S$ is given for the smallest order only as a list of elements (a,b) such that xya and xyb are blocks of S and $-S$, respectively, over all triples of the form $0xy$ in the mate. Checking that no repetition occurs among first coordinate entries or second coordinate entries establishes orthogonality of T with both S and $-S$.

$v = 187$:

{4,52,131}, {28,142,17}, {120,177,77}, {97,153,124}, {9,16,162}, {133,53,1}, {94,62,31}, {55,58,74}, {40,41,106}, {63,57,67}, {81,121,172}, {127,181,66}, {49,125,13}, {54,156,164}, {163,22,2}, {69,155,150}, {3,91,93}, {168,26,180}, {15,65,107}, {176,60,138}, {29,47,111}, {5,103,79}, {101,8,78}, {71,92,24}, {46,104,37}, {32,160,182}, {7,21,159}, {161,186,27}, {167,84,123}, {154,51,169}, {139,109,126};

{128,75,84}, {166,5,71}, {161,143,52}, {0,39,132}, {90,148,21}, {147,117,58}, {27,141,113}, {48,16,12}, {121,75,5}, {10,48,90}, {139,74,88}, {148,168,80}, {106,155,49}, {66,45,16}, {119,43,36}, {118,166,54}, {109,132,120}, {135,83,68}, {119,29,153}, {159,44,175}, {105,146,113}, {54,57,157}, {178,183,184}, {10,154,12}, {125,165,103}, {27,109,183}, {97,121,84}, {183,173,156}, {86,165,39}, {100,81,135}, {93,68,178}.

CERTIFICATE: $(14,76)$, $(157,92)$, $(16,19)$, $(27,91)$, $(24,16)$, $(141,75)$, $(118,129)$, $(134,167)$, $(173,27)$, $(109,62)$, $(13,41)$, $(138,11)$, $(166,10)$, $(45,144)$, $(60,136)$, $(120,135)$, $(104,54)$, $(76,72)$, $(61,139)$, $(154,78)$, $(90,39)$, $(145,161)$, $(108,107)$, $(160,67)$, $(164,48)$, $(8,155)$, $(96,90)$, $(175,130)$, $(97,94)$, $(29,36)$, $(87,171)$, $(161,105)$, $(172,52)$, $(22,117)$, $(146,120)$, $(42,114)$, $(135,44)$, $(38,181)$, $(92,71)$, $(110,6)$, $(63,116)$, $(169,97)$, $(70,145)$, $(32,37)$, $(158,119)$, $(159,12)$, $(57,157)$, $(4,172)$, $(56,165)$, $(102,50)$, $(106,82)$, $(17,51)$, $(181,43)$, $(147,122)$, $(65,66)$, $(131,83)$, $(151,65)$, $(43,45)$, $(112,134)$, $(117,110)$, $(183,53)$, $(44,69)$, $(26,186)$, $(46,60)$, $(115,169)$, $(103,68)$, $(71,127)$, $(5,178)$, $(100,80)$, $(156,177)$, $(142,133)$, $(55,85)$, $(51,154)$, $(184,88)$, $(113,158)$, $(180,58)$, $(137,42)$, $(49,95)$, $(152,46)$, $(133,180)$, $(126,111)$, $(3,147)$, $(39,141)$, $(23,21)$, $(7,25)$, $(129,40)$, $(186,174)$, $(107,96)$, $(68,5)$, $(136,149)$, $(77,170)$, $(33,102)$, $(130,49)$.

$v = 205$:

{118,21,66}, {102,168,140}, {13,161,31}, {12,146,47}, {177,138,95}, {65,176,169}, {148,182,80}, {200,136,74}, {129,62,14}, {181,35,194}, {185,50,175}, {183,88,139}, {134,56,15}, {198,77,135} {113,64,28}, {30,142,33}, {191,6,8}, {29,116,60}, {125,36,44}, {104,133,173} {72,76,57}, {32,9,164}, {196,17,197}, {159,153,98}, {10,143,52}, {117,149,144} {63,192,155}, {106,189,115}, {202,18,190}, {53,83,69}, {180,103,127} {162,151,97}, {4,109,92}, {188,141,81};

{60,15,90}, {13,199,133}, {76,198,88}, {99,177,109}, {167,27,173} {101,192,7}, {113,25,203}, {159,66,115}, {102,137,153}, {76,129,6}, {105,31,38} {112,141,55}, {111,8,70}, {118,55,5}, {143,134,37}, {193,97,21}, {29,57,109} {162,159,161}, {7,164,43}, {117,75,93}, {47,10,81}, {79,102,41}, {130,98,90} {149,203,160}, {163,63,150}, {17,42,183}, {170,201,11}, {42,98,38} {121,100,126}, {93,71,172}, {171,73,154}, {134,189,61}, {194,78,147}, {105,119,177}.

$v = 253$:

{203,175,128}, {123,237,146}, {114,151,241}, {99,47,107}, {210,149,147}, {226,111,169}, {202,168,136}, {120,224,162}, {3,102,148}, {80,64,109}, {78,156,19}, {100,57,96}, {205,2,46}, {22,42,189}, {115,8,130}, {247,89,170}, {174,90,242}, {73,190,243}, {39,110,104}, {251,71,184}, {30,172,51}, {155,158,193}, {124,69,60}, {166,92,248}, {219,35,252}, {28,211,14}, {49,62,142}, {84,173,249}, {16,103,134}, {86,186,234}, {34,106,113}, {132,40,81}, {245,244,17}, {77,94,82}, {140,116,250}, {118,167,221}, {101,61,91}, {157,37,59}, {25,176,52}, {144,229,133}, {204,223,79}, {88,218,200};

{214,130,36}, {195,135,110}, {99,202,224}, {22,98,138}, {197,65,32}, {201,245,44}, {111,234,43}, {210,28,106}, {72,71,108}, {7,218,62}, {213,156,203}, {159,90,164}, {171,185,206}, {68,29,251}, {206,80,152}, {64,14,251}, {72,153,201}, {245,53,10}, {61,37,10}, {174,68,178}, {7,20,178}, {109,29,26}, {239,50,186}, {240,248,54}, {136,143,234}, {139,30,204}, {84,142,11}, {199,60,106}, {125,127,157}, {34,197,46}, {11,177,100}, {212,171,197}, {219,213,202}, {131,32,140}, {202,101,146}, {91,110,5}, {77,195,223}, {198,221,87}, {58,248,24}, {64,225,176}, {171,209,56}, {191,71,211}.

$v = 265$:

{209,102,219}, {48,243,239}, {131,115,19}, {98,175,257}, {177,163,190}, {140,111,14}, {27,38,200}, {185,251,94}, {52,223,255}, {146,170,214}, {210,61,259}, {143,78,44}, {213,99,218}, {191,113,226}, {129,93,43}, {208,236,86}, {199,184,147}, {144,97,24}, {92,164,9}, {138,116,11}, {127,37,101}, {35,95,135}, {194,217,119}, {125,171,234}, {36,232,262}, {15,242,8}, {26,155,84}, {186,166,178}, {13,55,197}, {205,253,72}, {244,183,103}, {139,122,4}, {28,250,252}, {136,264,130}, {231,87,212}, {5,23,237}, {261,216,53}, {89,174,2}, {206,29,30}, {167,142,221}, {215,156,159}, {117,22,126}, {69,16,180}, {56,132,77};

{67,34,264}, {189,63,215}, {254,230,170}, {126,119,251}, {117,169,159}, {183,54,140}, {162,25,146}, {14,118,163}, {180,125,3}, {127,141,216}, {27,185,221}, {261,107,166}, {17,251,80}, {1,197,47}, {2,81,59}, {182,19,92}, {258,262,239}, {49,17,238}, {114,115,77}, {64,10,224}, {38,56,263}, {179,102,200}, {64,126,191}, {188,244,64}, {73,65,90}, {185,198,213}, {151,156,217}, {130,208,60}, {41,94,197}, {150,53,3}, {156,65,76}, {91,24,260}, {183,32,163}, {156,197,55}, {40,175,256}, {7,200,153}, {14,196,96}, {212,39,138}, {159,51,49}, {26,146,53}, {72,231,225}, {66,63,75}, {103,69,39}, {140,53,101}.

$v = 295$:

{158,206,226}, {262,261,67}, {216,72,7}, {62,132,101}, {73,33,189}, {120,152,23}, {109,56,130}, {172,105,18}, {93,276,221}, {293,249,48}, {282,3,10}, {134,17,144}, {5,288,2}, {183,12,100}, {202,175,213}, {70,292,228}, {260,266,64}, {245,98,247}, {176,268,146}, {236, 270,84}, {116,239,235}, {153,38,104}, {79,53, 163}, {248,16,31}, {199,274,117}, {69,149,77}, {133,147,15}, {86,182,27}, {190,227,173}, {29, 255,11}, {60,251,279}, {78,185,32}, {55,220,20}, {25,114,156}, {151,241,198}, {234,37,24}, {197, 257,136}, {4,242,49}, {165,140,285}, {99,210, 281}, {87,212,291}, {75,42,178}, {294,21,275}, {82,26,187}, {43,157,95}, {90,196,9}, {265,96, 229}, {208,203,179}, {92,169,34};

{244,208,234}, {179,90,132}, {289,84,248}, {255, 21,94}, {105,86,261}, {106,164,229}, {97,37,251}, {120,12,230}, {201,126,34}, {249,61,186}, {213, 284,160}, {86,276,98}, {14,198,80}, {281,292, 258}, {18,230,210}, {187,174,205}, {201,39,72}, {47,187,55}, {98,198,241}, {212,115,59}, {278,92, 143}, {248,180,239}, {174,203,53}, {192,12,293}, {13,209,20}, {234,240,194}, {19,219,276}, {117, 266,33}, {70,75,168}, {54,81,219}, {207,277,279}, {119,87,37}, {31,119,210}, {83,162,66}, {56,170, 214}, {138,89,163}, {171,23,290}, {263,285,5}, {2,50,5}, {122,200,106}, {125,177,212}, {184,15, 188}, {71,72,157}, {76,188,37}, {127,254,23}, {164,140,194}, {286,93,206}, {191,170,115}, {173, 240,14}.

$n = 319$:

{7,285,27}, {200,264,174}, {19,213,87}, {39,240,40}, {270,127,241}, {38,276,5}, {95,245,298}, {220,269,149}, {244,217,177}, {296,313,29}, {119,218,301}, {317,113,208}, {66,121,132}, {192,242,204}, {93,193,33}, {126,30,163}, {196,107,16}, {246,302,90}, {237,222,179}, {183,54,82}, {166,135,18}, {61,318,259}, {76,120,123}, {148,236,254}, {36,20,263}, {99,223,316}, {257,308,73}, {249,272,117}, {11,252,56}, {314,293,31}, {176,182,280}, {275,69,294}, {98,12,209}, {198,64,57}, {191,24,104}, {100,114,105}, {3,160,156}, {89,268,281}, {49,260,10}, {118,195,6}, {171,1,147}, {136,287,215}, {63,235,21}, {168,312,158}, {170,9,140}, {211,141,286}, {203,129,306}, {307,70,261}, {153,15,151}, {53,116,150}, {216,238,184}, {300,292,46}, {65,299,274};

{63,222,221}, {132,317,108}, {291,158,224}, {30,166,136}, {317,187,295}, {72,194,251}, {107,115,73}, {126,144,221}, {103,195,67}, {92,153,265}, {256,240,315}, {245,205,248}, {1,143,163}, {175,200,85}, {44,32,100}, {124,305,72}, {308,192,190}, {14,53,202}, {89,186,234}, {134,178,52}, {152,316,179}, {249,10,318}, {84,47,34}, {15,281,114}, {229,142,262}, {28,242,300}, {155,38,210}, {259,308,304}, {160,69,261}, {264,315,58}, {52,26,21}, {86,3,146}, {65,163,234}, {238,303,138}, {292,84,61}, {144,251,158}, {15,249,34}, {121,224,256}, {16,314,139}, {98,19,113}, {30,183,59}, {77,161,36}, {300,132,54}, {11,4,85}, {173,97,211}, {159,27,205}, {197,26,206}, {285,221,215}, {63,0,28}, {197,76,293}, {154,283,273}, {269,113,4}, {41,271,24}. □

Lemma 2.2. *There exist* 3OSTS *of orders* $v = 217, 247, 259$ *and* 301.

Proof. These constructions use the same idea as in the previous lemma, but computation is aided by the use of a multiplier ζ of order three in $\mathbf{Z}_v^*$. Each system is generated with multiplication by $\{1, \zeta, \zeta^2\}$ in addition to cyclic automorphism. So $\lfloor (v-1)/18 \rfloor + 0, 1$ or 2 zero-sum triples are given for both the opposite orthogonal basis and mate. Note up to two triples may be fixed under ζ, and these are given at the end of each list.

$v = 217,\ \zeta = 67$:

{204,179,51}, {184,135,115}, {185,154,95}, {175,172,87}, {94,79,44}, {195,174,65}, {133,66,18}, {150,144,140}, {199,159,76}, {116,99,2}, {145,71,1}, {117,61,39};

{99,79,39}, {196,161,77}, {132,44,41}, {87,69,61}, {214,143,77}, {133,74,10}, {96,86,35}, {211,184,39}, {164,136,134}, {186,147,101}, {122,48,47}, {142,53,22}.

$v = 247,\ \zeta = 87$:

{97,81,69}, {216,171,107}, {198,169,127}, {148,93,6}, {211,200,83}, {197,180,117}, {236,147,111}, {157,76,14}, {156,87,4}, {135,89,23}, {115,82,50}, {178,163,153}, {239,137,118}, {120,66,61}, {226,149,119};

{120,67,60}, {239,235,20}, {190,187,117}, {106,76,65}, {165,61,21}, {141,91,15}, {213,205,76}, {246,151,97}, {194,178,122}, {234,142,118}, {233,155,106}, {118,76,53}, {159,53,35}, {123,80,44}, {112,111,24}.

$v = 259,\ \zeta = 121$:

{232,146,140}, {151,107,1}, {240,158,120}, {136,68,55}, {204,35,20}, {198,49,12}, {229,194,95}, {205,171,142}, {183,69,7}, {242,239,37}, {258,189,71}, {127,84,48}, {195,58,6}, {196,168,154}, {237,187,94};

{252,243,23}, {224,182,112}, {137,64,58}, {210,178,130}, {214,43,2}, {241,206,71}, {190,36,33}, {219,214,85}, {163,49,47}, {245,229,44}, {191,64,4}, {245,244,29}, {248,227,43}, {256,157,105}, {225,30,4}.

$n = 301,\ \zeta = 135$:

{267,238,97}, {130,117,54}, {276,240,86}, {220,70,11}, {116,101,84}, {197,76,28}, {284,231,87}, {285,279,38}, {201,82,18}, {297,173,132}, {137,127,37}, {165,94,42}, {209,75,17}, {158,138,5}, {153,80,68}, {273,266,63}, {121,99,81}, {258,215,129};

{298,202,102}, {166,125,10}, {160,135,6}, {151,79,71}, {252,250,100}, {287,226,89}, {292,210,100}, {163,128,10}, {292,207,103}, {229,63,9}, {200,60,41}, {278,273,51}, {108,101,92}, {300,276,26}, {179,81,41}, {163,98,40}, {274,268,60}, {260,184,158}. □

We now turn to the direct construction of 3OSTS of orders $v \equiv 3 \pmod 6$. A common theme will be exploiting cube roots of unity with various automorphisms. An STS$(2m+1)$ with an automorphism consisting of two cycles of length m and a fixed point is called 2-*rotational*. The STS is determined completely by $(2m+1)/3$ base blocks. More concretely, we use the set of points $V = \mathbf{Z}_m \times \{0,1\} \cup \{\infty\}$, with generating automorphism $x_i \mapsto (x+1)_i,\ \infty \mapsto \infty$. The examples in the following lemma are found by choosing ω of order three in $\mathbf{Z}_m^*$ and hill-climbing to a set of base blocks so that the resulting design $(V, \mathscr{B})$ is orthogonal to $(V, \omega\mathscr{B})$, implying that $(V, \mathscr{B})$, $(V, \omega\mathscr{B})$ and $(V, \omega^2\mathscr{B})$ are 3OSTS.

Lemma 2.3. *There exist* 3OSTS *of order* $v = 87, 99, 123, 135,$ *and* 159.

Proof. The cube roots and base triples are given below. The orthogonality certificate is described by giving third elements c occurring in $\omega\mathscr{B}$ with all pairs xy that occur in $\mathscr{B}$ with 0_0, and with 0_1. Orthogonality is satisfied when these two lists consist of distinct elements not equal to 0_0 or 0_1, respectively. The block $\{\infty, 0_0, 0_1\}$ must also be avoided.

$v = 87,\ m = 43,\ \omega = 6$:

$\{0_1, 21_0, 25_0\}$, $\{0_1, 28_0, 41_0\}$, $\{4_1, 5_1, 34_1\}$, $\{0_1, 5_0, 40_0\}$, $\{0_0, 1_1, 27_1\}$, $\{0_1, 13_0, 18_0\}$, $\{2_1, 8_1, 33_1\}$, $\{0_0, 8_1, 29_1\}$, $\{0_1, 9_0, 23_0\}$, $\{0_1, 24_0, 36_0\}$, $\{19_1, 29_1, 38_1\}$, $\{0_1, 4_0, 19_0\}$, $\{0_0, 17_1, 33_1\}$, $\{8_1, 16_1, 19_1\}$, $\{0_1, 30_0, 32_0\}$, $\{0_0, 4_1, 32_1\}$, $\{0_0, 40_1, 42_1\}$, $\{0_0, 16_1, 23_1\}$, $\{0_0, 6_1, 26_1\}$, $\{17_0, 33_0, 36_0\}$,

$\{0_0, 37_1, 41_1\}$, $\{0_0, 31_1, 36_1\}$, $\{0_1, 31_0, 38_0\}$, $\{0_1, 8_0, 29_0\}$, $\{1_0, 18_0, 24_0\}$, $\{0_1, 15_0, 33_0\}$, $\{7_0, 39_0, 40_0\}$, $\{0_1, 0_0, 34_0\}$, $\{\infty, 0_0, 21_1\}$.

CERTIFICATE: (0_0) 9_0, 15_0, 18_1, 26_0, 16_0, 5_0, 12_1, 42_1, 39_0, 15_1, 38_1, 31_0, 0_1, 35_0, 35_1, 7_0, 27_0, 13_0, 36_0, 21_1, 34_1, 37_1, 31_1, 14_1, 6_1, 33_0, 17_0, 11_1, 9_1, 1_0, 22_1, 34_0, 40_0, 32_0, 14_0, 27_1, 11_0, 19_1, 29_0, 10_1, 1_1, 41_0, 24_0,
(0_1) 40_1, 6_1, 23_1, 28_0, 4_0, 9_1, 35_1, 30_1, 31_1, 26_1, 0_0, 20_1, 34_0, 1_0, 8_1, 16_1, 26_0, 2_0, 8_0, 29_0, 32_1, 31_0, 6_0, 18_1, 38_1, 21_1, 37_0, 3_0, 17_0, 43_0, 15_0, 3_1, 10_1, 7_1, 9_0, 27_1, 39_1, 14_1, 22_0, 25_1, 5_0, 7_0, 19_1.

$v = 99$, $m = 49$, $\omega = 18$:
$\{0_1, 12_0, 15_0\}$, $\{0_1, 38_0, 46_0\}$, $\{0_0, 1_1, 12_1\}$, $\{11_0, 43_0, 44_0\}$, $\{0_1, 7_0, 17_0\}$, $\{5_0, 10_0, 34_0\}$, $\{6_1, 7_1, 36_1\}$, $\{0_0, 7_1, 47_1\}$, $\{0_1, 30_0, 44_0\}$, $\{0_0, 23_1, 30_1\}$, $\{0_1, 23_0, 36_0\}$, $\{0_1, 21_0, 39_0\}$, $\{0_1, 0_0, 9_0\}$, $\{0_0, 15_1, 21_1\}$, $\{5_1, 13_1, 31_1\}$, $\{5_0, 16_0, 28_0\}$, $\{0_1, 18_0, 40_0\}$, $\{0_1, 25_0, 29_0\}$, $\{0_0, 4_1, 16_1\}$, $\{0_0, 8_1, 22_1\}$, $\{0_0, 41_1, 43_1\}$, $\{3_1, 7_1, 39_1\}$, $\{0_1, 4_0, 10_0\}$, $\{0_0, 2_1, 35_1\}$, $\{0_1, 1_0, 16_0\}$, $\{0_1, 5_0, 35_0\}$, $\{0_1, 13_0, 20_0\}$, $\{8_1, 18_1, 23_1\}$, $\{0_0, 17_1, 38_1\}$, $\{0_1, 22_0, 24_0\}$, $\{17_1, 39_1, 42_1\}$, $\{0_1, 3_0, 31_0\}$, $\{\infty, 0_0, 6_1\}$.

$v = 123$, $m = 61$, $\omega = 47$:
$\{6_1, 20_1, 35_1\}$, $\{0_0, 6_1, 42_1\}$, $\{0_1, 0_0, 13_0\}$, $\{0_0, 5_1, 18_1\}$, $\{0_1, 37_0, 47_0\}$, $\{0_0, 26_1, 32_1\}$, $\{0_1, 11_0, 14_0\}$, $\{22_0, 48_0, 52_0\}$, $\{0_1, 32_0, 46_0\}$, $\{13_1, 53_1, 56_1\}$, $\{0_0, 11_1, 33_1\}$, $\{0_1, 18_0, 24_0\}$, $\{0_0, 44_1, 56_1\}$, $\{0_0, 41_1, 60_1\}$, $\{0_0, 27_1, 55_1\}$, $\{6_1, 23_1, 32_1\}$, $\{0_1, 23_0, 45_0\}$, $\{0_0, 21_1, 59_1\}$, $\{0_0, 20_1, 25_1\}$, $\{16_0, 49_0, 57_0\}$, $\{0_1, 12_0, 31_0\}$, $\{0_1, 44_0, 59_0\}$, $\{0_1, 30_0, 42_0\}$, $\{0_0, 13_1, 54_1\}$, $\{11_0, 16_0, 34_0\}$, $\{0_1, 33_0, 57_0\}$, $\{0_1, 16_0, 27_0\}$, $\{6_1, 56_1, 60_1\}$, $\{0_1, 4_0, 25_0\}$, $\{0_0, 51_1, 53_1\}$, $\{7_1, 23_1, 31_1\}$, $\{0_0, 10_1, 40_1\}$, $\{0_1, 9_0, 38_0\}$, $\{5_0, 6_0, 50_0\}$, $\{0_0, 8_1, 9_1\}$, $\{0_0, 12_1, 46_1\}$, $\{0_1, 26_0, 60_0\}$, $\{0_0, 7_1, 58_1\}$, $\{37_0, 39_0, 46_0\}$, $\{0_1, 22_0, 58_0\}$, $\{\infty, 0_0, 22_1\}$.

$v = 135$, $m = 67$, $\omega = 37$:
$\{7_1, 17_1, 43_1\}$, $\{0_1, 18_0, 66_0\}$, $\{0_1, 25_0, 45_0\}$, $\{0_0, 7_1, 55_1\}$, $\{0_0, 25_1, 39_1\}$, $\{0_0, 11_1, 57_1\}$, $\{0_1, 9_0, 40_0\}$, $\{12_1, 23_1, 32_1\}$, $\{0_0, 0_1, 13_1\}$, $\{0_0, 17_1, 33_1\}$, $\{21_1, 48_1, 65_1\}$, $\{0_0, 34_1, 37_1\}$, $\{16_1, 53_1, 65_1\}$, $\{0_1, 53_0, 55_0\}$, $\{0_1, 36_0, 41_0\}$, $\{0_1, 4_0, 49_0\}$, $\{0_1, 5_0, 16_0\}$, $\{0_0, 45_1, 50_1\}$, $\{0_1, 11_0, 35_0\}$, $\{4_0, 18_0, 45_0\}$, $\{24_0, 49_0, 61_0\}$, $\{0_0, 29_1, 54_1\}$, $\{0_0, 40_1, 41_1\}$, $\{0_1, 2_0, 48_0\}$, $\{0_1, 6_0, 63_0\}$, $\{17_1, 55_1, 62_1\}$, $\{0_1, 1_0, 52_0\}$, $\{0_0, 28_1, 52_1\}$, $\{0_1, 21_0, 24_0\}$, $\{33_0, 39_0, 62_0\}$, $\{3_0, 7_0, 57_0\}$, $\{0_0, 3_1, 36_1\}$, $\{0_0, 8_1, 16_1\}$, $\{0_0, 6_1, 38_1\}$, $\{0_0, 21_1, 23_1\}$, $\{0_1, 37_0, 65_0\}$, $\{0_0, 47_1, 53_1\}$, $\{0_0, 9_1, 48_1\}$, $\{37_0, 45_0, 52_0\}$, $\{23_0, 55_0, 56_0\}$, $\{0_1, 8_0, 57_0\}$, $\{0_0, 60_1, 64_1\}$, $\{0_1, 23_0, 32_0\}$, $\{0_0, 5_1, 20_1\}$, $\{\infty, 0_0, 24_1\}$.

$v = 159$, $m = 79$, $\omega = 23$:
$\{23_1, 25_1, 31_1\}$, $\{0_0, 24_1, 47_1\}$, $\{16_0, 22_0, 41_0\}$, $\{0_1, 28_0, 73_0\}$, $\{0_1, 11_0, 43_0\}$, $\{0_1, 70_0, 72_0\}$, $\{0_0, 63_1, 78_1\}$, $\{7_0, 17_0, 55_0\}$, $\{0_1, 0_0, 12_0\}$, $\{0_0, 27_1, 71_1\}$, $\{0_1, 17_0, 69_0\}$, $\{0_0, 60_1, 72_1\}$, $\{0_0, 4_1, 33_1\}$, $\{35_1, 49_1, 74_1\}$, $\{0_0, 55_1, 77_1\}$,

$\{0_1, 48_0, 77_0\}$, $\{0_0, 3_1, 16_1\}$, $\{0_0, 13_1, 59_1\}$, $\{0_0, 42_1, 70_1\}$, $\{0_0, 12_1, 32_1\}$, $\{43_0, 50_0, 65_0\}$, $\{38_0, 58_0, 62_0\}$, $\{0_0, 23_1, 30_1\}$, $\{0_0, 20_1, 56_1\}$, $\{0_0, 44_1, 76_1\}$, $\{0_1, 53_0, 61_0\}$, $\{0_1, 29_0, 65_0\}$, $\{0_1, 34_0, 45_0\}$, $\{0_1, 14_0, 54_0\}$, $\{0_0, 21_1, 48_1\}$, $\{0_0, 11_1, 52_1\}$, $\{0_1, 41_0, 44_0\}$, $\{0_0, 17_1, 66_1\}$, $\{0_0, 40_1, 64_1\}$, $\{0_0, 5_1, 22_1\}$, $\{0_0, 39_1, 49_1\}$, $\{30_0, 56_0, 72_0\}$, $\{0_0, 53_1, 54_1\}$, $\{0_0, 43_1, 61_1\}$, $\{41_1, 57_1, 60_1\}$, $\{0_0, 1_1, 46_1\}$, $\{0_0, 37_1, 41_1\}$, $\{12_0, 25_0, 42_0\}$, $\{19_1, 67_1, 72_1\}$, $\{0_0, 19_1, 28_1\}$, $\{5_0, 23_0, 51_0\}$, $\{0_0, 15_1, 57_1\}$, $\{0_0, 58_1, 69_1\}$, $\{45_0, 54_0, 59_0\}$, $\{0_0, 8_1, 29_1\}$, $\{0_1, 4_0, 5_0\}$, $\{31_0, 52_0, 75_0\}$, $\{\infty, 0_0, 73_1\}$. □

The next construction also constructs 2-rotational 3OSTS. However in this case it is assumed that there are 19 infinite points (instead of one). So in these cases $v = 2m + 19$. None of these infinite points occur together in any blocks of the difference set so when the blocks below are developed via the group, there is a *hole* of size 19. This can then be filled in with the 3OSTS(19) on the infinite points $\{\infty_1, \infty_2, \ldots, \infty_{19}\}$.

Lemma 2.4. *There exist* 3OSTS *of order* $v = 141$ *and* 153.

Proof. The set D of mixed differences occurring with the fixed points was chosen before the hill-climb. Powers of a generator were chosen in D to ensure that D, ωD, and $\omega^2 D$ are pairwise disjoint, which will hold for $m > 3 \cdot 19$. Note that the 19 blocks containing the infinite points are given at the end of each list. In both cases these 19 blocks are: $\{\infty_j, 0_0, (2^j)_1\}$ for $0 \leqslant j < 19$.

$m = 61$, $\omega = 13$, 19 fixed points:

$\{9_1, 13_1, 39_1\}$, $\{0_0, 5_1, 50_1\}$, $\{0_1, 35_0, 46_0\}$, $\{35_1, 40_1, 47_1\}$, $\{0_1, 0_0, 10_0\}$, $\{0_1, 8_0, 42_0\}$, $\{27_1, 36_1, 59_1\}$, $\{0_1, 9_0, 21_0\}$, $\{1_0, 6_0, 54_0\}$, $\{12_0, 54_0, 56_0\}$, $\{0_1, 12_0, 32_0\}$, $\{11_0, 54_0, 57_0\}$, $\{0_0, 21_1, 42_1\}$, $\{0_1, 15_0, 31_0\}$, $\{12_1, 51_1, 59_1\}$, $\{4_1, 7_1, 50_1\}$, $\{0_0, 33_1, 43_1\}$, $\{0_0, 10_1, 38_1\}$, $\{4_0, 28_0, 29_0\}$, $\{0_0, 54_1, 60_1\}$, $\{0_0, 39_1, 58_1\}$, $\{0_1, 2_0, 24_0\}$, $\{7_0, 14_0, 40_0\}$, $\{0_0, 56_1, 57_1\}$, $\{0_0, 28_1, 41_1\}$, $\{0_1, 14_0, 54_0\}$, $\{1_1, 18_1, 42_1\}$, $\{0_1, 6_0, 36_0\}$, $\{0_1, 41_0, 47_0\}$, $\{0_1, 44_0, 48_0\}$, $\{27_0, 36_0, 59_0\}$, $\{31_1, 33_1, 58_1\}$, $\{0_0, 23_1, 34_1\}$, $\{0_1, 16_0, 30_0\}$, $\{\infty_j, 0_0, (2^j)_1\}$ for $0 \leqslant j < 19$.

$m = 67$, $\omega = 29$, 19 fixed points:

$\{29_0, 44_0, 61_0\}$, $\{0_0, 24_1, 47_1\}$, $\{28_1, 43_1, 63_1\}$, $\{0_1, 34_0, 42_0\}$, $\{0_1, 14_0, 41_0\}$, $\{0_1, 44_0, 60_0\}$, $\{2_0, 5_0, 60_0\}$, $\{12_1, 14_1, 41_1\}$, $\{0_1, 36_0, 40_0\}$, $\{0_0, 48_1, 57_1\}$, $\{14_1, 56_1, 64_1\}$, $\{0_0, 12_1, 60_1\}$, $\{0_0, 30_1, 52_1\}$, $\{0_0, 34_1, 45_1\}$, $\{38_1, 45_1, 51_1\}$, $\{0_0, 37_1, 49_1\}$, $\{6_0, 19_0, 42_0\}$, $\{37_0, 38_0, 59_0\}$, $\{0_0, 22_1, 46_1\}$, $\{0_1, 1_0, 38_0\}$, $\{0_1, 16_0, 64_0\}$, $\{0_0, 42_1, 58_1\}$, $\{0_1, 0_0, 5_0\}$, $\{32_1, 42_1, 60_1\}$, $\{7_1, 12_1, 48_1\}$, $\{43_1, 44_1, 47_1\}$, $\{0_0, 28_1, 65_1\}$, $\{0_1, 13_0, 54_0\}$, $\{0_1, 26_0, 28_0\}$, $\{0_0, 6_1, 59_1\}$, $\{35_0, 46_0, 53_0\}$, $\{0_0, 17_1, 63_1\}$, $\{0_1, 17_0, 46_0\}$, $\{0_1, 11_0, 53_0\}$, $\{0_0, 11_1, 44_1\}$, $\{32_0, 46_0, 56_0\}$, $\{0_1, 32_0, 52_0\}$, $\{24_0, 52_0, 58_0\}$, $\{\infty_j, 0_0, (2^j)_1\}$ for $0 \leqslant j < 19$. □

An STS($3m$) with an automorphism consisting of three cycles of length m is called 3-*cyclic*. Such a system is determined completely by $(3m-1)/2$ base triples. A computational construction for three mutually orthogonal 3-cyclic STS is given in Section 3.1 of [4]. The pointset $\mathbf{Z}_m \times \mathbf{Z}_3$, was used with generating automorphism $x_i \mapsto (x+1)_i$. Here, subscripts represent the second coordinate in the product. Define the map on orbits $\alpha : x_i \mapsto x_{i+1}$. A hill-climb was used to construct a set of base triples, which when developed form a set $\mathscr{B}$ of blocks such that $\mathscr{B}$, $\alpha\mathscr{B}$ and $\alpha^2\mathscr{B}$ are orthogonal in pairs. Larger m than could be handled in [4] are handled here using a multiplier ζ of order three in $\mathbf{Z}_m$. The non-abelian automorphism group generated by $x_i \mapsto (x+1)_i$ and $x_i \mapsto (\zeta x)_{i+1}$ is used in the next lemma. As expected, this cuts computation time roughly to one-third of that for the algorithm in [4].

Lemma 2.5. *There exist* 3OSTS *of order* $3m$ *for* $m = 31, 37, 39, 43, 49, 57, 61, 65, 67, 73, 79$, *i.e. there exist* 3OSTS *of order* $v = 93, 111, 117, 129, 147, 171, 183, 195, 201, 219, 237$.

Proof. We give $(m-1)/2$ base triples for one STS($3m$). In addition to the base triples given, the additional base triple $\{1_0, \zeta_1, \zeta_2^2\}$ (generating a short orbit) must be included in each case. The indicated multiplier of order three in $\mathbf{Z}_m$ is used in the non-abelian automorphism group described above to get a full set of blocks $\mathscr{B}$. As before, the three pairwise orthogonal systems are generated by the map α. The orthogonality certificate amounts to a single list of third elements c occurring in $\alpha\mathscr{B}$ with all pairs xy that occur in $\mathscr{B}$ with 0_0.

$m = 31$, $\zeta = 5$:

$\{0_2, 5_1, 10_1\}$, $\{0_0, 11_1, 11_2\}$, $\{0_1, 5_0, 7_0\}$, $\{0_2, 8_1, 12_1\}$, $\{0_0, 6_1, 28_2\}$, $\{0_2, 1_0, 28_0\}$, $\{0_1, 8_2, 17_2\}$, $\{7_1, 26_1, 29_1\}$, $\{0_2, 5_0, 8_0\}$, $\{0_0, 3_1, 10_2\}$, $\{0_0, 18_1, 19_1\}$, $\{0_0, 1_1, 17_2\}$, $\{14_1, 21_1, 27_1\}$, $\{0_1, 3_2, 11_2\}$, $\{0_2, 19_1, 21_1\}$.

CERTIFICATE: (0_0) 12_2, 11_0, 18_0, 5_1, 24_0, 15_0, 23_2, 17_1, 16_2, 24_1, 23_1, 25_2, 16_1, 19_2, 20_2, 0_1, 1_0, 17_0, 26_2, 11_1, 1_1, 9_2, 10_2, 6_1, 14_1, 12_0, 15_1, 26_0, 19_1, 3_0, 13_2, 20_1, 17_2, 24_2, 22_2, 6_2, 18_2, 9_0, 7_1, 9_1, 26_1, 4_0, 8_0, 30_0, 5_2, 22_0.

$m = 37$, $\zeta = 10$:

$\{0_0, 24_1, 10_2\}$, $\{0_2, 16_1, 29_1\}$, $\{0_2, 10_1, 12_1\}$, $\{0_0, 16_2, 22_2\}$, $\{0_0, 0_2, 24_2\}$, $\{0_0, 7_1, 36_2\}$, $\{0_0, 30_1, 23_2\}$, $\{0_0, 8_1, 26_1\}$, $\{0_1, 2_2, 12_2\}$, $\{0_0, 22_1, 31_1\}$, $\{0_0, 35_1, 11_2\}$, $\{0_0, 2_2, 13_2\}$, $\{9_0, 13_0, 15_0\}$, $\{0_0, 27_1, 33_1\}$, $\{0_1, 5_0, 25_0\}$, $\{0_0, 4_2, 32_2\}$, $\{23_2, 24_2, 27_2\}$, $\{0_2, 2_0, 30_0\}$.

$m = 39$, $\zeta = 16$:

$\{0_0, 1_1, 21_1\}$, $\{2_2, 9_2, 30_2\}$, $\{0_0, 3_1, 20_2\}$, $\{0_0, 6_1, 9_2\}$, $\{0_2, 5_1, 34_1\}$, $\{0_1, 25_0, 37_0\}$, $\{0_0, 10_1, 17_1\}$, $\{0_2, 3_0, 7_0\}$, $\{0_1, 0_2, 33_2\}$, $\{0_1, 1_0, 34_0\}$, $\{0_2, 8_0, 11_0\}$, $\{0_0, 33_1, 13_2\}$, $\{0_0, 18_1, 19_1\}$, $\{0_0, 13_1, 35_2\}$, $\{0_1, 2_0, 10_0\}$, $\{17_1, 29_1, 33_1\}$, $\{0_1, 11_2, 12_2\}$, $\{0_0, 27_2, 29_2\}$, $\{0_2, 16_0, 29_0\}$.

$m = 43,\ \zeta = 6$:

$\{0_1, 1_0, 40_0\}$, $\{0_2, 27_0, 42_0\}$, $\{0_2, 12_0, 34_0\}$, $\{0_2, 5_1, 30_1\}$, $\{0_2, 19_1, 26_1\}$, $\{0_2, 8_1, 39_1\}$, $\{0_0, 18_2, 23_2\}$, $\{0_0, 0_2, 12_2\}$, $\{0_0, 19_1, 25_1\}$, $\{0_0, 9_1, 15_2\}$, $\{0_0, 7_1, 40_1\}$, $\{6_2, 39_2, 41_2\}$, $\{0_2, 5_0, 29_0\}$, $\{0_2, 10_0, 30_0\}$, $\{0_1, 14_2, 27_2\}$, $\{0_0, 16_1, 17_2\}$, $\{0_0, 6_1, 28_1\}$, $\{0_0, 18_1, 41_1\}$, $\{0_1, 9_2, 15_2\}$, $\{0_1, 21_0, 31_0\}$, $\{0_2, 9_1, 20_1\}$.

$m = 49,\ \zeta = 18$:

$\{21_1, 30_1, 47_1\}$, $\{0_0, 20_1, 22_1\}$, $\{0_1, 26_0, 48_0\}$, $\{0_0, 14_2, 16_2\}$, $\{3_2, 4_2, 42_2\}$, $\{0_2, 15_0, 29_0\}$, $\{0_0, 4_1, 23_2\}$, $\{0_1, 24_2, 42_2\}$, $\{0_0, 46_1, 46_2\}$, $\{0_0, 35_2, 38_2\}$, $\{0_0, 16_1, 44_1\}$, $\{0_0, 18_1, 21_2\}$, $\{18_1, 33_1, 47_1\}$, $\{0_0, 7_2, 47_2\}$, $\{0_0, 8_2, 12_2\}$, $\{0_0, 24_1, 40_2\}$, $\{0_2, 2_1, 18_1\}$, $\{0_0, 12_1, 2_2\}$, $\{0_2, 43_1, 48_1\}$, $\{0_1, 39_0, 40_0\}$, $\{0_0, 37_1, 47_1\}$, $\{0_1, 4_0, 21_0\}$, $\{0_0, 40_1, 18_2\}$, $\{0_2, 36_0, 44_0\}$.

$m = 57,\ \zeta = 7$:

$\{0_1, 4_0, 30_0\}$, $\{0_0, 9_1, 12_1\}$, $\{0_0, 21_1, 51_2\}$, $\{18_0, 45_0, 53_0\}$, $\{0_0, 51_1, 40_2\}$, $\{0_0, 16_1, 40_1\}$, $\{0_0, 48_1, 17_2\}$, $\{0_0, 4_1, 35_1\}$, $\{0_0, 18_1, 34_2\}$, $\{6_0, 52_0, 56_0\}$, $\{0_0, 3_1, 56_2\}$, $\{8_0, 24_0, 25_0\}$, $\{0_0, 26_1, 36_2\}$, $\{0_0, 36_1, 11_2\}$, $\{0_0, 11_1, 13_2\}$, $\{0_0, 17_1, 55_2\}$, $\{30_0, 36_0, 48_0\}$, $\{8_0, 52_0, 55_0\}$, $\{0_0, 22_1, 10_2\}$, $\{0_0, 13_1, 56_1\}$, $\{0_0, 28_1, 41_2\}$, $\{0_1, 3_0, 8_0\}$, $\{1_0, 15_0, 43_0\}$, $\{0_1, 11_0, 49_0\}$, $\{0_0, 24_1, 30_1\}$, $\{0_1, 32_0, 55_0\}$, $\{0_0, 15_1, 19_1\}$, $\{0_0, 1_1, 1_2\}$.

$m = 61,\ \zeta = 13$:

$\{0_0, 57_2, 59_2\}$, $\{0_1, 19_2, 53_2\}$, $\{0_1, 35_2, 41_2\}$, $\{0_2, 29_0, 34_0\}$, $\{0_0, 8_2, 30_2\}$, $\{0_1, 11_2, 46_2\}$, $\{0_0, 43_1, 49_1\}$, $\{0_1, 9_2, 38_2\}$, $\{0_2, 30_0, 54_0\}$, $\{0_2, 50_0, 59_0\}$, $\{0_0, 20_2, 38_2\}$, $\{0_1, 39_0, 55_0\}$, $\{0_0, 54_1, 54_2\}$, $\{0_0, 23_1, 38_1\}$, $\{19_0, 20_0, 22_0\}$, $\{0_0, 4_2, 28_2\}$, $\{0_0, 7_1, 44_1\}$, $\{0_0, 8_1, 50_1\}$, $\{7_2, 22_2, 32_2\}$, $\{0_0, 16_1, 48_1\}$, $\{0_2, 13_1, 14_1\}$, $\{0_1, 29_0, 58_0\}$, $\{0_0, 14_1, 34_2\}$, $\{0_2, 22_0, 35_0\}$, $\{0_0, 21_1, 41_1\}$, $\{0_1, 8_0, 14_0\}$, $\{0_0, 4_1, 19_2\}$, $\{0_2, 43_0, 47_0\}$, $\{0_0, 33_1, 36_2\}$, $\{0_1, 4_2, 57_2\}$.

$m = 65,\ \zeta = 16$:

$\{0_0, 10_1, 36_2\}$, $\{5_0, 9_0, 51_0\}$, $\{0_0, 29_1, 13_2\}$, $\{0_1, 0_0, 49_0\}$, $\{0_1, 18_0, 42_0\}$, $\{0_1, 32_0, 60_0\}$, $\{29_0, 42_0, 59_0\}$, $\{0_0, 39_1, 50_1\}$, $\{0_0, 63_1, 17_2\}$, $\{0_0, 55_1, 56_2\}$, $\{0_1, 30_0, 40_0\}$, $\{0_0, 56_1, 4_2\}$, $\{0_0, 37_1, 28_2\}$, $\{0_0, 20_1, 58_1\}$, $\{0_0, 6_1, 34_1\}$, $\{0_1, 24_0, 63_0\}$, $\{0_0, 22_1, 27_1\}$, $\{0_0, 17_1, 45_2\}$, $\{0_1, 34_0, 37_0\}$, $\{0_1, 8_0, 44_0\}$, $\{0_0, 32_1, 46_1\}$, $\{0_0, 3_1, 49_1\}$, $\{0_0, 11_1, 44_1\}$, $\{0_0, 8_1, 30_2\}$, $\{1_0, 13_0, 51_0\}$, $\{0_0, 12_1, 48_1\}$, $\{0_1, 27_0, 61_0\}$, $\{18_0, 23_0, 24_0\}$, $\{0_0, 30_1, 9_2\}$, $\{0_0, 59_1, 53_2\}$, $\{6_0, 13_0, 46_0\}$, $\{0_0, 43_1, 45_1\}$.

$m = 67,\ \zeta = 29$:

$\{0_2, 7_1, 37_1\}$, $\{0_2, 11_0, 64_0\}$, $\{0_0, 63_1, 38_2\}$, $\{0_0, 42_1, 61_1\}$, $\{0_2, 26_1, 62_1\}$, $\{0_1, 0_2, 6_2\}$, $\{0_0, 52_1, 41_2\}$, $\{0_2, 2_0, 37_0\}$, $\{0_2, 5_0, 52_0\}$, $\{1_0, 31_0, 35_0\}$, $\{0_2, 1_1, 55_1\}$, $\{0_2, 3_1, 24_1\}$, $\{0_2, 61_0, 66_0\}$, $\{0_2, 43_1, 53_1\}$, $\{0_0, 57_2, 59_2\}$, $\{0_2, 33_0, 36_0\}$, $\{0_0, 31_1, 55_1\}$, $\{0_1, 44_2, 57_2\}$, $\{2_2, 11_2, 54_2\}$, $\{0_0, 45_2, 52_2\}$, $\{0_1, 15_2, 52_2\}$, $\{0_2, 1_0, 25_0\}$, $\{0_0, 18_1, 21_2\}$, $\{0_0, 32_1, 53_2\}$, $\{0_0, 39_1, 43_1\}$,

$\{0_1, 7_2, 49_2\}$, $\{0_1, 16_2, 62_2\}$, $\{0_0, 7_2, 12_2\}$, $\{0_1, 10_2, 32_2\}$, $\{0_0, 36_2, 64_2\}$, $\{0_1, 2_2, 18_2\}$, $\{0_1, 23_2, 26_2\}$, $\{0_1, 9_2, 40_2\}$.

$m = 73$, $\zeta = 8$:

$\{0_1, 50_0, 57_0\}$, $\{0_1, 43_0, 65_0\}$, $\{0_0, 38_2, 39_2\}$, $\{0_2, 5_0, 18_0\}$, $\{28_2, 53_2, 65_2\}$, $\{0_1, 11_0, 28_0\}$, $\{0_1, 23_0, 53_0\}$, $\{0_1, 31_2, 66_2\}$, $\{0_0, 41_1, 57_1\}$, $\{0_2, 8_1, 28_1\}$, $\{0_2, 44_0, 45_0\}$, $\{0_0, 37_1, 54_2\}$, $\{0_1, 34_2, 60_2\}$, $\{0_2, 40_0, 72_0\}$, $\{0_0, 9_1, 50_2\}$, $\{0_2, 0_1, 12_1\}$, $\{5_2, 20_2, 48_2\}$, $\{0_0, 67_1, 24_2\}$, $\{5_0, 69_0, 72_0\}$, $\{0_0, 2_1, 5_1\}$, $\{0_1, 11_2, 67_2\}$, $\{0_0, 14_1, 60_2\}$, $\{0_2, 60_1, 65_1\}$, $\{0_2, 27_0, 47_0\}$, $\{0_2, 21_1, 67_1\}$, $\{0_2, 7_0, 62_0\}$, $\{0_2, 43_0, 59_0\}$, $\{0_1, 27_0, 58_0\}$, $\{0_2, 6_0, 46_0\}$, $\{0_2, 2_1, 41_1\}$, $\{0_0, 42_1, 69_2\}$, $\{0_2, 30_0, 66_0\}$, $\{0_0, 15_2, 57_2\}$, $\{0_0, 25_1, 51_1\}$, $\{0_2, 8_0, 56_0\}$, $\{0_0, 39_1, 35_2\}$.

$m = 79$, $\zeta = 23$:

$\{0_2, 5_1, 67_1\}$, $\{0_2, 18_0, 34_0\}$, $\{0_1, 7_2, 72_2\}$, $\{0_2, 10_0, 68_0\}$, $\{5_0, 25_0, 49_0\}$, $\{0_0, 37_1, 68_2\}$, $\{0_0, 39_1, 43_1\}$, $\{0_0, 8_2, 72_2\}$, $\{0_2, 2_1, 44_1\}$, $\{0_2, 9_1, 33_1\}$, $\{4_2, 20_2, 55_2\}$, $\{0_1, 39_0, 49_0\}$, $\{0_0, 17_2, 26_2\}$, $\{0_0, 54_1, 64_1\}$, $\{0_0, 6_1, 42_1\}$, $\{0_2, 4_0, 72_0\}$, $\{0_0, 50_1, 16_2\}$, $\{0_0, 45_1, 20_2\}$, $\{0_0, 11_1, 66_2\}$, $\{0_0, 66_1, 6_2\}$, $\{0_0, 15_1, 49_1\}$, $\{0_1, 5_2, 60_2\}$, $\{0_0, 36_1, 39_2\}$, $\{0_1, 2_0, 47_0\}$, $\{17_1, 64_1, 77_1\}$, $\{0_0, 44_1, 14_2\}$, $\{0_2, 0_1, 77_1\}$, $\{0_2, 23_0, 51_0\}$, $\{0_2, 54_0, 61_0\}$, $\{0_1, 23_0, 62_0\}$, $\{0_2, 41_0, 55_0\}$, $\{0_0, 68_1, 55_2\}$, $\{0_2, 31_1, 49_1\}$, $\{29_1, 62_1, 67_1\}$, $\{0_0, 35_1, 65_1\}$, $\{0_0, 12_1, 23_2\}$, $\{3_1, 25_1, 51_1\}$, $\{0_2, 6_0, 38_0\}$, $\{0_0, 25_1, 34_2\}$. □

We now summarize the results from the direct constructions given in this subsection.

Proposition 2.6. *There exist* 3OSTS(v) *with* $v \equiv 1 \pmod 6$ *for*

$v \in \{187, 205, 217, 247, 253, 259, 265, 295, 301, 319\}$.

Proposition 2.7. *There exist* 3OSTS(v) *with* $v \equiv 3 \pmod 6$ *for*

$v \in \{87, 93, 99, 111, 117, 123, 129, 135, 141, 147, 153, 159, 171, 183, 195, 201, 219, 237\}$.

2.2. 3OGDD

For a wide class of designs, the existence of small frames is a crucial ally for recursive constructions. As in earlier OSTS work [2], it is desirable to find sets of pairwise orthogonal 3-GDDs: here we need three sets, while in [2] only two were required. Not surprisingly, the available computational methods become more diverse with the 3OSTS problem. The first approach we give involves an initial hill-climb to a random cyclic 3-GDD, followed by a backtrack search of all possible orthogonal mates (also required to be cyclic). Then all pairs of mates are checked to see if any happens to form an orthogonal pair. Because of the (possibly several) exhaustive searches, this method is currently only feasible for less than about ten base blocks.

Lemma 2.8. *There exist* 3OGDD *of types* 2^{13} *and* 6^9.

Proof. For each type g^u, we use $\mathbf{Z}_{gu}$ as a pointset with group partition defined by cosets of $\langle u \rangle$. The $g(u-1)/6$ base blocks for each of the three systems are presented. Note the orthogonality certificates (relative to the point 0) must consist of distinct elements not in $\langle u \rangle$, the subgroup of order g.

2^{13}: I: {0,7,1}, {0,4,16}, {0,17,2}, {0,21,3},
II: {0,4,14}, {0,5,3}, {0,19,1}, {0,6,17},
III: {0,16,15}, {0,5,14}, {0,20,2}, {0,7,3}.

CERTIFICATES: I/II (0) 18,20,15,2,7,8,4,21,12,24,6,10
II/III (0) 3,1,9,23,2,11,25,18,21,7,6,17,
III/I (0) 17,5,8,25,24,15,16,20,22,19,23,14.

6^9: I: {0,40,39}, {0,50,21}, {0,30,37}, {0,23,3}, {0,2,46}, {0,41,19}, {0,5,11}, {0,12,28},
II: {0,21,2}, {0,39,38}, {0,13,47}, {0,40,29}, {0,4,30}, {0,22,12}, {0,31,48}, {0,49,3},
III: {0,38,37}, {0,21,46}, {0,4,19}, {0,13,10}, {0,40,28}, {0,22,20}, {0,23,47}, {0,5,48}. □

A few more 3OGDD were constructed by hill-climbing to a 3-cyclic GDD orthogonal to its images under the order three orbit map α, as was done in Lemma 2.5, except here the automorphism ζ is not needed.

Lemma 2.9. *There exist* 3OGDD *of types* 2^9, 2^{12}, 3^{11}, 6^8 *and* 6^{12}.

Proof. For type g^u, we use the group $\mathbf{Z}_{gu/3} \times \mathbf{Z}_3$. Action on the second coordinate generates the three orthogonal systems. The group partition $\mathscr{G}$ is described, followed by $g(u-1)/2$ base triples. An orthogonality certificate again accompanies the smallest example with the same convention as in Lemma 2.5.

2^9: $\mathscr{G} = \{\{x_i, (3+x)_i\}\colon x = 0,1,2, i = 0,1,2\}$,
$\{0_1, 5_2, 3_2\}$, $\{0_0, 5_1, 1_1\}$, $\{0_0, 4_2, 3_2\}$, $\{0_0, 4_1, 2_0\}$, $\{0_0, 2_2, 0_1\}$, $\{0_0, 1_2, 3_1\}$, $\{0_1, 0_2, 5_1\}$, $\{0_0, 0_2, 1_0\}$.

CERTIFICATE: (0_0) $3_2, 4_0, 1_0, 2_2, 4_1, 5_1, 1_2, 4_2$,
(0_1) $4_1, 1_1, 1_2, 5_2, 5_1, 1_0, 4_2, 3_0$,
(0_2) $2_0, 4_2, 4_0, 1_1, 5_1, 0_0, 5_0, 3_1$.

2^{12}: $\mathscr{G} = \{\{x_i, (4+x)_i\} : x = 0, 1, 2, 3, i = 0, 1, 2\}$,
$\{0_0, 3_2, 2_2\}$, $\{0_0, 1_2, 1_1\}$, $\{0_1, 7_2, 5_2\}$, $\{0_0, 4_2, 6_1\}$, $\{0_0, 7_2, 3_1\}$, $\{0_1, 2_2, 7_1\}$, $\{0_0, 6_2, 5_1\}$, $\{0_0, 6_0, 5_0\}$, $\{0_0, 2_1, 0_1\}$, $\{0_0, 5_2, 0_2\}$, $\{0_0, 7_1, 4_1\}$.

3^{11}: $\mathscr{G} = \{\{x\} \times \mathbf{Z}_3 : x = 0, 1, \ldots, 10\}$,
$\{0_0, 9_1, 8_1\}$, $\{0_0, 9_2, 3_2\}$, $\{0_1, 6_1, 3_2\}$, $\{0_0, 10_1, 5_2\}$, $\{0_0, 5_1, 1_0\}$, $\{0_0, 2_2, 1_1\}$, $\{0_0, 8_2, 6_2\}$, $\{0_1, 9_2, 2_1\}$, $\{0_0, 2_1, 1_2\}$, $\{0_1, 5_2, 3_1\}$, $\{0_0, 6_0, 2_0\}$, $\{0_2, 4_2, 3_2\}$, $\{0_0, 7_1, 3_1\}$, $\{0_0, 10_2, 6_1\}$, $\{0_0, 7_2, 3_0\}$.

6^8: $\mathscr{G} = \{\{x, x+8\} \times \mathbf{Z}_3 : x = 0, 1, \ldots, 7\}$,
$\{0_1, 12_1, 1_1\}$, $\{0_0, 14_1, 9_0\}$, $\{0_0, 4_0, 11_1\}$, $\{0_0, 11_2, 9_2\}$, $\{0_0, 5_0, 2_1\}$, $\{0_0, 15_1, 9_1\}$, $\{0_0, 3_2, 2_0\}$, $\{0_1, 13_2, 9_1\}$, $\{0_0, 3_0, 10_2\}$, $\{0_0, 13_2, 3_1\}$, $\{0_1, 7_2, 14_2\}$, $\{0_1, 5_2, 11_2\}$, $\{0_0, 15_2, 12_2\}$, $\{0_2, 12_2, 11_2\}$, $\{0_0, 4_1, 10_0\}$, $\{0_1, 2_1, 1_2\}$, $\{0_0, 6_2, 1_0\}$, $\{0_0, 4_2, 1_1\}$, $\{0_1, 3_1, 9_2\}$, $\{0_0, 14_2, 12_1\}$, $\{0_0, 6_1, 2_2\}$.

6^{12}: $\mathscr{G} = \{\{x, x+12\} \times \mathbf{Z}_3 : x = 0, 1, \ldots, 11\}$,
$\{0_1, 3_1, 14_2\}$, $\{0_0, 5_1, 4_2\}$, $\{0_1, 4_1, 2_2\}$, $\{0_0, 22_1, 4_1\}$, $\{0_0, 23_1, 6_1\}$, $\{0_0, 9_1, 18_2\}$, $\{0_1, 10_2, 2_1\}$, $\{0_0, 15_0, 2_0\}$, $\{0_0, 18_1, 13_1\}$, $\{0_0, 8_1, 3_2\}$, $\{0_0, 14_2, 1_2\}$, $\{0_0, 10_2, 5_2\}$, $\{0_0, 11_1, 1_1\}$, $\{0_0, 19_2, 2_1\}$, $\{0_0, 11_2, 8_2\}$, $\{0_2, 16_2, 2_2\}$, $\{0_0, 22_2, 19_1\}$, $\{0_0, 9_2, 17_1\}$, $\{0_0, 6_2, 13_2\}$, $\{0_0, 7_0, 3_0\}$, $\{0_0, 20_1, 2_2\}$, $\{0_0, 17_2, 16_2\}$, $\{0_0, 21_1, 14_0\}$, $\{0_0, 21_2, 3_1\}$, $\{0_0, 15_1, 1_0\}$, $\{0_1, 21_2, 1_2\}$, $\{0_0, 6_0, 16_1\}$, $\{0_1, 20_2, 5_2\}$, $\{0_1, 7_2, 13_2\}$, $\{0_1, 4_2, 13_1\}$, $\{0_0, 19_0, 15_2\}$, $\{0_1, 16_1, 15_1\}$, $\{0_0, 8_0, 7_2\}$. $\square$

Next, we use a variant of the previous method in which odd pure and/or mixed differences are artificially pre-covered in the hill-climb. These differences are stored as occurring in blocks with "infinite" points in a shorter orbit. Orthogonality checking is somewhat more delicate in this case. Due to this and a relatively large number of base triples, several days of CPU time were expended (on a parallel machine) on the next design.

Lemma 2.10. *There exists a* 3OGDD *of type* $6^{10}2^1$.

Proof. The points and groups are the same as in the examples of type 6^u in Lemma 2.9, except with an additional group of 2 extra points. The generating automorphism acts in orbits of length two on these points: ∞ and ∞'. This means for example that the blocks generated by the base block $\{\infty, 0_1, 3_2\}$ are as follows: $\{\infty', 1_1, 4_2\}$, $\{\infty, 2_1, 5_2\}$, $\{\infty', 3_1, 6_2\}, \ldots \{\infty', 19_1, 2_2\}$. Note the orthogonality certificate must contain pairs occurring with a representative from these short orbits as well.

$6^{10}2^1$: $\{0_0, 18_2, 5_1\}$, $\{0_0, 5_2, 12_2\}$, $\{0_1, 5_2, 4_2\}$, $\{0_0, 6_2, 1_2\}$, $\{0_2, 14_2, 12_2\}$, $\{0_0, 4_1, 2_2\}$, $\{0_1, 2_1, 11_2\}$, $\{0_0, 4_0, 1_0\}$, $\{0_0, 19_1, 16_1\}$, $\{0_0, 19_2, 8_2\}$, $\{0_0, 8_0, 17_2\}$, $\{0_0, 6_1, 13_2\}$, $\{0_0, 3_1, 12_1\}$, $\{0_0, 8_1, 7_1\}$, $\{0_0, 9_1, 17_1\}$, $\{0_1, 5_1, 1_2\}$, $\{0_0, 5_0, 14_0\}$,

$\{0_1, 6_1, 14_2\}$, $\{0_0, 15_1, 2_0\}$, $\{0_0, 7_2, 4_2\}$, $\{0_0, 14_2, 2_1\}$, $\{0_1, 16_1, 15_2\}$, $\{0_0, 16_2, 13_0\}$, $\{0_1, 6_2, 2_2\}$, $\{0_0, 14_1, 11_2\}$, $\{0_0, 18_1, 11_1\}$, $\{\infty, 0_0, 1_1\}$, $\{\infty, 0_1, 3_2\}$, $\{\infty, 0_2, 5_0\}$.

CERTIFICATE: (0_0) 3_1, 14_1, 2_0,15_2, 17_1,12_2, 17_2,15_1, 16_1,19_2, ∞', 14_2, 18_1, 11_1, 6_0, 6_1, 7_0, 5_1, 19_0, 5_0, 8_1, 5_2, 2_1, 8_0, 3_2, 14_0, 16_0, 18_0,
(0_1) 19_0, 17_1, 14_1, 19_2, 16_2, 4_0, 9_0, 1_1, 8_0, 19_1, 5_2, 6_0, 15_1, 6_2, 4_2, ∞', 13_2, 5_0, 13_0, 14_0, 12_2, 6_1, 1_0, 15_2, 5_1, 11_2, 2_1, 4_1,
(0_2) ∞, 1_2, 9_2, 17_0, 14_0, 19_2, 11_2, 3_0, 9_1, 16_0, 15_1, 4_2, 7_1, 16_2, 5_0, 15_2, 13_1, 17_1, 12_2, 18_1, 13_2, 2_1, 12_1, 8_0, 11_0, 4_0, 2_2, 18_2,
(∞) $9_0, 12_2, 4_0 + (0, 2, 4, \ldots, 18)$. □

A somewhat more subtle use of infinite points is required in the next construction. With the same 3-cyclic structure as in the last two lemmas, we artificially cover an odd pure difference with an orbit (∞, ∞'), and two mixed differences in the other two orbits with a fixed ∞ and ∞'. Orthogonality is an even tighter constraint here, so the following search took several hours on a parallel computer.

Lemma 2.11. *There exists* 3OGDD *of type* $6^8 2^1$.

Proof. The base blocks and orthogonality certificate are presented below. The block at the end of the list, given as $\{\infty | \infty', 0_0, 1_0\}$, is shorthand notation for the following set of blocks: $\{\{\infty, 0_0, 1_0\}, \{\infty', 1_0, 2_0\}, \{\infty, 2_0, 3_0\}, \ldots, \{\infty, 14_0, 15_0\}, \{\infty', 15_0, 0_0\}\}$. The two different actions on ∞ and ∞' contribute four (rather than two) elements in the orthogonality certificates corresponding to 0_0, and 0_2, given at the end of each list.

$6^8 2^1$: $\{0_0, 4_1, 3_0\}$, $\{0_0, 4_0, 11_2\}$, $\{0_0, 11_0, 9_1\}$, $\{0_0, 11_1, 1_2\}$, $\{0_0, 15_1, 13_1\}$, $\{0_1, 13_2, 4_2\}$, $\{0_0, 15_2, 9_2\}$, $\{0_1, 11_2, 10_1\}$, $\{0_0, 14_2, 3_2\}$, $\{0_0, 5_2, 2_2\}$, $\{0_1, 15_2, 3_2\}$, $\{0_0, 12_1, 10_0\}$, $\{0_1, 10_2, 9_2\}$, $\{0_0, 14_0, 5_1\}$, $\{0_1, 12_1, 1_1\}$, $\{0_0, 6_2, 10_1\}$, $\{0_0, 6_1, 3_1\}$, $\{0_1, 7_2, 9_1\}$, $\{0_0, 12_2, 10_2\}$, $\{0_0, 13_2, 9_0\}$, $\{\infty, 0_1, 5_2\}$, $\{\infty', 0_1, 2_2\}$, $\{\infty | \infty', 0_0, 1_0\}$.

CERTIFICATE: (0_0) 13_0, 11_0, 5_0, ∞, 14_0, 10_1, 14_2, 4_2, 10_0, 2_2, 15_1, 15_0, 9_1, 4_0, 11_1, 4_1, 7_2, 13_1, 3_1, 3_0, 12_2, 13_2, 10_2, 15_2,
(0_1) 10_1, 4_1, 9_0, 12_0, 13_2, 11_0, 7_1, 7_0, 4_2, 5_1, 5_2, 15_1, 10_2, ∞, 6_0, 5_0, 6_1, 15_0, 14_1, 9_1, 10_0, 4_0,
(0_2) 6_2, 12_0, 6_1, 11_2, 3_2, 12_2, 1_2, 9_0, ∞', 4_2, 13_2, 7_2, 10_2, 1_1, 13_0, 14_2, 6_0, 3_1, 15_0, 15_2, 10_1, 13_1, 12_1, 15_1,
(∞) $7_2 + (0, 2, 4, \ldots, 14)$, $14_1 + (0, 1, 2, \ldots, 15)$,
(∞') $6_2 + (0, 2, 4, \ldots, 14)$, $6_1 + (0, 1, 2, \ldots, 15)$. □

One important ingredient, a 3OGDD of type 6^{11}, cannot exist with either a cyclic or 3-cyclic automorphism group (due to an odd number of odd differences). However, this design can be found quickly with a 6-cyclic automorphism together with a multiplier to reduce the number of base triples. As with the past several methods, an order 3 map on the orbits generates the three mutually orthogonal systems. Extremely rapid success was observed when applying this method to 3OGDD of type 6^q for larger $q \equiv 3 \pmod 4$. 3OGDD for many of these orders were found, however, since we will not need them in our subsequent recursive constructions they are not presented here.

Lemma 2.12. *There exists* 3OGDD *of type* 6^{11}.

Proof. The group $\mathbf{Z}_{11} \times \mathbf{Z}_6$ is used, with the second coordinate again represented by subscripts. The multiplier μ of order 5 is applied to the first coordinates, and the 60 resulting base blocks are developed additively over $\mathbf{Z}_{11}$. The groups of the GDD are $\{x_i\colon\ i = 0, \ldots, 5\}$ for a given x, and an order three map $0 \mapsto 2 \mapsto 4 \mapsto 0, 1 \mapsto 3 \mapsto 5 \mapsto 1$ on the subscripts generates the three pairwise orthogonal systems. For the orthogonality certificate, we present for each of the six additive orbits only one representative under multiplication by μ. No two elements (with identical subscripts) in the same coset of $\langle \mu \rangle$ may appear in a list.

6^{11}, $\mu = 4$: $\{0_0, 1_1, 6_0\}$, $\{0_1, 1_2, 8_4\}$, $\{0_3, 1_2, 5_5\}$, $\{0_1, 1_3, 8_2\}$, $\{0_2, 1_4, 8_5\}$, $\{0_3, 1_1, 10_4\}$, $\{0_1, 1_5, 2_1\}$, $\{0_3, 1_4, 2_5\}$, $\{0_5, 1_5, 9_0\}$, $\{0_0, 1_4, 7_4\}$, $\{0_3, 1_3, 2_0\}$, $\{0_2, 1_2, 2_0\}$.

CERTIFICATE: (0_0) $8_4, 5_5, 5_2, 2_2, 5_1, 10_3$ (0_1) $3_2, 5_4, 7_2, 1_5, 7_4, 5_0$ (0_2) $3_5, 2_4, 8_3, 2_1, 10_0, 5_0$ (0_3) $1_5, 3_1, 8_0, 7_3, 6_5, 8_4$ (0_4) $10_1, 5_5, 1_3, 1_0, 9_1, 8_3$ (0_5) $10_2, 3_4, 9_2, 2_0, 5_3, 2_1$. □

We conclude our presentation of ingredients with two more applications of infinite points. In each case, the non-abelian action is used as in Lemma 2.5, but on the first two coordinates of the pointset $\mathbf{Z}_m \times \mathbf{Z}_3 \times \{0, 1\}$. We represent a point (a, b, c) simply as abc for convenience. As before, certain carefully chosen differences are pre-covered with fixed infinite points ∞_i, and action on the $\mathbf{Z}_3$ coordinate generates the three orthogonal systems. These hill-climbs ran relatively fast due to the abundant algebraic structure.

Lemma 2.13. *There exist* 3OGDD *of types* $6^7 2^1$, 6^{13} *and* 6^{14}.

Proof. The (noninfinite) groups are simply $\{x\} \times \mathbf{Z}_3 \times \{0, 1\}$, for $x \in \mathbf{Z}_m$. Base blocks and one orthogonality certificate are presented below. Note that in the 3OGDD of type $6^7 2^1$ there is one base block which generates a short orbit, this block is given at the end of the list. The automorphisms allow for checking merely two lists in the certificate.

$6^7 2^1$, $\zeta = 2$:

$\{000, 421, 200\}$, $\{000, 611, 100\}$, $\{001, 421, 201\}$, $\{001, 510, 220\}$,
$\{001, 621, 101\}$, $\{000, 501, 201\}$, $\{001, 620, 321\}$, $\{000, 610, 411\}$,

$\{011,321,120\}$, $\{010,620,120\}$, $\{010,611,320\}$, $\{\infty_1,100,211\}$, $\{\infty_2,101,210\}$, $\{100,210,420\}$,

CERTIFICATE: (000) 300, 610, 420, 200, 220, 310, 121, 311, ∞_1, 111, 301, 110, 601, 221, 210, 211, 201, 400, 520,
(001) 100, 511, 110, 301, 300, 120, 601, 420, 201, 121, 410, 210, 221, 421, ∞_2, 621, 610, 521, 401.

6^{13}, $\zeta=3$: $\{000,1210,1020\}$, $\{011,821,720\}$, $\{000,611,220\}$, $\{011,1221,921\}$, $\{001,601,320\}$, $\{001,501,420\}$, $\{000,1111,110\}$, $\{001,920,811\}$, $\{001,921,120\}$, $\{000,700,101\}$, $\{000,410,200\}$, $\{000,801,400\}$, $\{000,821,711\}$, $\{001,721,201\}$, $\{001,1011,820\}$, $\{011,1020,320\}$, $\{000,721,510\}$, $\{000,1220,121\}$, $\{010,720,211\}$, $\{001,710,301\}$, $\{010,411,310\}$, $\{000,1010,610\}$, $\{001,821,421\}$, $\{010,1121,611\}$.

6^{14}, $\zeta=3$: $\{000,1021,911\}$, $\{000,1120,301\}$, $\{001,1211,221\}$, $\{000,710,500\}$, $\{011,721,220\}$, $\{011,620,211\}$, $\{011,921,421\}$, $\{010,1020,920\}$, $\{011,411,221\}$, $\{000,1020,321\}$, $\{010,1220,821\}$, $\{001,810,211\}$, $\{010,610,110\}$, $\{010,1121,921\}$, $\{000,901,801\}$, $\{010,811,310\}$, $\{010,1221,221\}$, $\{000,620,521\}$, $\{000,1011,401\}$, $\{001,710,411\}$, $\{000,1110,120\}$, $\{000,1221,510\}$, $\{\infty_1,100,011\}$, $\{\infty_2,100,211\}$, $\{\infty_3,100,711\}$, $\{\infty_4,101,010\}$, $\{\infty_5,101,210\}$, $\{\infty_6,101,710\}$. □

Lemma 2.14. *There exists* 3OGDD *of type* $1^{42}3^1$.

$m=7$, $\zeta=2$: $\{000,220,300\}$, $\{000,211,201\}$, $\{001,621,121\}$, $\{001,520,510\}$, $\{001,020,601\}$, $\{001,411,410\}$, $\{011,420,020\}$, $\{001,611,310\}$, $\{020,421,321\}$, $\{010,320,111\}$, $\{010,120,310\}$, $\{000,611,301\}$, $\{100,210,420\}$, $\{101,211,421\}$, $\{\infty_1,101,010\}$, $\{\infty_2,100,211\}$, $\{\infty_3,100,411\}$.

CERTIFICATE: (000) 201, 520, 320, 100, 321, ∞_2, 610, 411, 300, 101, 121, 011, 621, 410, 210, 001, 020, 110, 200, 400, 500, 420, (001) 200, 520, 600, 510, 110, 601, ∞_3, 401, 420, 201, 620, ∞_1, 221, 021, 211, 410, 621, 411, 301, 501, 101, 511.

We conclude this subsection with a summary of the 3OGDD constructed in the previous lemmas. We note in passing that other 3OGDDs were constructed that are not presented here. These include 3OGDDs of types $3^{13}, 6^{11}2^1, 6^{12}4^1, 6^{13}2^1, 6^{13}4^1, 1^{78}3^1$ and $1^{104}3^1$. These are not needed in our subsequent recursive constructions.

Theorem 2.15. *There exist* 3OGDD *for each of the following types*: $2^9, 2^{12}, 2^{13}$, 3^{11}, 6^n *for* $n\in\{8,9,11,12,13,14\}$, 6^n2^1 *for* $n\in\{7,8,10\}$ *and* $1^{42}3^1$.

3. 3OSTS with $v \equiv 1 \pmod 6$

In this section we will prove that 3OSTS(v) exist for all $v \equiv 1 \pmod 6$ with $v \geqslant 19$. It is convenient to write $v = 6n + 1$ and base our analysis on the values of n. So for the remainder of this section we assume that $v = 6n + 1$.

We begin by constructing 3OSTS for several small values not covered by any previous direct construction or by the subsequent more general recursive constructions.

Proposition 3.1. *There exists* 3OSTS *of order* $v = 6n + 1$ *for* $n = 39, 54, 59$ *and* 119.

Proof. For $n = 39$, begin with a transversal design TD(9,13) and give every point weight 2. Now, use Wilson's fundamental construction (Theorem 1.3) with the ingredient a 3OGDD of type 2^9 (which exists by Lemma 2.9) to construct a 3OGDD of type 26^9. Add a point at infinity and fill in the groups with 3OSTS(27) (Theorem 1.1) to construct the desired 3OSTS(v) for $v = 6 \times 39 + 1$.

For $n = 54$, begin with a transversal design TD(8,7) and give weight 6 to every point in the first 7 groups. Give weight 2 to three of the points in the last group and weight 6 to the remaining four points. Now, use Wilson's fundamental construction with ingredients 3OGDDs of type 6^8 and $6^7 2^1$ (which exists by Lemma 2.15) to construct a 3OGDD of type $42^7 30^1$. Add a point at infinity and fill in the groups with 3OSTS(31) and 3OSTS(43) to construct the desired 3OSTS(v) for $v = 6 \times 54 + 1$.

For $n = 59$, begin with a transversal design TD(8, 8) and delete one point to form an 8-GDD of type 7^9. Now, in all but the last group of this GDD, give the points weight 6. In the last group, give one point weight 6 and each of the remaining six points weight 2. Now inflate using the 3OGDDs of types $6^7 2^1$ and 6^8 (these exist by Theorem 2.15). This produces a 3OGDD of type $42^8 18^1$. Finally, add one infinite point to this 3OGDD and fill in the groups with 3OSTS(43) and 3OSTS(19) to get a 3OSTS of order $355 = 6 \times 59 + 1$.

The case $n = 119$ is similar to the case of $n = 54$. Begin with a transversal design TD(11,11) and use the ingredients 3OGDDs of type 6^{11} and $6^{10} 2^1$ to construct a 3OGDD of type $66^{10} 54^1$. Then fill in the groups with 3OSTS(67) and 3OSTS(55) to complete the construction. □

From Theorem 1.1 and Propositions 2.6 and 3.1 we have 3OSTS($6n + 1$) for the following for values of $n \leqslant 119$. We use the notation $a \dots b$ to denote all integers n with $a \leqslant n \leqslant b$.

Proposition 3.2. *There exists* 3OSTS($6n + 1$) *for* $n \in \{3 \dots 63, 66, 68, 70, 72, 73, 76, 77, 81, 83, 119\}$.

The following recursive construction covers many cases. It utilizes the fact that there exist 3OGDDs of types $6^8, 6^9, 6^7 2^1$ and $6^8 2^1$. All of these are given in the previous section.

Theorem 3.3. *Assume there exists a transversal design* $TD(9,g)$ *and there exist* $3OSTS(6s+1)$ *for all* $3 \leqslant s \leqslant g$*, then there exists* $3OSTS(6n+1)$ *for* $n = 7g + \lfloor g/3 \rfloor + x$ *and for all* $7g + \lfloor g/3 \rfloor + x + 2 \leqslant n \leqslant 9g$*, where* $x = 0, 1$*, or* 2 *with* $x \equiv g \pmod 3$.

Proof. Begin with a transversal design $TD(9,g)$ and give every point in the first 7 groups weight 6. In group 8 give $3a$ points weight 2 and the remaining $g - 3a$ points weight 6. In group 9 give b points weight 6 and the remaining points weight 0. Now, use Wilson's fundamental construction (Theorem 1.3) with ingredient 3OGDDs of types $6^8, 6^9, 6^7 2^1$ and $6^8 2^1$ (which all exist by Theorem 2.15) to construct a 3OGDD of type $(6g)^7(6g-12a)^1 b^1$. Now, add a point at infinity and fill in the groups with $3OSTS(6g+1)$, $3OSTS(6(g-2a)+1)$, and $3OSTS(6b+1)$. This results in a 3OSTS of order $v = 6(7g + (g-2a) + b) + 1$.

Now, we consider the range of orders for the resultant 3OSTS. The minimum such order occurs when $b = 0$ and a is as large as possible, i.e. when $a = \lfloor g/3 \rfloor = \alpha$. At that value of a the construction yields a $3OSTS(6n+1)$ with $n = 7g + \alpha + x$ where x is defined in the statement of this theorem. Let $a = \alpha - 1$ to get a $3OSTS(6n+1)$ with $n = 7g + \alpha + x + 2$. Now let b vary from 3 to g to obtain $3OSTS(6n+1)$ for all $7g + \alpha + x + 3 \leqslant n \leqslant 7g + \alpha + x + g$. Finally, let $a = 0$ and b vary from 3 to g to get $3OSTS(6n+1)$ with for all $8g + 3 \leqslant n \leqslant 9g$, completing the range. Note that since $g \geqslant 8$ in order for the $TD(9,g)$ to exist, then after weighting the points, the size of the 8th group will be at least 18 (when $g = 9$). Hence the required 3OSTS will exist by hypothesis. □

Using Theorem 3.3 we can construct $3OSTS(6n+1)$ for many relatively small values of n.

Proposition 3.4. *There exist* $3OSTS(6n+1)$ *for* $n \in \{62 \ldots 82,\ 84 \ldots 118, 120 \ldots 801\}$.

Proof. In the table below we apply Theorem 3.3 for many values of g noting that in each case there exists a $TD(9,g)$ and there is a $3OSTS(6g+1)$. In the column labeled n range, we give the values from $7g + \lfloor g/3 \rfloor + x + 2$ to $9g$, where $x = 0, 1$, or 2 with $x \equiv g \pmod 3$. Where necessary we also list $n = 7g + \lfloor g/3 \rfloor + x$. It is easy to verify that all necessary ingredients exist for all of these applications of this construction.

g	n range	g	n range
8	62–72	31	230–279
9	68–81	37	274–333
11	82, 84–99	43	318–387
13	98–117	49	362–441
16	118, 120–144	53	392–477
19	142–171	61	450–549
23	172–207	73	538–657
27	200–243	89	656–801

As a result of Propositions 3.2 and 3.4 we have the following result.

Proposition 3.5. *There exists* 3OSTS$(6n+1)$ *for all* $3 \leqslant n \leqslant 801$.

We now complete the spectrum for 3OSTS(v) with $v \equiv 1 \pmod 6$. We will use the 3OGDDs of type $6^{11}, 6^{12}, 6^{13}$ and 6^{14} as the main ingredients in this recursive construction.

Theorem 3.6. *There exists* 3OSTS$(6n+1)$ *if and only if* $n \geqslant 3$.

Proof. Assume there exists a transversal design TD$(14, m)$. Now truncate points in the last three groups and note that each block contains either 11, 12, 13 or 14 points. Give each remaining point weight 6 and replace each block with a 3OGDD of type $6^{11}, 6^{12}, 6^{13}$ and 6^{14}, whichever is appropriate. These 3OGDD exist by Theorem 2.15. This yields a 3OGDD of type $(6m)^{11}(6a)^1(6b)^1(6c)^1$, where $0 \leqslant a, b, c \leqslant m$. We add a point at infinity and can fill in the groups if there exists 3OSTS of orders $6m+1, 6a+1, 6b+1$, and $6c+1$.

Assuming for the moment that there exist 3OSTS$(6s+1)$ for all $3 \leqslant s \leqslant m$, then we can complete this construction for all $3 \leqslant a, b, c \leqslant m$ as well as for $a = 0$, $b = 0$ or $c = 0$. Therefore, by appropriate choices of a, b and c, if there exists a TD$(14, m)$, then there exists a 3OSTS$(6n+1)$ for all $11m + 3 \leqslant n \leqslant 14m$.

To finish the proof, now assume that $n > 801$ and proceed by induction, i.e. we assume that there exists a 3OSTS$(6t+1)$ for all $3 \leqslant t < n$. From the construction above we know that if there exists a TD$(14, m)$ and a 3OSTS$(6m+1)$ and if $11m + 3 \leqslant n \leqslant 14m$, then there is a 3OSTS$(6n+1)$. The inequality gives a bound on m, namely that $n/14 \leqslant m \leqslant (n-3)/11$. Now since $n > 801$, we have that $m \geqslant 58$, and hence that there exists a 3OSTS$(6m+1)$, by induction. Also, again since $n > 801$ the range of possible values for m is at least $(801-3)/11 - 801/14 > 15$. One can look at the table of lower bounds for MOLS given in [1] and easily note that from 58 through 7288, there is at least one value of m with $N(m) \geqslant 13$ in any string of 15 consecutive integers, hence there is a TD$(14, m)$. Once $m \geqslant 7289$ there is a TD$(14, m)$ for all m [1]. Hence from the construction and with the existence of all the ingredients, we have that there exists a 3OSTS$(6n+1)$. This completes the case when $n > 801$. When $3 \leqslant n \leqslant 801$ the existence result follows from Proposition 3.5. The required nonexistence of 3OSTS$(6n+1)$ for $n = 1$ and 2 is given in Theorem 1.2. $\square$

4. 3OSTS with $v \equiv 3 \pmod 6$

This case is similar to the case of $v \equiv 1 \pmod 6$. Unfortunately, we will not be able to find 3OSTS(v) for every value of $v \equiv 3 \pmod 6$, but we will only leave 26 possible exceptional cases, which is still pretty good. It is convenient now for the remainder of this section to write $v = 6n + 3$ and again base our analysis on the values of n. So for the remainder of this section we assume that $v = 6n + 3$. From Theorem 1.1 and Proposition 2.7 we have 3OSTS$(6n+3)$ for the following values of $n \leqslant 39$.

Proposition 4.1. *There exists a* 3OSTS$(6n+3)$ *for* $n \in \{4 \dots 16, 18 \dots 26, 28, 30, 32, 33, 36, 39\}$.

We now construct 3OSTS$(6n+3)$ for several small values of n not covered by any previous direct construction or by the more general recursive construction that follows.

Proposition 4.2. *There exists* 3OSTS$(6n+3)$ *for* $n \in \{52, 53, 54, 55, 56, 60, 62, 82, 118\}$.

Proof. All of these designs result from an application of the Wilson's fundamental construction followed by filling in the holes, Theorems 1.3 and 1.4.

For $n = 52$, start with a TD(13, 13) and delete 12 points from a block to form a $\{12, 13\}$–GDD of type $12^{12}13^1$. In this GDD, give all of the points weight two. Since there exists a 3OGDD of type 2^{12} by Lemma 2.9 and a 3OGDD of type 2^{13} by Lemma 2.8, this construction produces a 3OGDD of type $24^{12}26^1$. Finally, add one infinite point to this 3OGDD and fill in the groups using 3OSTS(25) and 3OSTS(27) to get the desired 3OSTS of order $315 = 6 \times 52 + 3$.

For $n = 53$ and 55, begin with a TD(8,7) and give weight 6 to every point in the first 7 groups. In the last group give either 4 points weight 2 and 3 points weight 6, or give 1 point weight 2 and 6 points weight 6. Now inflate using 3OGDDs of types $6^7 2^1$ and 6^8 (these exist by Theorem 2.15) to obtain 3OGDDs of types $42^7 26^1$ and $42^7 38^1$. Now fill in the groups to obtain 3OSTS of order $321 = 6 \times 53 + 3$ and $333 = 6 \times 55 + 3$.

For $n = 54$, again begin with a TD(8,7) and give weight 6 to every point in the first 7 groups. In the last group give three points weight 2 and four points weight 6. Now inflate using 3OGDDs of types $6^7 2^1$ and 6^8 to obtain a 3OGDD of type $42^7 30^1$. We now fill in the groups using Theorem 1.4(c). Add three points at infinity and in each the first 7 groups plus the infinity elements, place a 3OGDD of type $1^{42}3^1$ with the hole of size three on the infinite points. On the last group plus the infinite points put a 3OSTS(33). The result is a 3OSTS of order $327 = 6 \times 54 + 3$.

For $n = 56$, begin with a TD(13,13) and give weight 2 to every point. Now inflate using a 3OGDD of type 2^{13} (this exists by Theorem 2.15) to obtain a 3OGDD of type 26^{13}. Now add one infinite point and fill in the groups with 3OSTS(27) to obtain a 3OSTS of order $339 = 6 \times 56 + 3$.

For $n = 60$, begin with a transversal design TD(11,11) and give each point weight 3. Then inflate using a 3OGDD of type 3^{11} to obtain a 3OGDD of type 33^{11}. Fill in the groups with a 3OSTS(33) to obtain a 3OSTS of order $363 = 6 \times 60 + 3$.

For $n = 62$, begin with a TD(8, 8) and delete one point to form an 8-GDD of type 7^9. In all but the last group of this GDD, give the points weight 6. In the last group, give one point weight 2 and each of the remaining six points weight 6. Now inflate using the 3OGDDs of types $6^7 2^1$ and 6^8 (these exist by Theorem 2.15). This produces a 3OGDD of type $42^8 38^1$. Finally, add one infinite point to this 3OGDD and fill in the groups with 3OSTS(43) and 3OSTS(39) to get a 3OSTS of order $375 = 6 \times 62 + 3$.

For $n = 82$, begin with a TD(13,19) and give weight 2 to every point and inflate using a 3OGDD of type 2^{13} to obtain a 3OGDD of type 38^{13}. Fill in each group with a 3OSTS(39) to obtain a 3OSTS of order $495 = 6 \times 82 + 3$.

Finally, for $n = 118$ begin with a TD(11,11). Give weight 6 to every point in the first ten groups and also to seven points in the last group. The remaining four points in the last group get weight 2. Now inflate using 3OGDDs of types $6^{10}2^1$ and 6^{11} to obtain a 3OGDD of type $66^{10}50^1$. Add a point at infinity and fill in the groups with 3OSTS(67) and 3OSTS(51) to obtain a 3OSTS or order $711 = 6 \times 118 + 3$. □

The following is the main recursive construction. It is very similar to Theorem 3.3, except the upper and lower bounds will not be determined as specifically at first. This is due to the fact that more small values $v \equiv 3 \pmod 6$ are missing compared to the 1 modulo 6 case. Before stating the construction a definition is needed.

We will need to fill in one of the groups with a 3OSTS(v) with $v \equiv 3 \pmod 6$, however there will be bounds on how small and large this 3OSTS can be (as a function of g). We let n_1 be the smallest value such that there exists a 3OSTS($6 \times n_1 + 3$) with $n_1 \geqslant \lfloor (g-1)/3 \rfloor + x = \gamma$ where $x = 0, 1,$ or 2 with $x \equiv g - 1 \pmod 3$ and with $n_1 \equiv \gamma \pmod 2$. And we let n_2 be the largest value such that there exists a 3OSTS($6 \times n_2 + 3$) with $n_2 \leqslant g - 1$.

Theorem 4.3. *Assume there exists a transversal design* TD$(9, g)$ *and let* n_1 *and* n_2 *be as defined above. Then there exists* 3OSTS$(6n + 3)$ *for* $n = 7g + n_1$ *and for all* $7g + n_1 + 2 \leqslant n \leqslant 8g + n_2$.

Proof. Begin with a transversal design TD(9,g) and give every point in the first 7 groups weight 6. In group 8 give $3a + 1$ points weight 2 and the remaining $g - 3a - 1$ points weight 6. In group 9 give b points weight 6 and the remaining points weight 0. Use Wilson's fundamental construction (Theorem 1.3) with ingredient 3OGDDs of types $6^8, 6^9, 6^72^1$ and 6^82^1 to construct a 3OGDD of type $(6g)^7(6g - 12a - 4)^1b^1$. Now, add a point at infinity and fill in the groups with 3OSTS$(6g + 1)$, 3OSTS$(6(g - 2a - 1) + 3)$ (if it exists), and 3OSTS$(6b + 1)$. This results in a 3OSTS of order $v = 6(7g + (g - 2a - 1) + b) + 3$.

The range of orders in the resultant 3OSTS by this construction can be shown to be $v = 6n + 3$ for $n = 7g + n_1$ and for all $7g + n_1 + 2 \leqslant n \leqslant 8g + n_2$. The proof is analogous to the proof given in Theorem 3.3. □

Using the theorem above, the next proposition gives 3OSTS$(6n + 3)$ for many values of n.

Proposition 4.4. *There exist* 3OSTS$(6n + 3)$ *for* $n = 61$ *and all* $63 \leqslant n \leqslant 1790$.

Proof. In the table below we apply Theorem 4.3 for many values of g noting that in each case there exists a TD$(9, g)$ (and there is a 3OSTS$(6g + 1)$). In the columns labeled n_1 and n_2 we give these values as defined above. Note that there must exist 3OSTS$(6n_1+3)$ and 3OSTS$(6n_2+3)$ and that there are bounds given above for the values of n_1 and n_2.

In the column labeled n range, we give the values from $7g+n_1+2$ to $8g+n_2$. Where necessary we also list $n=7g+n_1$. Thus there exists 3OSTS$(6n+3)$ for all n in this range.

g	n_1	n_2	n range
8	5	7	61, 63–71
9	4	8	69–80
11	4	10	81, 83–98
13	4	12	97–116
16	5	15	117, 119–143
19	6	18	141–170
23	8	20	171–204
27	10	24	201–240
31	10	30	229–278
37	12	36	273–332
43	14	36	317–380
49	16	36	361–428
53	18	36	391–460
59	20	54	435–526
64	21	63	471–575
73	24	72	537–656
89	30	88	655–800
103	36	102	759–926
121	54	120	903–1088
143	54	142	1057–1286
163	56	162	1199–1466
199	66	198	1461–1790

The only values of $n \geqslant 63$ missing from the list above are $n=82$ and 118. These were covered in Theorem 4.2. □

We are now in position to close out the spectrum.

Proposition 4.5. *There exist* 3OSTS$(6n+3)$ *for all* $n \geqslant 1791$.

Proof. To finish the proof, now assume that $n \geqslant 1791$ and proceed by induction assuming that there exists a 3OSTS$(6s+3)$ for all $63 \leqslant s < n$.

From Theorem 4.3 we know that there is a 3OSTS$(6n+3)$, if there exists a TD$(9,g)$, a 3OSTS$(6g+1)$, a 3OSTS$(6n_1+3)$ and a 3OSTS$(6n_2+3)$ and if $7g+n_1 \leqslant n \leqslant 8g+n_2$. From Theorem 3.6 there exists a 3OSTS$(6g+1)$, for all $g \geqslant 3$. Now, remember that n_1 is the smallest value such that there exists a 3OSTS$(6 \times n_1 + 3)$ with $n_1 \geqslant \lfloor (g-1)/3 \rfloor + x = \gamma$ where $x = 0, 1$, or 2 with $x \equiv g - 1 \pmod 3$ and with $n_1 \equiv \gamma \pmod 2$ and n_2 is the largest value such that there exists a 3OSTS$(6 \times n_2 + 3)$ with $n_2 \leqslant g - 1$. So since $n \leqslant 8g + n_2 < 9g$, then $g > n/9$. It follows since $n \geqslant 1791$, that $g \geqslant 200$. Without actually picking g at this point we can still note that whatever g is picked, there exist a 3OSTS$(6n_1 + 3)$ and a 3OSTS$(6n_2 + 3)$ since $g - 1 \geqslant n_2 > n_1 > g/3 \geqslant 67$. Hence, by the induction hypothesis, there exists a 3OSTS$(6n_1 + 3)$ and a 3OSTS$(6n_2 + 3)$. It remains only to find an appropriate value for g.

The bound $n \geqslant 7g + (g-1)/3$ implies that $g \leqslant (3n+1)/22$, while the bound $n \leqslant 9g - 1$ implies that $g > (n+1)/9$. Hence we have a range on the possible values of g, namely that $(n+1)/9 < g < (3n+1)/22$. Now since $n > 1791$, this range for possible values of g is at least $(3 \times 1791 + 1)/22 - 1792/9 \geqslant 52$. One can look at the table of lower bounds for MOLS given in [1] and easily note that from 200 through 2774, there is at least one value of g with $N(g) \geqslant 8$ in any string of 52 consecutive integers, hence there is a TD$(9, g)$. There exists a TD$(9, g)$ for all $g \geqslant 2775$ [1]. Hence from the construction and with the existence of all the ingredients, we have that there exists a 3OSTS$(6n + 3)$. This completes the proof. □

We now present our final theorem with regard to 3OSTS$(6n + 3)$. It follows immediately from Propositions 4.1, 4.2, 4.4 and 4.5 and Theorem 1.2.

Theorem 4.6. *There exists a* 3OSTS$(6n + 3)$ *if and only if* $n \geqslant 3$ *with the possible exception of* $n \in \{3, 17, 27, 29, 31, 34, 35, 37, 38, 40 \ldots 51, 57, 58, 59\}$.

5. Conclusion

In this paper we have used direct hill-climbing constructions to find 3OSTSs and 3OGDDs of many small orders. We then employed recursive constructions to prove that there exist three pairwise orthogonal Steiner triple systems, 3OSTS, of order v for all $v \equiv 1 \pmod 6$, with $v \geqslant 19$ and for all $v \equiv 3 \pmod 6$, with $v \geqslant 27$ except for 24 possible exceptions, the largest of which has order $v = 357$. We have little doubt that 3OSTS exist for all $v \geqslant 27$. We are not willing to conjecture as to whether or not a 3OSTS(21) exists. We believe this to be a very interesting question.

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