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Study of convergence of homotopy perturbation method for systems of partial differential equations

Jafar Biazar [a,*], Hossein Aminikhah [b]

[a] Department of Mathematics, Faculty of Sciences, The University of Guilan, P.O. Box 1914, P.C. 41938, Rasht, Iran

[b] Department of Mathematics, School of Mathematical Sciences, Shahrood University of Technology, P.O. Box 316, Shahrood, Iran

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ABSTRACT

The aim of this paper is convergence study of homotopy perturbation method for systems of nonlinear partial differential equations. The sufficient condition for convergence of the method is addressed. Since mathematical modeling of numerous scientific and engineering experiments lead to Brusselator and Burgers' system of equations, it is worth trying new methods to solve these systems. We construct a new efficient recurrent relation to solve nonlinear Burgers' and Brusselator systems of equations. Comparison of the results obtained by homotopy perturbation method with those of Adomian's decomposition method and dual-reciprocity boundary element method leads to significant consequences. Two standard problems are used to validate the method.

1. Introduction

Reaction–diffusion Brusselator prepares a useful model for studying the cooperative processes in chemical kinetics. Such a trimolecular reaction step arises in the formation of ozone by atomic oxygen via a triple collision. This system governs also in enzymatic reactions, in plasma and laser physics, and in multiple couplings between certain modes [1,2]. Burgers' equation is used to describe various kinds of phenomena such as turbulence and the approximation theory of flow through a shock wave traveling in a viscous fluid [3,4]. Numerical methods which are commonly used such as finite difference, finite element or characteristics method need large size of computational works and usually the effect of round-off error causes loss of accuracy in the results. Analytical methods commonly used for solving these equations are very restricted and can be used in very special cases, so they cannot be used to solve equations of numerous realistic scenarios.

The homotopy perturbation method was introduced by He [5–10] in the year 1998. In this method the solution is considered as the summation of an infinite series which converges rapidly to the exact solutions. This technique has been employed to solve a large variety of linear and nonlinear equations. This scheme is used for solving nonlinear boundary value problems [11], nonlinear fractional partial differential equations [12], and nonlinear Hirota–Satsuma coupled KdV partial differential equations [13]. This method is also adopted for solving the pure strong nonlinear second-order differential equations [14]. Also this author employed the homotopy-perturbation method for solving the complex-valued differential equations with strong cubic nonlinearity [15]. Some other applications of this method are as follows: application of He's homotopy perturbation method is described to solve nonlinear integro-differential equations [16], for traveling wave solutions of nonlinear wave equations [17], nonlinear convective–radioactive cooling equation, nonlinear heat equations (porous media equation) and nonlinear heat equations with cubic nonlinearity [18]. The authors of [19] employed He's homotopy perturbation method to compute an approximation to the solution of the system of nonlinear ordinary differential

* Corresponding author.
E-mail addresses: biazar@guilan.ac.ir (J. Biazar), aminikhah@shahroodut.ac.ir, hossein.aminikhah@gmail.com (H. Aminikhah).

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equations governing the problem of the spread of a nonfatal disease in a population which is assumed to have constant size over the period of the epidemic. In general, this method has been successfully applied to solve many types of linear and nonlinear problems in science and engineering by many authors [20–30].

The rest of this paper is organized as follows:

Section 2 is assigned to a brief introduction and convergence of the homotopy perturbation method. In Section 3, Brusselator and Burgers' equations have been solved by the proposed method. To illustrate and show the efficiency of the method two examples are presented in Section 4. And conclusions will appear in Section 5.

2. Homotopy perturbation method and convergence of the method

The essential idea of this method is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When $p = 0.$, the system of equations is in sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of "deformation", the solution of each of which is "close" to that at the previous stage of "deformation". Eventually at $p = 1$, the system takes the original form of equation and the final stage of "deformation" gives the desired solution.

To illustrate the basic concept of homotopy perturbation method, consider the following nonlinear system of differential equations

$$A(\mathbf{U}) = f(\mathbf{r}), \quad \mathbf{r} \in \Omega, \tag{1}$$

with boundary conditions

$$B\left(\mathbf{U}, \frac{\partial \mathbf{U}}{\partial n}\right) = 0, \quad \mathbf{r} \in \Gamma,$$

where A is a differential operator, B is a boundary operator, $f(\mathbf{r})$ is a known analytic function, and Γ is the boundary of the domain Ω. Generally speaking the operator A can be divided into two parts L and N, where L is a linear, and N is a nonlinear operator. Eq. (1), therefore, can be rewritten as follows:

$$L(\mathbf{U}) + N(\mathbf{U}) - f(\mathbf{r}) = 0.$$

We construct a homotopy $\mathbf{V}(\mathbf{r}, p) : \Omega \times [0, 1] \to \mathbb{R}^n$, which satisfies

$$H(\mathbf{V}, p) = (1 - p)\,[L(\mathbf{V}) - L(\mathbf{U}_0)] + p\,[A(\mathbf{V}) - f(\mathbf{r})] = 0, \quad p \in [0, 1],\ \mathbf{r} \in \Omega,$$

or equivalently,

$$H(\mathbf{V}, p) = L(\mathbf{V}) - L(\mathbf{U}_0) + pL(\mathbf{U}_0) + p[N(\mathbf{V}) - f(\mathbf{r})] = 0, \tag{2}$$

where $\mathbf{U}_0$ is an initial approximation of Eq. (1). In this method, using the homotopy parameter p, we have the following power series presentation for $\mathbf{V}$,

$$\mathbf{V} = \mathbf{V}_0 + p\mathbf{V}_1 + p^2\mathbf{V}_2 + \cdots.$$

The approximate solution can be obtained by setting $p = 1$, i.e.

$$\mathbf{U} = \lim_{p \to 1} \mathbf{V} = \mathbf{U}_0 + \mathbf{U}_1 + \mathbf{U}_2 + \cdots.$$

Convergence

Let us write Eq. (2) in the following form

$$L(\mathbf{V}) = L(\mathbf{U}_0) + p\,[f(\mathbf{r}) - N(\mathbf{V}) - L(\mathbf{U}_0)]. \tag{3}$$

Applying the inverse operator, L^{-1} to both sides of Eq. (3), we obtain

$$\mathbf{V} = \mathbf{U}_0 + p\left[L^{-1}f(\mathbf{r}) - L^{-1}N(\mathbf{V}) - \mathbf{U}_0\right]. \tag{4}$$

Suppose that

$$\mathbf{V} = \sum_{i=0}^{\infty} p^i \mathbf{v}_i, \tag{5}$$

substituting (5) into the right-hand side of Eq. (4), we have Eq. (4) in the following form

$$\mathbf{V} = \mathbf{U}_0 + p\left[L^{-1}f(\mathbf{r}) - \left(L^{-1}N\right)\left[\sum_{i=0}^{\infty} p^i \mathbf{v}_i\right] - \mathbf{U}_0\right].$$

If $p \to 1$, the exact solution may be obtained by using

$$\begin{aligned}\mathbf{U} &= \lim_{p\to 1}\mathbf{V}\\ &= L^{-1}\left(f(\mathbf{r})\right) - \left(L^{-1}N\right)\left[\sum_{i=0}^{\infty}\mathbf{v}_i\right]\\ &= L^{-1}\left(f(\mathbf{r})\right) - \sum_{i=0}^{\infty}\left(L^{-1}N\right)(\mathbf{v}_i).\end{aligned}$$

To study the convergence of the method let us state the following Theorem.

Theorem (*Sufficient Condition of Convergence*). *Suppose that X and Y are Banach spaces and $N : X \to Y$ is a contractive nonlinear mapping, that is*

$$\forall \mathbf{w},\quad \mathbf{w}^* \in X;\ \|N(\mathbf{w}) - N(\mathbf{w}^*)\| \le \gamma\|\mathbf{w} - \mathbf{w}^*\|,\quad 0 < \gamma < 1.$$

Then according to Banach's fixed point theorem N has a unique fixed point u, that is $N(\mathbf{u}) = \mathbf{u}$.

Assume that the sequence generated by homotopy perturbation method can be written as

$$\mathbf{W}_n = N(\mathbf{W}_{n-1}),\qquad \mathbf{W}_{n-1} = \sum_{i=0}^{n-1}\mathbf{w}_i,\quad n = 1, 2, 3, \ldots,$$

and suppose that $\mathbf{W}_0 = \mathbf{w}_0 \in B_r(\mathbf{w})$ where $B_r(\mathbf{w}) = \{\mathbf{w}^* \in X \mid \|\mathbf{w}^* - \mathbf{w}\| < r\}$, then we have

(i) $\mathbf{W}_n \in B_r(\mathbf{w})$,
(ii) $\lim_{n\to\infty}\mathbf{W}_n = \mathbf{w}$.

Proof. (i) By inductive approach, for $n = 1$ we have

$$\|\mathbf{W}_1 - \mathbf{w}\| = \|N(\mathbf{W}_0) - N(\mathbf{w})\| \le \gamma\|\mathbf{w}_0 - \mathbf{w}\|.$$

Assume that $\|\mathbf{W}_{n-1} - \mathbf{w}\| \le \gamma^{n-1}\|\mathbf{w}_0 - \mathbf{w}\|$, as induction hypothesis, then

$$\|\mathbf{W}_n - \mathbf{w}\| = \|N(\mathbf{W}_{n-1}) - N(\mathbf{w})\| \le \gamma\|\mathbf{W}_{n-1} - \mathbf{w}\| \le \gamma^n\|\mathbf{w}_0 - \mathbf{w}\|.$$

Using (i), we have

$$\|\mathbf{W}_n - \mathbf{w}\| \le \gamma^n\|\mathbf{w}_0 - \mathbf{w}\| \le \gamma^n r < r \Rightarrow \mathbf{W}_n \in B_r(\mathbf{w}).$$

(ii) Because of $\|\mathbf{W}_n - \mathbf{w}\| \le \gamma^n\|\mathbf{w}_0 - \mathbf{w}\|$ and $\lim_{n\to\infty}\gamma^n = 0$, $\lim_{n\to\infty}\|\mathbf{W}_n - \mathbf{w}\| = 0$, that is,

$$\lim_{n\to\infty}\mathbf{W}_n = \mathbf{w}.\quad\square$$

3. Method of solution

3.1. Two-dimensional Burgers' equation

Consider the following system of two-dimensional Burgers' equations [4].

$$\begin{aligned}&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\\ &\frac{\partial u}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),\end{aligned}\tag{6}$$

subject to the initial conditions:

$$\begin{aligned}&u(x, y, 0) = f(x, y),\quad (x, y) \in \Omega,\\ &v(x, y, 0) = g(x, y),\quad (x, y) \in \Omega,\end{aligned}\tag{7}$$

and the boundary conditions

$$\begin{aligned}&u(x, y, t) = f_1(x, y, t),\quad x, y \in \Gamma,\ t > 0,\\ &v(x, y, t) = f_2(x, y, t),\quad x, y \in \Gamma,\ t > 0,\end{aligned}\tag{8}$$

where $\Omega = \{(x, y) \mid a \le x \le b, a \le y \le b\}$ and Γ is its boundary, $u(x, y, t)$ and $v(x, y, t)$ are the velocity components to be determined, f, g, f_1 and f_2 are known functions and R is the Reynolds number.

To solve Eq. (6) with initial conditions (7), according to the homotopy perturbation, we construct the following homotopy:

$$\begin{aligned}&(1-p)\left(\frac{\partial U^*}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial U^*}{\partial t}+U^*\frac{\partial U^*}{\partial x}+V^*\frac{\partial U^*}{\partial y}-\frac{1}{R}\left(\frac{\partial^2 U^*}{\partial x^2}+\frac{\partial^2 U^*}{\partial y^2}\right)\right)=0,\\&(1-p)\left(\frac{\partial V^*}{\partial t}-\frac{\partial v_0}{\partial t}\right)+p\left(\frac{\partial V^*}{\partial t}+U^*\frac{\partial V^*}{\partial x}+V^*\frac{\partial V^*}{\partial y}-\frac{1}{R}\left(\frac{\partial^2 V^*}{\partial x^2}+\frac{\partial^2 V^*}{\partial y^2}\right)\right)=0.\end{aligned}\tag{9}$$

Suppose the solution of Eq. (9) has the form

$$\begin{aligned}&U^*=U_0^*+pU_1^*+p^2U_2^*+\cdots,\\&V^*=V_0^*+pV_1^*+p^2V_2^*+\cdots.\end{aligned}\tag{10}$$

Substituting (10) into (9), and comparing coefficients of the terms with the identical powers of p, lead to

$$p^0:\begin{cases}\dfrac{\partial U_0^*}{\partial t}-\dfrac{\partial u_0}{\partial t}=0,\\ \dfrac{\partial V_0^*}{\partial t}-\dfrac{\partial v_0}{\partial t}=0,\end{cases}$$

$$p^1:\begin{cases}\dfrac{\partial U_1^*}{\partial t}+\dfrac{\partial u_0}{\partial t}+U_0^*\dfrac{\partial U_0^*}{\partial x}+V_0^*\dfrac{\partial U_0^*}{\partial y}-\dfrac{1}{R}\left(\dfrac{\partial^2 U_0^*}{\partial x^2}+\dfrac{\partial^2 U_0^*}{\partial y^2}\right)=0, & U_1^*(x,y,0)=0,\\ \dfrac{\partial V_1^*}{\partial t}+\dfrac{\partial v_0}{\partial t}+U_0^*\dfrac{\partial V_0^*}{\partial x}+V_0^*\dfrac{\partial V_0^*}{\partial y}-\dfrac{1}{R}\left(\dfrac{\partial^2 V_0^*}{\partial x^2}+\dfrac{\partial^2 V_0^*}{\partial y^2}\right)=0, & V_1^*(x,y,0)=0,\end{cases}$$

$$\vdots$$

$$p^j:\begin{cases}\dfrac{\partial U_j^*}{\partial t}+\displaystyle\sum_{k=0}^{j-1}\left(U_k^*\dfrac{\partial U_{j-k-1}^*}{\partial x}+V_k^*\dfrac{\partial U_{j-k-1}^*}{\partial y}\right)-\dfrac{1}{R}\left(\dfrac{\partial^2 U_{j-1}^*}{\partial x^2}+\dfrac{\partial^2 U_{j-1}^*}{\partial y^2}\right)=0, & U_j^*(x,y,0)=0,\\ \dfrac{\partial V_j^*}{\partial t}+\displaystyle\sum_{k=0}^{j-1}\left(U_k^*\dfrac{\partial V_{j-k-1}^*}{\partial x}+V_k^*\dfrac{\partial V_{j-k-1}^*}{\partial y}\right)-\dfrac{1}{R}\left(\dfrac{\partial^2 V_{j-1}^*}{\partial x^2}+\dfrac{\partial^2 V_{j-1}^*}{\partial y^2}\right)=0, & V_j^*(x,y,0)=0.\end{cases}$$

For the sake of simplicity we take

$$U_0^*=u_0=f(x,y),\qquad V_0^*=v_0=g(x,y).\tag{11}$$

And we derive the following recurrent equations

$$\begin{aligned}U_j^*&=\frac{1}{R}\int_0^t\left(\frac{\partial^2 U_{j-1}^*}{\partial x^2}+\frac{\partial^2 U_{j-1}^*}{\partial y^2}\right)\mathrm{d}t-\int_0^t\sum_{k=0}^{j-1}\left(U_k^*\frac{\partial U_{j-k-1}^*}{\partial x}+V_k^*\frac{\partial U_{j-k-1}^*}{\partial y}\right)\mathrm{d}t,\quad j=1,2,\ldots\\V_j^*&=\frac{1}{R}\int_0^t\left(\frac{\partial^2 V_{j-1}^*}{\partial x^2}+\frac{\partial^2 V_{j-1}^*}{\partial y^2}\right)\mathrm{d}t-\int_0^t\sum_{k=0}^{j-1}\left(U_k^*\frac{\partial V_{j-k-1}^*}{\partial x}+V_k^*\frac{\partial V_{j-k-1}^*}{\partial y}\right)\mathrm{d}t,\quad j=1,2,\ldots.\end{aligned}\tag{12}$$

The approximate solution of (6) can be obtained by setting $p=1$,

$$\begin{aligned}&u=\lim_{p\to1}U^*=U_0^*+U_1^*+U_2^*+\cdots,\\&v=\lim_{p\to1}V^*=V_0^*+V_1^*+V_2^*+\cdots.\end{aligned}\tag{13}$$

3.2. The reaction–diffusion Brusselator system

Consider the following system of two-dimensional Brusselator system [2].

$$\begin{cases}\dfrac{\partial u}{\partial t}=B+u^2v-(A+1)u+\alpha\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)\\ \dfrac{\partial v}{\partial t}=Au-u^2v+\alpha\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right).\end{cases}\tag{14}$$

For $u(x,y,t)$ and $v(x,y,t)$ in a two-dimensional region Ω bounded by a simple closed curve Γ subject to the initial conditions:

$$(u(x,y,t),v(x,y,t))=(f(x,y),g(x,y))\quad\text{for }(x,y)\in\Omega,\tag{15}$$

and the boundary conditions:

$$(u(x,y,t),v(x,y,t))=(w(x,y,t),z(x,y,t))\quad\text{for }(x,y)\in\Gamma_1\text{ and }t>0,\tag{16}$$

$$\left(\frac{\partial u}{\partial n},\frac{\partial v}{\partial n}\right)=(p(x,y,t),q(x,y,t))\quad\text{for }(x,y)\in\Gamma_2\text{ and }t>0,\tag{17}$$

where A, B and α are suitable given constants, f, g, w, z, p and q are suitably prescribed functions, Γ_1 and Γ_2 are nonintersecting curves such that $\Gamma_1 \cup \Gamma_2 = C$, $\frac{\partial u}{\partial n} = \vec{n}.\nabla v$ and $\vec{n}$ is the unit normal outward vector R at the point (x, y) on Γ.

To solve Eq. (14) with initial condition (15), according to the homotopy perturbation, we construct the following homotopy

$$\begin{cases} (1-p)\left(\dfrac{\partial U^*}{\partial t} - \dfrac{\partial u_0}{\partial t}\right) + p\left(\dfrac{\partial U^*}{\partial t} - B - U^{*2}V^* + (A+1)U^* - \alpha\left(\dfrac{\partial^2 U^*}{\partial x^2} + \dfrac{\partial^2 U^*}{\partial y^2}\right)\right) = 0, \\ (1-p)\left(\dfrac{\partial V^*}{\partial t} - \dfrac{\partial v_0}{\partial t}\right) + p\left(\dfrac{\partial V^*}{\partial t} - AU^* + U^{*2}V^* - \alpha\left(\dfrac{\partial^2 U^*}{\partial x^2} + \dfrac{\partial^2 U^*}{\partial y^2}\right)\right) = 0 \end{cases} \tag{18}$$

or equivalently

$$\begin{cases} \dfrac{\partial U^*}{\partial t} - \dfrac{\partial u_0}{\partial t} + p\left(-B - U^{*2}V^* + (A+1)U^* - \alpha\left(\dfrac{\partial^2 U^*}{\partial x^2} + \dfrac{\partial^2 U^*}{\partial y^2}\right) + \dfrac{\partial u_0}{\partial t}\right) = 0, \\ \dfrac{\partial V^*}{\partial t} - \dfrac{\partial v_0}{\partial t} + p\left(-AU^* + U^{*2}V^* - \alpha\left(\dfrac{\partial^2 U^*}{\partial x^2} + \dfrac{\partial^2 U^*}{\partial y^2}\right) + \dfrac{\partial v_0}{\partial t}\right) = 0. \end{cases} \tag{19}$$

Suppose that the solution of Eq. (19) has the form (10), substituting (10) into (19), and comparing coefficients of the terms with the identical powers of p, lead to

$$p^0: \begin{cases} \dfrac{\partial U_0^*}{\partial t} - \dfrac{\partial u_0}{\partial t} = 0, \\ \dfrac{\partial V_0^*}{\partial t} - \dfrac{\partial v_0}{\partial t} = 0, \end{cases}$$

$$p^1: \begin{cases} \dfrac{\partial U_1^*}{\partial t} - \dfrac{\partial u_0}{\partial t} - B - U_0^{*2}V_0^* + (A+1)U_0^* - \alpha\left(\dfrac{\partial^2 U_0^*}{\partial x^2} + \dfrac{\partial^2 U_0^*}{\partial y^2}\right) = 0, \qquad U_1^*(x,y,0) = 0, \\ \dfrac{\partial V_1^*}{\partial t} - \dfrac{\partial v_0}{\partial t} - AU_0^* + U_0^{*2}V_0^* - \alpha\left(\dfrac{\partial^2 U_0^*}{\partial x^2} + \dfrac{\partial^2 U_0^*}{\partial y^2}\right) = 0, \qquad V_1^*(x,y,0) = 0, \end{cases}$$

$$p^2: \begin{cases} \dfrac{\partial U_2^*}{\partial t} - 2U_0^*U_1^*V_0^* - U_0^2V_1 + (A+1)U_1^* - \alpha\left(\dfrac{\partial^2 U_1^*}{\partial x^2} + \dfrac{\partial^2 U_1^*}{\partial y^2}\right) = 0, \qquad U_2^*(x,y,0) = 0, \\ \dfrac{\partial V_2^*}{\partial t} - AU_1^* + 2U_0^*U_1^*V_0^* + U_0^{*2}V_1^* - \alpha\left(\dfrac{\partial^2 U_1^*}{\partial x^2} + \dfrac{\partial^2 U_1^*}{\partial y^2}\right) = 0, \qquad V_2^*(x,y,0) = 0, \end{cases}$$

$$\vdots$$

$$p^j: \begin{cases} \dfrac{\partial U_j^*}{\partial t} - \displaystyle\sum_{i=0}^{j}\sum_{k=0}^{j-1} U_i^*U_k^*V_{j-k-i-1}^* + (A+1)U_{j-1}^* - \alpha\left(\dfrac{\partial^2 U_{j-1}^*}{\partial x^2} + \dfrac{\partial^2 U_{j-1}^*}{\partial y^2}\right) = 0, \qquad U_j^*(x,y,0) = 0, \\ \dfrac{\partial V_j^*}{\partial t} - AU_{j-1}^* + \displaystyle\sum_{i=0}^{j}\sum_{k=0}^{j-1} U_i^*U_k^*V_{j-k-i-1}^* - \alpha\left(\dfrac{\partial^2 U_{j-1}^*}{\partial x^2} + \dfrac{\partial^2 U_{j-1}^*}{\partial y^2}\right) = 0, \qquad V_j^*(x,y,0) = 0. \end{cases}$$

For simplicity we take

$$u_0 = U_0^* = f(x,y) + Bt, \qquad v_0 = V_0^* = g(x,y). \tag{20}$$

And we have the following recurrent equations

$$\begin{aligned} U_j^* &= \int_0^t \left[-(A+1)U_{j-1}^* + \alpha\left(\frac{\partial^2 U_{j-1}^*}{\partial x^2} + \frac{\partial^2 U_{j-1}^*}{\partial y^2}\right) + \sum_{i=0}^{j}\sum_{k=0}^{j-1} U_i^*U_k^*V_{j-k-i-1}^*\right] \mathrm{d}t, \quad j = 1, 2, 3, \ldots, \\ V_j^* &= \int_0^t \left[-AU_{j-1}^* + \alpha\left(\frac{\partial^2 U_{j-1}^*}{\partial x^2} + \frac{\partial^2 U_{j-1}^*}{\partial y^2}\right) - \sum_{i=0}^{j}\sum_{k=0}^{j-1} U_i^*U_k^*V_{j-k-i-1}^*\right] \mathrm{d}t, \quad j = 1, 2, 3, \ldots. \end{aligned} \tag{21}$$

The approximate solution of (14) can be obtained by setting $p = 1$,

$$\begin{aligned} u &= \lim_{p\to 1} U^* = U_0^* + U_1^* + U_2^* + \cdots, \\ v &= \lim_{p\to 1} V^* = V_0^* + V_1^* + V_2^* + \cdots. \end{aligned} \tag{22}$$

4. Test problems

To illustrate the method and to show the ability of the method two examples are presented here.

Example 1. Consider the two-dimensional Burgers' equations (6), ($R = 1$) with the following initial conditions:

$$u(x, y, 0) = x + y, \qquad v(x, y, 0) = x - y \quad \text{and } (x, y, t) \in \mathbb{R}^2 \times \left[0, \frac{1}{\sqrt{2}}\right).$$

By using He's homotopy perturbation method, we have

$$\frac{\partial U^*}{\partial t} - \frac{\partial u_0}{\partial t} = p\left(\left(\frac{\partial^2 U^*}{\partial x^2} + \frac{\partial^2 U^*}{\partial y^2}\right) - U^* \frac{\partial U^*}{\partial x} - V^* \frac{\partial U^*}{\partial y} - \frac{\partial u_0}{\partial t}\right),$$
$$\frac{\partial V^*}{\partial t} - \frac{\partial v_0}{\partial t} = p\left(\left(\frac{\partial^2 V^*}{\partial x^2} + \frac{\partial^2 V^*}{\partial y^2}\right) - U^* \frac{\partial V^*}{\partial x} - V^* \frac{\partial V^*}{\partial y} - \frac{\partial v_0}{\partial t}\right).$$

Starting with

$$u_0 = U_0^* = x + y,$$
$$v_0 = V_0^* = x - y,$$

from (12), we obtain the following recurrent relations:

$$U_j^* = \int_0^t \left(\frac{\partial^2 U_{j-1}^*}{\partial x^2} + \frac{\partial^2 U_{j-1}^*}{\partial y^2} - \sum_{k=0}^{j-1}\left(U_k^* \frac{\partial U_{j-k-1}^*}{\partial x} + V_k^* \frac{\partial U_{j-k-1}^*}{\partial y}\right)\right) \mathrm{d}t, \quad j = 1, 2, 3, \ldots$$
$$V_j^* = \int_0^t \left(\frac{\partial^2 V_{j-1}^*}{\partial x^2} + \frac{\partial^2 V_{j-1}^*}{\partial y^2} - \sum_{k=0}^{j-1}\left(U_k^* \frac{\partial V_{j-k-1}^*}{\partial x} + V_k^* \frac{\partial V_{j-k-1}^*}{\partial y}\right)\right) \mathrm{d}t, \quad j = 1, 2, 3, \ldots.$$

Then we derive the following results

$$\begin{array}{ll}
U_1^*(x, t) = -2xt, & V_1^*(x, t) = -2yt, \\
U_2^*(x, t) = 2xt^2 + 2yt^2, & V_2^*(x, t) = 2xt^2 - 2yt^2, \\
U_3^*(x, t) = -4xt^3, & V_3^*(x, t) = -2yt^3, \\
U_4^*(x, t) = 2xt^4 + 2yt^4, & V_4^*(x, t) = 2xt^4 - 2yt^4, \\
U_5^*(x, t) = -8xt^5, & V_5^*(x, t) = -8xt^5, \\
\vdots & \vdots
\end{array}$$

Suppose that

$$\mathbf{W}_n = \mathcal{N}(\mathbf{W}_{n-1}), \qquad \mathbf{W}_{n-1} = (W_{n-1}^*, W_{n-1}^{**}),$$
$$W_0^* = U_0^* + U_1^*, \qquad W_n^* = \sum_{j=0}^{2n+1} \int_0^t \left(\frac{\partial^2 U_{j-1}^*}{\partial x^2} + \frac{\partial^2 U_{j-1}^*}{\partial y^2} - \sum_{k=0}^{j-1}\left(U_k^* \frac{\partial U_{j-k-1}^*}{\partial x} + V_k^* \frac{\partial U_{j-k-1}^*}{\partial y}\right)\right) \mathrm{d}t, \quad n = 1, 2, \ldots,$$
$$W_0^{**} = V_0^* + V_1^*, \qquad W_n^{**} = \sum_{j=0}^{2n+1} \int_0^t \left(\frac{\partial^2 V_{j-1}^*}{\partial x^2} + \frac{\partial^2 V_{j-1}^*}{\partial y^2} - \sum_{k=0}^{j-1}\left(U_k^* \frac{\partial V_{j-k-1}^*}{\partial x} + V_k^* \frac{\partial V_{j-k-1}^*}{\partial y}\right)\right) \mathrm{d}t, \quad n = 1, 2, \ldots$$

and $t \le \sqrt{\frac{\gamma}{2}}$, $0 < \gamma < 1$.

According to the Theorem for the nonlinear mapping $\mathcal{N}$, a sufficient condition for convergence of the homotopy perturbation method is the strict contraction of $\mathcal{N}$. Therefore we have

$$\|W_0^* - u\| = \left\|\frac{-2t^2}{1 - 2t^2}(x + y - 2xt)\right\|,$$
$$\|W_1^* - u\| = \left\|\frac{-4t^2}{1 - 2t^2}(x + y - 2xt)\right\| \le 2\left(\sqrt{\frac{\gamma}{2}}\right)^2 \left\|\frac{2t^2}{1 - 2t^2}(x + y - 2xt)\right\| = \gamma\|W_0^* - u\|,$$
$$\|W_2^* - u\| = \left\|\frac{-8t^2}{1 - 2t^2}(x + y - 2xt)\right\| \le 4\left(\sqrt{\frac{\gamma}{2}}\right)^4 \left\|\frac{2t^2}{1 - 2t^2}(x + y - 2xt)\right\| = \gamma^2\|W_0^* - u\|,$$
$$\vdots$$
$$\|W_n^* - u\| = \left\|\frac{-2^{n+1}t^{2n+2}}{1 - 2t^2}(x + y - 2xt)\right\| \le 2^n\left(\sqrt{\frac{\gamma}{2}}\right)^{2n} \left\|\frac{2t^2}{1 - 2t^2}(x + y - 2xt)\right\| = \gamma^n\|W_0^* - u\|.$$

Therefore, $\lim_{n\to\infty}\|W_n^* - u\| \le \lim_{n\to\infty}\gamma^n\|W_0 - u\| = 0$, that is $u(x,y,t) = \lim_{n\to\infty} W_n^* = \frac{x+y-2xt}{1-2t^2}$, which is an exact solution.

Similarly,

$$\|W_n^{**} - v\| = \left\|\frac{-2^{n+1}t^{2n+2}}{1-2t^2}(x+y-2yt)\right\| \le 2^n\left(\sqrt{\frac{\gamma}{2}}\right)^{2n}\left\|\frac{2t^2}{1-2t^2}(x+y-2yt)\right\| = \gamma^n\|W_0^{**} - v\|.$$

Therefore, $v(x,y,t) = \lim_{n\to\infty} W_n^{**} = \frac{x+y-2yt}{1-2t^2}$, which is an exact solution.

Example 2. Consider the two-dimensional Brusselator system [2]:

$$\begin{aligned} \frac{\partial u}{\partial t} &= u^2 v - 2u + \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\ \frac{\partial v}{\partial t} &= u - u^2 v + \frac{1}{4}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \end{aligned} \tag{23}$$

subject to the initial conditions

$$u(x,y,0) = \exp[-x-y],\ v(x,y,0) = \exp[x+y] \quad \text{and } (x,y,t) \in \mathbb{R}^2 \times [0,2].$$

The exact solution is $u(x,y,t) = \exp\left[-x-y-\frac{t}{2}\right]$, $v(x,y,t) = \exp\left[x+y+\frac{t}{2}\right]$.

By using the homotopy perturbation method, we have

$$\begin{aligned} &(1-p)\left(\frac{\partial U^*}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial U^*}{\partial t} - U^{*2}V^* + 2\,U^* - \frac{1}{4}\left(\frac{\partial^2 U^*}{\partial x^2} + \frac{\partial^2 U^*}{\partial y^2}\right)\right) = 0, \\ &(1-p)\left(\frac{\partial V^*}{\partial t} - \frac{\partial v_0}{\partial t}\right) + p\left(\frac{\partial V^*}{\partial t} - U^* + U^{*2}V^* - \frac{1}{4}\left(\frac{\partial^2 V^*}{\partial x^2} + \frac{\partial^2 V^*}{\partial y^2}\right)\right) = 0. \end{aligned}$$

Starting with $u_0 = U_0^* = \exp[-x-y]$ and $v_0 = V_0^* = \exp[x+y]$, from (21), we obtain the recurrence relation

$$\begin{aligned} U_j^* &= \int_0^t \left[-2\,U_{j-1}^* + \frac{1}{4}\left(\frac{\partial^2 U_{j-1}^*}{\partial x^2} + \frac{\partial^2 U_{j-1}^*}{\partial y^2}\right) + \sum_{i=0}^{j}\sum_{k=0}^{j-1} U_i^* U_k^* V_{j-k-i-1}^*\right] \mathrm{d}t, \quad j = 1,2,3,\ldots, \\ V_j^* &= \int_0^t \left[U_{j-1}^* + \frac{1}{4}\left(\frac{\partial^2 V_{j-1}^*}{\partial x^2} + \frac{\partial^2 V_{j-1}^*}{\partial y^2}\right) - \sum_{i=0}^{j}\sum_{k=0}^{j-1} U_i^* U_k^* V_{j-k-i-1}^*\right] \mathrm{d}t, \quad j = 1,2,3,\ldots. \end{aligned}$$

Then we drive the following results

$$\begin{aligned} &U_1^*(x,t) = \frac{1}{2}t\exp[-x-y] = \frac{1}{2^1 1!}t\exp[-x-y], && V_1^*(x,t) = \frac{1}{2}t\exp[x+y] = \frac{1}{2^1 1!}t\exp[x+y], \\ &U_2^*(x,t) = \frac{1}{8}t^2\exp[-x-y] = \frac{1}{2^2 2!}t^2\exp[-x-y], && V_2^*(x,t) = \frac{1}{8}t^2\exp[x+y] = \frac{1}{2^2 2!}t^2\exp[x+y], \\ &U_3^*(x,t) = \frac{1}{48}t^3\exp[-x-y] = \frac{1}{2^3 3!}t^3\exp[-x-y], && V_3^*(x,t) = \frac{1}{48}t^3\exp[x+y] = \frac{1}{2^3 3!}t^3\exp[x+y], \\ &U_4^*(x,t) = \frac{1}{384}t^4\exp[-x-y] = \frac{1}{2^4 4!}t^4\exp[-x-y], && V_4^*(x,t) = \frac{1}{384}t^4\exp[x+y] = \frac{1}{2^4 4!}t^4\exp[x+y], \\ &\vdots \end{aligned}$$

Suppose that

$$\begin{aligned} &\mathbf{W}_n = N(\mathbf{W}_{n-1}), \qquad \mathbf{W}_{n-1} = (U_{n-1}, V_{n-1}), \\ &U_0 = U_0^* = u_0, \qquad U_n = \sum_{j=0}^{n}\int_0^t \left[-2\,U_j^* + \frac{1}{4}\left(\frac{\partial^2 U_j^*}{\partial x^2} + \frac{\partial^2 U_j^*}{\partial y^2}\right) + \sum_{i=0}^{j}\sum_{k=0}^{j} U_i^* U_k^* V_{j-k-i}^*\right] \mathrm{d}t, \quad n = 1,2,\ldots, \\ &V_0 = V_0^* = v_0, \qquad V_n = \sum_{j=0}^{n}\int_0^t \left[U_j^* + \frac{1}{4}\left(\frac{\partial^2 V_j^*}{\partial x^2} + \frac{\partial^2 V_j^*}{\partial y^2}\right) - \sum_{i=0}^{j}\sum_{k=0}^{j} U_i^* U_k^* V_{j-k-i}^*\right] \mathrm{d}t, \quad n = 1,2,\ldots. \end{aligned}$$

According to the Theorem for the nonlinear mapping $\mathcal{N}$, a sufficient condition for convergence of the homotopy perturbation method is the strict contraction of $\mathcal{N}$. Therefore we have

$$\|u_0 - u\| = \left\|\exp[-x-y]\left(1 - \exp\left[-\frac{t}{2}\right]\right)\right\|,$$

$$\begin{aligned}\|U_1-u\| &= \left\|\exp[-x-y]\left(1-\frac{t}{2}-\exp\left[-\frac{t}{2}\right]\right)\right\|\\ &\le \left\|\exp[-x-y]\left(1-\exp\left[-\frac{t}{2}\right]\right)\right\|\left\|1-\frac{t}{2\left(1-\exp\left[-\frac{t}{2}\right]\right)}\right\|.\end{aligned}$$

Because, for all $t\in[0,2]$ we have, $\left\|1-\frac{t}{2\left(1-\exp\left[-\frac{t}{2}\right]\right)}\right\|\le\gamma=0.582<1$, it follows that,

$$\begin{aligned}\|U_1-u\| &\le \gamma\left\|\exp[-x-y]\left(1-\exp\left[-\frac{t}{2}\right]\right)\right\| = \gamma\|v_0-u\|,\\ \|U_2-u\| &= \left\|\exp[-x-y]\left(1-\frac{t}{2}+\frac{t^2}{8}-\exp\left[-\frac{t}{2}\right]\right)\right\|\\ &\le \left\|\exp[-x-y]\left(1-\frac{t}{2}-\exp\left[-\frac{t}{2}\right]\right)\right\|\left\|\left(1+\frac{t^2}{8\left(1-\frac{t}{2}-\exp\left[-\frac{t}{2}\right]\right)}\right)\right\|.\end{aligned}$$

$\forall t\in[0,2]$, $\left\|\left(1+\frac{t^2}{8\left(1-\frac{t}{2}-\exp\left[-\frac{t}{2}\right]\right)}\right)\right\|\le 0.359<\gamma$, thus, $\|U_2-u\|\le\gamma^2\|u_0-u\|$.

$$\begin{aligned}\|U_3-u\| &= \left\|\exp[-x-y]\left(1-\frac{t}{2}+\frac{t^2}{8}-\frac{t^3}{48}-\exp\left[-\frac{t}{2}\right]\right)\right\|\\ &\le \left\|\exp[-x-y]\left(1-\frac{t}{2}+\frac{t^2}{8}-\exp\left[-\frac{t}{2}\right]\right)\right\|\left\|\left(1-\frac{t^3}{48\left(1-\frac{t}{2}+\frac{t^2}{8}-\exp\left[-\frac{t}{2}\right]\right)}\right)\right\|.\end{aligned}$$

$\forall t\in[0,2]$, $\left\|\left(1-\frac{t^3}{48\left(1-\frac{t}{2}+\frac{t^2}{8}-\exp\left[-\frac{t}{2}\right]\right)}\right)\right\|\le 0.261<\gamma$, thus,

$$\begin{aligned}&\|U_3-u\| \le \gamma^3\|u_0-u\|\\ &\vdots\\ &\|U_n-u\| \le \gamma^n\|u_0-u\|.\end{aligned}$$

Therefore, $\lim_{n\to\infty}\|U_n-u\|\le\lim_{n\to\infty}\gamma^n\|u_0-u\|=0$, that is

$$u(x,y,z,t)=\lim_{n\to\infty}U_n=\exp\left[-x-y-\frac{t}{2}\right].$$

Also,

$$\begin{aligned}\|v_0-v\| &= \left\|\exp[x+y]\left(1-\exp\left[\frac{t}{2}\right]\right)\right\|,\\ \|V_1-v\| &= \left\|\exp[x+y]\left(1+\frac{t}{2}-\exp\left[\frac{t}{2}\right]\right)\right\|\\ &\le \left\|\exp[x+y]\left(1-\exp\left[\frac{t}{2}\right]\right)\right\|\left\|1+\frac{t}{2\left(1-\exp\left[\frac{t}{2}\right]\right)}\right\|.\end{aligned}$$

For all $t\in[0,2]$ we have $\left\|1+\frac{t}{2\left(1-\exp\left[\frac{t}{2}\right]\right)}\right\|\le\gamma=0.418<1$, therefore,

$$\begin{aligned}\|V_1-v\| &\le \gamma\left\|\exp[x+y]\left(1-\exp\left[\frac{t}{2}\right]\right)\right\| = \gamma\|v_0-v\|,\\ \|V_2-v\| &= \left\|\exp[x+y]\left(1+\frac{t}{2}+\frac{t^2}{8}-\exp\left[\frac{t}{2}\right]\right)\right\|\\ &\le \left\|\exp[x+y]\left(1+\frac{t}{2}-\exp\left[\frac{t}{2}\right]\right)\right\|\left\|\left(1+\frac{t^2}{8\left(1+\frac{t}{2}-\exp\left[\frac{t}{2}\right]\right)}\right)\right\|.\end{aligned}$$

Also $\forall t \in [0, 2]$, $\left\| \left(1 + \frac{t^2}{8\left(1+\frac{t}{2}-\exp\left[\frac{t}{2}\right]\right)}\right) \right\| \leq 0.304 < \gamma$, thus, $\|V_2 - v\| \leq \gamma^2 \|v_0 - v\|$.

$$\begin{aligned}\|V_3 - v\| &= \left\| \exp[x+y] \left(1 + \frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{48} - \exp\left[\frac{t}{2}\right]\right) \right\| \\ &\leq \left\| \exp[x+y] \left(1 + \frac{t}{2} + \frac{t^2}{8} - \exp\left[\frac{t}{2}\right]\right) \right\| \left\| \left(1 + \frac{t^3}{48\left(1+\frac{t}{2}+\frac{t^2}{8}-\exp\left[\frac{t}{2}\right]\right)}\right) \right\|.\end{aligned}$$

Also $\forall t \in [0, 2]$, $\left\| \left(1 + \frac{t^3}{48\left(1+\frac{t}{2}+\frac{t^2}{8}-\exp\left[\frac{t}{2}\right]\right)}\right) \right\| \leq 0.236 < \gamma$, thus,

$$\begin{aligned}&\|V_3 - v\| \leq \gamma^3 \|v_0 - v\| \\ &\vdots \\ &\|V_n - v\| \leq \gamma^n \|v_0 - v\|.\end{aligned}$$

Therefore, $\lim_{n\to\infty} \|V_n - v\| \leq \lim_{n\to\infty} \gamma^n \|v_0 - v\| = 0$, that is

$$v(x, y, t) = \lim_{n\to\infty} V_n = \exp\left[x + y + \frac{t}{2}\right].$$

5. Conclusion

In this work, we use homotopy perturbation method for solving Brusselator and Burgers' equations. He's homotopy perturbation method is a powerful straightforward method. One important objective of our research is the examination of the convergence of homotopy perturbation method. By using this method we obtain a new efficient recurrent relation to solve nonlinear Brusselator and Burgers' equations. The results show that homotopy perturbation method is a powerful mathematical tool for solving systems of nonlinear partial differential equations having wide applications in science and engineering. In comparison with Adomian decomposition method [1,4], in the present method there is no need to calculate Adomian polynomials. In comparison with boundary element method [2], the size of computational work has been reduced and rapid convergence has been guaranteed. Authors are working on the convergence of HPM, when applying for solving other functional equations. The computations are done using Maple 10.

References

[1] Abdul-Majid Wazwaz, The decomposition method applied to systems of partial differential equations and to the reaction–diffusion Brusselator model, Applied Mathematics and Computation 110 (2000) 251–264.
[2] Whye-Teong Ang, The two-dimensional reaction–diffusion Brusselator system: A dual-reciprocity boundary element solution, Engineering Analysis with Boundary Elements 27 (2003) 897–903.
[3] J.M. Burger, A Mathematical Model Illustrating the Theory of Turbulence, Academic Press, New York, 1948.
[4] A. Refik Bahadir, A fully implicit finite-difference scheme for two dimensional Burgers' equations, Applied Mathematics and Computation 137 (2003) 131–137.
[5] J.H. He, Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering 178 (1999) 257–262.
[6] J.H. He, A coupling method of homotopy technique and perturbation technique for nonlinear problems, International Journal of Non-Linear Mechanics 35 (1) (2000) 37–43.
[7] J.H. He, Comparison of homotopy perturbation method and homotopy analysis method, Applied Mathematics and Computation 156 (2004) 527–539.
[8] J.H. He, Homotopy perturbation method: A new nonlinear analytical technique, Applied Mathematics and Computation 135 (2003) 73–79.
[9] J.H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, Applied Mathematics and Computation 151 (2004) 287–292.
[10] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons and Fractals 26 (2005) 695–700.
[11] J.H. He, Homotopy perturbation method for solving boundary value problems, Physics Letters A 350 (2006) 87–88.
[12] Q. Wang, Homotopy perturbation method for fractional KdV-Burgers equation, Chaos, Solitons and Fractals 190 (2007) 1795–1802.
[13] D.D. Ganji, M. Rafei, Solitary wave solutions for a generalized Hirota–Satsuma coupled KdV equation by homotopy perturbation method, Physics Letters A 356 (2006) 131–137.
[14] L. Cveticanin, Homotopy perturbation method for pure nonlinear differential equation, Chaos, Solitons and Fractals 30 (2006) 1221–1230.
[15] L. Cveticanin, The homotopy-perturbation method applied for solving complex-valued differential equations with strong cubic nonlinearity, Journal of Sound and Vibration 285 (4–5) (2005) 1171–1179.
[16] J. Biazar, H. Ghazvini, M. Eslami, He's homotopy perturbation method for systems of integro-differential equations, Chaos, Solitons and Fractals 39 (3) (2009) 1253–1258.
[17] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons and Fractals 26 (2005) 695–700.
[18] D.D. Ganji, A. Sadighi, Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations, Journal of Computational and Applied Mathematics 34 (2007) 1003–1016.
[19] Q. Wang, Homotopy perturbation method for fractional KdV-Burgers equation, Chaos, Solitons and Fractals 190 (2007) 1795–1802.
[20] J. Biazar, H. Ghazvini, He's homotopy perturbation method for solving systems of Volterra integral equations of the second kind, Chaos, Solitons and Fractals 39 (2) (2009) 770–777.
[21] J. Biazar, H. Ghazvini, Exact solutions for non-linear Schrödinger equations by He's homotopy perturbation method, Physics Letters A 366 (2007) 79–84.
[22] D.D. Ganji, The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer, Physics Letters A 355 (2006) 337–341.
[23] A.M. Siddiqui, R. Mahmood, Q.K. Ghori, Homotopy perturbation method for thin film flow of a fourth grade fluid down a vertical cylinder, Physics Letters A 352 (2006) 404–410.

[24] Z.M. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order, International Journal of Nonlinear Sciences and Numerical Simulation 7 (1) (2006) 27–34.
[25] N. Bildik, A. Konuralp, The use of variational iteration method differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations, International Journal of Nonlinear Sciences and Numerical Simulation 7 (1) (2006) 65–70.
[26] P.D. Ariel, T. Hayat, S. Asghar, Homotopy perturbation method and axisymmetric flow over a stretching sheet, International Journal of Nonlinear Sciences and Numerical Simulation 7 (4) (2006) 399–406.
[27] J. Biazar, H. Ghazvini, Numerical solution for special non-linear Fredholm integral equation by HPM, Applied Mathematics and Computation 195 (2008) 681–687.
[28] J.H. He, Homotopy perturbation method for bifurcation of nonlinear problems, International Journal of Nonlinear Sciences and Numerical Simulation 6 (2) (2005) 207–208.
[29] X.C. Cai, W.Y. Wu, M.S. Li, Approximate period solution for a kind of nonlinear oscillator by He's perturbation method, International Journal of Nonlinear Sciences and Numerical Simulation 7 (1) (2006) 109–112.
[30] J. Biazar, H. Ghazvini, Homotopy perturbation method for solving hyperbolic partial differential equations, Computers and Mathematics with Applications 54 (7–8) (2007) 1047–1054.