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Note

A new proof of Vázsonyi's conjecture ☆

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Abstract

The diameter graph G of n points in Euclidean 3-space has a bipartite, centrally symmetric double covering on the sphere. Three easy corollaries follow: (1) A self-contained proof of Vázsonyi's conjecture that G has at most $2n-2$ edges, which avoids the ball polytopes used in the original proofs given by Grünbaum, Heppes and Straszewicz. (2) G can be embedded in the projective plane. (3) Any two odd cycles in G intersect [V.L. Dol'nikov, Some properties of graphs of diameters, Discrete Comput. Geom. 24 (2000) 293–299].

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Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, and $\mathbb{S}^{d-1}$ the unit $(d-1)$-sphere in $\mathbb{R}^d$ centred at the origin. Let S be a set of n points of diameter D in $\mathbb{R}^d$. Define the *diameter graph* on S by joining all *diameters*, i.e., point pairs at distance D. The following theorem was conjectured by Vázsonyi, as reported in [2]. It was subsequently independently proved by Grünbaum [3], Heppes [4] and Straszewicz [9].

Theorem 1. *The number of edges in a diameter graph on $n \geqslant 4$ points in $\mathbb{R}^3$ is at most $2n-2$.*

All three proofs (see [7, Theorem 13.14]) use the ball polytope obtained by taking the intersection of the balls of radius D centred at the points. However, ball polytopes do not behave the same as ordinary polytopes already in $\mathbb{R}^3$, where their graphs need not be 3-connected. See the detailed study of Kupitz, Martini and Perles in [6]. The proof presented here avoids their use.

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Theorem 2. *Any diameter graph in $\mathbb{R}^3$ has a bipartite double covering that has a centrally symmetric drawing on $\mathbb{S}^2$.*

In fact, each point $x \in S$ will correspond to an antipodal pair of points x_r and x_b on the sphere, with x_r coloured red and x_b blue. Each edge xy of the diameter graph will correspond to two antipodal edges $x_r y_b$ and $x_b y_r$ on $\mathbb{S}^2$, giving a properly 2-coloured graph on $2n$ vertices. The drawing will be made such that no edges cross. By Euler's formula there are at most $4n-4$ edges, and thus at most $2n-2$ edges in the original diameter graph, which proves Theorem 1.

The proof of Theorem 2 follows after the following two easy corollaries.

Corollary 3. *Any diameter graph in $\mathbb{R}^3$ can be embedded in the projective plane such that all odd cycles are noncontractible.*

Proof. Identify opposite points of $\mathbb{S}^2$ to obtain the projective plane $\mathbb{P}^2$. The centrally symmetric drawing of Theorem 2 becomes a drawing of the original diameter graph in $\mathbb{P}^2$. Since the drawing on $\mathbb{S}^2$ is 2-coloured, an odd cycle of length k in the diameter graph corresponds to a centrally symmetric cycle of length $2k$ on $\mathbb{S}^2$. Such a cycle on the sphere corresponds to a noncontractible closed Jordan curve in $\mathbb{P}^2$. $\square$

Corollary 4. *(See Dol'nikov [1].) Any two odd cycles in a diameter graph on a finite set in $\mathbb{R}^3$ intersect.*

Proof. As noted in the previous proof, an odd cycle corresponds to a centrally symmetric cycle in the double covering. Two centrally symmetric closed Jordan curves on $\mathbb{S}^2$ clearly intersect. $\square$

Proof of Theorem 2. Without loss of generality, assume that $D=1$. Repeatedly remove all vertices of degree at most 1 in the diameter graph. Since such vertices and their incident edges can easily be drawn later, this is no loss of generality. For each $x \in S$, let $R(x)$ be the intersection of $\mathbb{S}^2$ with the cone generated by $\{y-x\colon\ xy \text{ is a diameter}\}$. Each $R(x)$ is a convex spherical polygon with great circular arcs as edges. (If x has degree 2 then $R(x)$ is an arc.) Colour $R(x)$ red and $B(x) := -R(x)$ blue. Assume for the moment the following two properties of these polygons:

Lemma 1. *If $x \neq y$, then $R(x)$ and $R(y)$ are disjoint.*

Lemma 2. *If $R(x)$ and $B(y)$ intersect, then xy is a diameter and $R(x) \cap B(y) = \{y-x\}$.*

For each $x \in S$ choose an arbitrary point x_r in the interior of $R(x)$ and let $x_b = -x_r$. (If $R(x)$ is an arc, let x_r be in its relative interior.) Draw Jordan arcs inside $R(x)$ from x_r to all the vertices of $R(x)$, as well as antipodal arcs from x_b to the vertices of $B(x)$. This gives a centrally symmetric drawing of a 2-coloured double covering of the diameter graph. By Lemmas 1 and 2 no edges cross, and the theorem follows. $\square$

The following proofs of Lemmas 1 and 2 are dimension independent, and thus give a natural double covering on $\mathbb{S}^{d-1}$ of any diameter graph in $\mathbb{R}^d$.

Lemma 3. *Let $x_1, \ldots, x_k$ and $\sum_{i=1}^k \lambda_i x_i$ be unit vectors in $\mathbb{R}^d$, with all $\lambda_i \geqslant 0$. Let $y \in \mathbb{R}^d$. Suppose that $\|y-x_i\| \leqslant 1$ for all $i=1,\ldots,k$. Then $\|y-\sum_{i=1}^k \lambda_i x_i\| \leqslant 1$.*

Proof. By the triangle inequality,

$$1 = \left\| \sum_{i=1}^{k} \lambda_i x_i \right\| \leqslant \sum_{i=1}^{k} \lambda_i. \tag{1}$$

Expanding $\|y - x_i\|^2 \leqslant 1$ by inner products,

$$-2\langle x_i, y\rangle \leqslant -\|y\|^2. \tag{2}$$

Therefore,

$$\begin{aligned}\left\| y - \sum_{i=1}^{k} \lambda_i x_i \right\|^2 &= \|y\|^2 - 2\sum_{i=1}^{k} \lambda_i \langle x_i, y\rangle + 1 \\ &\leqslant \left(1 - \sum_{i=1}^{k} \lambda_i\right)\|y\|^2 + 1 \quad \text{by (2)} \\ &\leqslant 1 \quad \text{by (1)}. \qquad \square\end{aligned}$$

Proof of Lemma 1. Let the neighbours of x be $x + x_i$, and the neighbours of y be $y + y_j$, where all x_i and y_j are unit vectors. Suppose that

$$\sum_i \lambda_i x_i = \sum_j \mu_j y_j \in R(x) \cap R(y) \quad \text{with } \lambda_i, \mu_j \geqslant 0.$$

Since $\|x + x_i - y\| \leqslant 1$ for all i, Lemma 3 gives

$$\left\| x + \sum_i \lambda_i x_i - y \right\| \leqslant 1.$$

Similarly, Lemma 3 applied to $\|x - y - y_j\| \leqslant 1$ gives

$$\left\| x - y - \sum_j \mu_j y_j \right\| \leqslant 1.$$

By the triangle inequality,

$$\begin{aligned} 2 &= \left\| 2\sum_i \lambda_i x_i \right\| \\ &= \left\| \left(x + \sum_i \lambda_i x_i - y\right) - \left(x - y - \sum_j \mu_j y_j\right) \right\| \\ &\leqslant \left\| x + \sum_i \lambda_i x_i - y \right\| + \left\| x - y - \sum_j \mu_j y_j \right\| \\ &\leqslant 2. \end{aligned}$$

Consequently there is equality throughout. Since then $x + \sum_i \lambda_i x_i - y$ and $-x + y + \sum_j \mu_j y_j$ are unit vectors in the same direction, they are equal, which gives $x = y$. $\square$

Proof of Lemma 2. Since $\|x_i - x_j\| \leqslant 1$ for all i and j, $R(x)$ is contained in an open hemisphere of $\mathbb{S}^{d-1}$, hence $R(x) \cap B(x) = \emptyset$. Thus without loss of generality, $x \neq y$. As before, let the neighbours of x be $x + x_i$, and the neighbours of y be $y + y_j$, with all x_i and y_j unit vectors. Suppose that $\sum_i \lambda_i x_i = -\sum_j \mu_j y_j \in R(x) \cap B(y)$ with $\lambda_i, \mu_j \geqslant 0$. For a fixed j, $\|x + x_i - y - y_j\| \leqslant 1$ for all i. Lemma 3 then gives

$$\left\| x + \sum_i \lambda_i x_i - y - y_j \right\| \leqslant 1 \quad \text{for all } j.$$

Again by Lemma 3,

$$\left\| x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j \right\| \leqslant 1.$$

By the triangle inequality,

$$\begin{aligned} 2 &= \left\| 2 \sum_i \lambda_i x_i \right\| \\ &= \left\| \left(x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j \right) + (y - x) \right\| \\ &\leqslant \left\| x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j \right\| + \|y - x\| \\ &\leqslant 2. \end{aligned}$$

As in the previous proof, $x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j = y - x$. Consequently, $y - x = \sum_i \lambda_i x_i$, a unit vector, and $R(x) \cap B(y) = \{y - x\}$. □

Remarks

Perlstein and Pinchasi [8] independently obtained a similar proof of Vázsonyi's conjecture. They proved a more general result and found a connection to a theorem of Katchalski and Last [5] and Valtr [10].

The following example of Rom Pinchasi (personal communication) shows that diameter graphs need not be planar. Choose $\alpha, \beta, \gamma > 0$ with $\alpha > \beta$. Let $x_k = (\alpha \cos \frac{2\pi k}{5}, \alpha \sin \frac{2\pi k}{5}, 0)$. Then $A = \{x_k\colon 0 \leqslant k \leqslant 4\}$ is the set of vertices of a regular pentagon P in the xy-plane. Let $y_k = (\beta \cos \frac{2\pi k}{5}, \beta \sin \frac{2\pi k}{5}, \gamma)$. Then $B = \{y_k\colon 0 \leqslant k \leqslant 4\}$ is a smaller copy of the pentagon lifted by a distance γ in the direction of the z-axis. The values of α, β, γ can be fixed such that

$$\|x_i - x_{i+2}\| = \|x_i - x_{i+3}\| = \|x_i - y_{i+2}\| = \|x_i - y_{i+3}\| = D$$

for all i taken modulo 5, where D is the diameter of $S := A \cup B$. Thus the diameter graph of S is a subdivision of the complete graph on 5 vertices, which is not planar.

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