Calculating conjugacy classes in Sylow $p$-subgroups of finite Chevalley groups

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Abstract

In [S.M. Goodwin, On the conjugacy classes in maximal unipotent subgroups of simple algebraic groups, Transform. Groups 11 (1) (2006) 51–76, §8], the first author outlined an algorithm for calculating a parametrization of the conjugacy classes in a Sylow $p$-subgroup $U(q)$ of a finite Chevalley group $G(q)$, valid when $q$ is a power of a good prime for $G(q)$. In this paper we develop this algorithm and discuss an implementation in the computer algebra language GAP. Using the resulting computer program we are able to calculate the parametrization of the conjugacy classes in $U(q)$, when $G(q)$ is of rank at most 6. In these cases, we observe that the number of conjugacy classes of $U(q)$ is given by a polynomial in $q$ with integer coefficients.

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1. Introduction

Let $U_n(q)$ be the subgroup of $GL_n(q)$ consisting of upper unitriangular matrices. A long-standing conjecture, attributed to G. Higman [12] states that the number of conjugacy classes of
$U_n(q)$ is given by a polynomial in $q$ with integer coefficients. This has been verified by computer calculation for $n \leq 13$ in the work of A. Vera-López and J.M. Arregi, see [19]. This conjecture has generated a great deal of interest, see for example [17] and [18].

The equivalent problem of counting the number of (complex) irreducible characters of $U_n(q)$ has also attracted a lot of attention, see for example [13], [14] and [15]. Thanks to work of M. Isaacs [13], the degrees of the irreducible characters of $U_n(q)$ are known to all be powers $q^d$ of $q$ and all exponents $d$ occur for $0 \leq d \leq \mu(n)$, where $\mu(n)$ is an explicit upper bound due to work by G.I. Lehrer [15]. It was conjectured by Lehrer [15] that the number of irreducible characters of $U_n(q)$ of degree $q^d$ is a polynomial in $q$ with integer coefficients only depending on $n$ and $d$; this conjecture clearly implies Higman’s conjecture.

It is natural to consider the analogue of Higman’s conjecture for other finite groups of Lie type. Below we introduce some notation in order to discuss this analogue for the case of finite Chevalley groups.

Let $G$ be a split simple algebraic group defined over the finite field $\mathbb{F}_p$ of $p$ elements, and assume that $p$ is good for $G$. For a power $q$ of $p$ we write $G(q)$ for the finite group of $\mathbb{F}_q$-rational points of $G$; this is a finite Chevalley group. Let $U$ be a maximal unipotent subgroup of $G$ defined over $\mathbb{F}_p$, so that $U(q)$ is a Sylow $p$-subgroup of $G(q)$.

In this paper, we describe an algorithm that calculates a parametrization of the conjugacy classes of $U(q)$. We have implemented this algorithm in the computer algebra language GAP [5]. The algorithm is based on the outline given by the first author in [8]. The output of the computer program allows one to calculate the number $k(U(q))$ of conjugacy classes of $U(q)$. Using our computer program we have proved the following theorem.

**Theorem 1.1.** Let $G$ be a split simple algebraic group defined over $\mathbb{F}_p$, where $p$ is good for $G$. Let $U$ be a maximal unipotent subgroup of $G$ defined over $\mathbb{F}_p$. Let $q$ be a power of $p$. If the rank of $G$ is at most 6, then the number of conjugacy classes of $U(q)$ is given by a polynomial in $q$ with integer coefficients (and the polynomial itself is independent of $p$).

We have explicitly calculated the polynomial $k(U(q))$ for $G(q)$ of rank at most 5; these polynomials are presented in Table 1 in Section 4. For $G(q)$ of rank 6, the output of the computer program describing the parametrization of the conjugacy classes of $U(q)$ is long and rather complicated. It is possible to check from this output in a short amount of time that $k(U(q))$ is given by a polynomial in $q$; however, a great deal of calculation would be required to explicitly compute this polynomial. We explain this in more detail in Section 4.

Our computer program makes calculations using a $\mathbb{Z}$-form of the Lie algebra $u$ of $U$. In these calculations “implicit divisions” are made, which lead to the output not being valid for a finite number of primes $p$ that are recorded within the program. In all cases that we have run the program the only primes that are output are bad primes for $G$. A simple modification could insist on not dividing by certain prime numbers, and in this way it is possible to deal with the situation if any good primes were output. We note that the results on which our algorithm is based require the assumption that $p$ is a good prime for $G$, so the output of our algorithm is not valid for bad primes even if there have been no implicit divisions.

We have adapted the computer program so that it is also possible to use it to calculate the number of $U(q)$-conjugacy classes in certain subquotients of $U(q)$. More precisely, let $B = N_G(U)$, be the Borel subgroup of $G$ corresponding to $U$, then for normal subgroups $N \supseteq M$ of $B$ that are contained in $U$, we can calculate a parametrization of the $U(q)$-conjugacy classes in the quotient $N(q)/M(q)$. We have made a number of such calculations in case $G$ is of rank
greater than 6. In all cases where we have calculated the number of \( U(q) \)-conjugacy classes in \( N(q)/M(q) \), we observe that it is given by a polynomial in \( q \). In Table 2 in Section 4, we give the number \( k(U(q), U^{(l)}(q)) \) of \( U(q) \)-conjugacy classes in the \( l \)th term \( U^{(l)}(q) \) of the descending central series of \( U(q) \) for \( G \) of exceptional type and certain \( l \).

Generalizing a theorem of J. Alperin [1], the authors showed in [11, Theorem 4.6] that if the center of \( G \) is connected, then the number \( k(U(q), G(q)) \) of conjugacy classes of \( U(q) \) in all of \( G(q) \) is a polynomial in \( q \) with integer coefficients (in case \( G(q) \) has a simple component of type \( E_8 \), we require two polynomials depending on the congruence of \( q \) modulo 3). The theorem of Alperin [1] can be viewed as support for Higman’s conjecture. Analogously, [11, Theorem 4.6] suggests that, for \( G \) not of type \( E_8 \), the number of conjugacy classes of \( U(q) \) is given by a polynomial in \( q \). The results of our computer calculations give supporting evidence for this behavior, and we thus propose the following analogue and extension of Higman’s conjecture for arbitrary finite Chevalley groups.

**Conjecture 1.2.** Let \( G \) be a split simple algebraic group defined over \( \mathbb{F}_p \), where \( p \) is good for \( G \). Let \( U \) be a maximal unipotent subgroup of \( G \) defined over \( \mathbb{F}_p \). Let \( q \) be a power of \( p \). If \( G \) is not of type \( E_8 \), then \( k(U(q)) \) is given by a polynomial in \( q \) with integer coefficients (and the polynomial itself is independent of \( p \)). If \( G \) is of type \( E_8 \), then \( k(U(q)) \) is given by one of two polynomials depending on the congruence class of \( q \) modulo 3.

The dependence of \( k(U(q)) \) on the congruence class of \( q \) modulo 3 in the \( E_8 \) case in Conjecture 1.2 is suggested by the \( E_8 \) case in [11, Theorem 4.5(iii)]; though we do not wish to rule out the possibility that there is just one polynomial. As indicated above, due to the complexity of the computation, it is not feasible to run our computer program in case \( G \) is of type \( E_8 \). In fact, as shown in Table 2, at present we have only been able to calculate \( k(U(q), U^{(l)}(q)) \) explicitly for \( l \geq 10 \); we have \( \text{dim} \, U^{(10)} = 52 \) and \( \text{dim} \, U = 120 \) demonstrating the difficulty of running our program for \( E_8 \).

As mentioned above our algorithm is not valid for bad primes. It is possible to calculate the \( U(q) \)-conjugacy classes for \( G \) of type \( B_2 \) and \( p = 2 \) by hand. In this case we have \( k(U(q)) = 5(q - 1)^2 + 4(q - 1) + 1 \), which is a different polynomial to the one for good primes given in Table 1; this is due to degeneracies in the Chevalley commutator relations. In addition, [11, Theorem 4.6] is only valid for good primes, so we choose not to make any conjecture for bad primes.

From our calculations we can observe that each polynomial \( k(U(q)) \), for \( G \) of rank 5 or less, when written as a polynomial in \( q - 1 \) has non-negative integer coefficients, see Table 1. For \( G \) of type \( A_r \) and \( r \leq 12 \) this was already observed in the explicit results of Vera-López–Arregi [19]. It would be interesting to have a geometric interpretation of this positivity behavior. In Section 4, we give a reason why these positivity phenomena hold for the cases that we have calculated. This is done by analyzing the calculations made by the computer program. We expect that if Conjecture 1.2 is true, then the coefficients in \( k(U(q)) \) when written as a polynomial in \( q - 1 \) are always non-negative.

In the cases where we have calculated \( k(U(q)) \), we have observed that \( k(U(q)) \) always has constant term equal to 1 when written as a polynomial in \( q - 1 \). In Section 4, we explain why this is necessarily the case whenever \( k(U(q)) \) is a polynomial in \( q \).

Another observation is that the polynomial \( k(U(q)) \) is the same for \( G \) of type \( B_r \) and \( C_r \), for \( r = 3, 4, 5 \). It is likely that this is always the case for any \( r \). We expect that this should be explained by the duality of the underlying root systems.
Although $k(U(q), G(q))$ is always a polynomial in $q$ (assuming that $G$ has connected center and taking into account that for $G$ of type $E_8$ there are two polynomials) and our calculations here tell us that $k(U(q))$ is a polynomial for low rank, the number $k(U(q), B(q))$ of $U(q)$-conjugacy classes in $B(q)$ is not always a polynomial in $q$. Indeed for $G$ of type $G_2$ it is shown in [9, Example 4.8] that the number $k(B(q), U(q))$ of $B(q)$-conjugacy classes in $U(q)$ is given by two polynomials: $q + 15$ if $q$ is congruent to $-1$ mod $3$; and $q + 17$ if $q$ is congruent to $1$ mod $3$ (we are assuming $p$ is good for $G$). A general argument considering the commuting variety $\mathcal{C}(B, U) = \{(b, u) \in B \times U \mid bu = ub\}$ shows that we always have $k(U(q), B(q)) = (q - 1)^2 k(B(q), U(q))$. Thus $k(U(q), B(q))$ is not a polynomial in $q$ for $G$ of type $G_2$.

Our algorithm calculates a family of varieties $X_c$ that parameterize the conjugacy classes of $U$; moreover, these varieties are defined over $\mathbb{F}_p$. The algorithm determines the polynomials defining the $X_c$ as locally closed subsets of $(\mathbb{F}_p^*)^{mc}$ for certain $m_c \in \mathbb{Z}_{\geq 0}$. The varieties $X_c$ are determined with a backtrack algorithm using a depth-first search. The conjugacy classes of $U(q)$ are parameterized by the $\mathbb{F}_q$-rational points of the varieties $X_c$ and it is possible to count these points.

The idea behind the algorithm is similar to that for the algorithm used by Bürgstein and Hesselink in [2] for calculating the adjoint orbits of $B$ in $u = \text{Lie} U$; we remark that the algorithm in [2] was not written to give a complete description of the $B$-orbits in $u$. In addition, our algorithm generalizes the one used in the work of Vera-López and Arregi for the type $A$ situation, see for example [19]. Finally, we remark that the algorithm of this paper uses ideas from the computer program described in [6] in previous work of the first author.

We now give a brief outline of the structure of this paper. In Section 2, we introduce the notation that we require and recall the relevant results from [8] and [9]. Then in Section 3 we describe the algorithm and its implementation in GAP. Finally, in Section 4 we discuss the results of our computations. In particular, we present explicit values for $k(U(q))$ for $G$ of rank at most $5$ (Table 1) and the values of $k(U(q), U^0(q))$ for some cases where $G$ is of exceptional type (Table 2).

As general references for algebraic groups defined over finite fields we refer the reader to the books by Carter [3] and Digne and Michel [4].

2. Notation and recollection

Let $p$ be a prime and let $G$ be a split simple algebraic group defined over the finite field of $p$ elements $\mathbb{F}_p$. We assume throughout that $p$ is good for $G$ and we write $k$ for the algebraic closure of $\mathbb{F}_p$.

Fix a split maximal torus $T$ of $G$ and let $\Phi$ be the root system of $G$ with respect to $T$. For a root $\alpha \in \Phi$ we choose a generator $e_{\alpha}$ for the corresponding root subspace $g_{\alpha}$ of $g = \text{Lie} G$. Let $B \supseteq T$ be a Borel subgroup of $G$ that is defined over $\mathbb{F}_p$. Let $U$ be the unipotent radical of $B$ and let $u = \text{Lie} U$. Let $\Phi^+$ be the system of positive roots of $\Phi$ determined by $B$. The partial order on $\Phi$ determined by $\Phi^+$ is denoted by $\leq$.

For a power $q$ of $p$, we write $G(q)$ and $U(q)$ for the finite groups of $\mathbb{F}_q$-rational points of $G$ and $U$ respectively. We write $u(q)$ for the Lie algebra of $\mathbb{F}_q$-rational points of $u$.

We now recall some results from [8] and [9] on which our algorithm for calculating the conjugacy classes of $U(q)$ is based. Thanks to [7, Theorem 1.1], there are generalizations of some results in [8], as explained in [7, §6]; below we state the general versions without further comment. For the remainder of this section we fix a power $q$ of $p$. 
The theory of Springer isomorphisms can be used to show that the conjugacy classes of $U(q)$ are in bijective correspondence with the adjoint $U(q)$-orbits in $u(q)$, see for example [8, Proposition 6.2]. For the remainder of this paper, we will consider the adjoint orbits of $U$ in $u$ rather than the conjugacy classes of $U$, as this is more convenient for our purposes.

Next we discuss the notion of minimal representatives of $U$-orbits in $u$, and how they are used to partition the set of $U$-orbits in $u$. The reader is referred to [8, §5 and §6] and [9, §3 and §4] for full details.

We fix an enumeration of the set of positive roots $\Phi^+ = \{\beta_1, \ldots, \beta_N\}$, such that $i \leq j$ whenever $\beta_i \preceq \beta_j$, and define the sequence of $B$-submodules

$$u = m_0 \supseteq \cdots \supseteq m_N = \{0\}$$

of $u$ by $m_i = \sum_{j=i+1}^N \mathfrak{g}_{\beta_j}$. We consider the action of $U$ on successive quotients $u_i = u/m_i$ induced from the adjoint action of $U$ on $u$. We note that the parametrization of the adjoint $U$-orbits described below depends on the choice of the enumeration of $\Phi^+$.

Let $x \in u$ and consider the set

$$x + ke_{\beta_i} + m_i = \{x + \lambda e_{\beta_i} + m_i \mid \lambda \in k\} \subseteq u_i.$$

By [8, Lemma 5.1], for $x \in u$ either:

(I) all elements of $x + ke_{\beta_i} + m_i$ are $U$-conjugate; or
(R) no two elements of $x + ke_{\beta_i} + m_i$ are $U$-conjugate.

We say that

- $i$ is an inert point of $x$ if (I) holds;
- $i$ is a ramification point of $x$ if (R) holds.

An element $x + m_i = \sum_{j=1}^i x_j e_{\beta_j} + m_i$ of $u_i$ is said to be the minimal representative of its $U$-orbit in $u_i$ if $x_j = 0$ whenever $j$ is an inert point of $x$. It follows from [8, Proposition 5.4 and Lemma 5.5] that each $U$-orbit in $u_i$ contains a unique minimal representative; in particular, this holds for the action of $U$ on $u$.

Thanks to [8, Proposition 4.2 and Lemma 5.7], we have that $i$ is an inert point of $x \in u$ if and only if $\dim c_u(x + m_i) = \dim c_u(x + m_{i-1}) - 1$; if $i$ is a ramification point of $x$, then we have $\dim c_u(x + m_i) = \dim c_u(x + m_{i-1})$. Here $c_u(x + m_i)$ is the centralizer of $x + m_i$ for the action of $u$ on $u_i$ induced from the adjoint action of $u$ on itself.

The above discussion implies that the adjoint orbits of $U$ in $u$ are parameterized by their minimal representatives. Further, the set of minimal representatives can be partitioned into sets $X_c$ for $c \in \{I, R\}^N$: the set $X_c$ is defined to consist of the minimal representatives $x \in u$ of the $U$-orbits in $u$ such that for all $i = 1, \ldots, N$ we have that $i$ is an inert point of $x$ if and only if $c_i = I$. Thanks to [9, Proposition 2.4], each of the sets $X_c$ is a locally closed subset of $u$, and therefore has the structure of an algebraic variety.

The above partition of the $U$-orbits in $u$ can be refined to be indexed by $N$-tuples $c \in \{I, R_n, R_0\}^N$ as follows. For $c \in \{I, R_n, R_0\}^N$, the set $X_c$ is defined to consist of the minimal representatives $x = \sum x_i e_{\beta_i} \in u$ of the $U$-orbits in $u$ such that for all $i = 1, \ldots, N$ we have that $i$ is an inert point of $x$ if and only if $c_i = I$; and if $c_i \neq I$, then $x_i = 0$ if and only if $c_i = R_0$. Thanks
representatives are partitioned into the sets \( X_c(q) \) and \( U(q) \) parametrization of the \( G \) if the definition of \[8, \text{Lemma 6.3}\], the orbit \( U \) to \[9, \text{Lemma 4.2}\], each of the sets \( X_c \) to \( 3326 \) S.M. Goodwin, G. Röhrle / Journal of Algebra 321 (2009) 3321–3334
ture of an algebraic variety. In fact, \( X_c \) is a subvariety of \( \{(x_j)_{c_j=R_n} \mid x_j \in k^\times \} \cong (k^\times)^{m_c} \), where \( m_c = \{|i \mid c_i = R_n\}| \).

We now explain how the above parametrization of the \( U \)-orbits in \( u \) descends to give a parametrization of the \( U(q) \)-orbits in \( u(q) \). The reader is referred to \[8, \S 6\] for further details.

Thanks to \[8, \text{Proposition 4.5}\], we have that for \( x \in u \) the centralizer \( C_U(x) \) of \( x \) in \( U \) is connected. This implies that the \( U(q) \)-orbits in \( u(q) \) correspond bijectively to the \( U \)-orbits in \( u \) that are defined over \( F_q \). Let \( x \in u \) be the minimal representative of its \( U \)-orbit. Then, by \[8, \text{Lemma 6.3}\], the orbit \( U \cdot x \) is defined over \( F_q \) if and only if \( x \in u(q) \). We require that the definition of \( G \) over \( F_q \) is split for this last assertion, and this is the reason for this assumption. If the definition of \( G \) over \( F_q \) is not split, then it is a non-trivial task to determine which minimal representatives correspond to orbits defined over \( F_q \).

It follows from the above discussion that the adjoint orbits of \( U(q) \) in \( u(q) \) are parameterized by the minimal representatives of the \( U \)-orbits in \( u \) that lie in \( u(q) \). In turn these minimal representatives are partitioned into the sets \( X_c(q) \) of \( F_q \)-rational points of the varieties \( X_c \), for \( c \in \{I, R_n, R_0\}^N \).

3. The algorithm

In this section we develop the algorithm outlined in \[8\] for calculating the parametrization of the adjoint \( U \)-orbits in \( u \). The idea is to calculate the polynomials defining the varieties \( X_c \) for \( c \in \{I, R_n, R_0\}^N \) as locally closed subsets of \( (k^\times)^{m_c} \). We present the algorithm, and then explain why the algorithm does indeed calculate a parameterization of the adjoint \( U \)-orbits in \( u \). Next we discuss two modifications that are used in the GAP implementation, before briefly explaining the implementation. Finally, we explain how the output of the computer program is used to calculate \( k(U(q)) \).

In order to explain the algorithm we have to introduce some more notation; we continue to use the notation given in the previous section.

We wish to consider all primes \( p \) simultaneously, so we need a \( Z \)-form of \( g \). Let \( g_C \) be the complex simple Lie algebra of the same type as \( g \). Fix a Chevalley basis of \( g_C \) and let \( g_Z \) be the corresponding \( Z \)-form of \( g_C \). We let

\[ m := \max\{m_c \mid c \in \{I, R_n, R_0\}^N, \ X_c \neq \emptyset\}, \]

where \( m_c = |\{i \mid c_i = R_n\}| \), as defined earlier. We define

\[ \tilde{u} = u_Z \otimes_Z Z[t_1, \ldots, t_m], \]

where \( Z[t_1, \ldots, t_m] \) is the polynomial ring in \( m \) indeterminates \( t_1, \ldots, t_m \). We denote by \( e_{\beta_1}, \ldots, e_{\beta_N} \) the elements of the Chevalley basis of \( g_Z \) corresponding to \( \Phi^+ \), which is enumerated as in the previous section. These elements form a \( Z \)-basis of \( u_Z \), and by a minor abuse we also consider them as elements of both \( u \) and \( \tilde{u} \).

Let \( c \in \{I, R_n, R_0\}^l \) for some \( l \leq N \). For \( j = 1, \ldots, m_c \), we define \( \beta_{c,j} \in \Phi^+ \) by setting \( \beta_{c,j} = \beta_{l_j} \), where \( l \) is the \( j \)th smallest element in \( \{h \mid c_h = R_n\} \). We associate to \( c \) the element

\[ x_c(t) = \sum_{j=1}^{m_c} t_j e_{\beta_{c,j}} \in \tilde{u}. \]
Given $\tau = (\tau_1, \ldots, \tau_{mc}) \in k^{mc}$, we write $x_c(\tau)$ for the element of $u$ obtained by substituting $t_j = \tau_j$ in $x_c(t)$, i.e.

$$x_c(\tau) = \sum_{j=1}^{mc} \tau_j e_{\beta_{c,j}}.$$ 

The variety $X_c$ is a locally closed subset of $\{x_c(\tau) \mid \tau \in (k^\times)^{mc}\} \cong (k^\times)^{mc}$. Therefore, there are subsets $A^i_c$, $B^i_c$ of $k[t_1, \ldots, t_{mc}] \subseteq k[t_1, \ldots, t_m]$ such that $X_c$ is the disjoint union of the sets

$$X^i_c = \{x_c(\tau) \mid f(\tau) = 0 \text{ for all } f \in A^i_c \text{ and } g(\tau) \neq 0 \text{ for all } g \in B^i_c\},$$

for $i = 1, \ldots, l_c$. In fact the polynomials in the sets $A^i_c$ and $B^i_c$ can be taken to have integer coefficients; this is due to the integrality of the Chevalley commutator relations. The purpose of our algorithm is to determine certain choices for the sets $A^i_c$ and $B^i_c$.

We note here that it is most often the case that we can take $l_c = 1$ and $A^1_c = B^1_c = \emptyset$. The values of $c$ for which this is not the case in some sense explain why the determination of the conjugacy classes of $U(q)$ is complicated in general. We also remark that it is often the case that $X_c = \emptyset$, which corresponds to the case $l_c = 0$.

We now introduce some notation needed in order to say how the sets $A^i_c$ and $B^i_c$ are determined in the algorithm. Let $y_1, \ldots, y_N$ be variables. We may write:

$$\left[ \sum_{j=1}^N y_j e_{\beta_j}, x_c(t) \right] = \sum_{j=1}^N \sum_{k=1}^N P^c_{jk}(t) y_k e_{\beta_j},$$

where each $P^c_{jk}(t) \in \mathbb{Z}[t_1, \ldots, t_m]$ is linear: this is easily achieved using the Chevalley commutator relations for $g_C$. It is then the case that $\dim c_u(x_c(\tau) + m_i)$ is the dimension of the solution space of the system of linear equations:

$$\sum_{k=1}^N P^c_{jk}(\tau) y_k = 0,$$

for $j = 1, \ldots, i$.

We are now in a position to describe our algorithm. It calculates sets of polynomials that determine certain choices of the varieties $X^i_c$. These polynomials are calculated using the $\mathbb{Z}$-form $u_\mathbb{Z}$ of $u$ and they have rational coefficients. Observe that in step (2)(a) of the algorithm there may be “implicit divisions” by certain primes for which the output will not be valid. This is discussed in more detail later in this section.

The algorithm uses a backtrack algorithm with a depth-first search to calculate certain choices of sets $A^i_c$ and $B^i_c$ that determine the varieties $X^i_c$ for each $c \in \{I, R_n, R_0\}^N$, and $i = 1, \ldots, l_c$. In the algorithm we require a total order on $\mathbb{Z}[t_1, \ldots, t_m]$; we use the order defined in precedence by the number of terms, total degree and leading coefficient (with respect to the degree then lexicographic order on monomials).

The variables used in the algorithm are:
the “current string” $c$ is an element of $\{I, R_n, R_0\}^i$ for some $i$ and determines $x_c(t) \in \tilde{u}$ as above;

- the set of “satisfied” polynomials $A$ is a subset of $\mathbb{Z}[t_1, \ldots, t_m]$;
- the set of “unsatisfied” polynomials $B$ is a subset of $\mathbb{Z}[t_1, \ldots, t_m]$;
- the matrix $Q(t)$ is an element of $\text{Mat}_{i,N}(\mathbb{Z}[t_1, \ldots, t_m])$, which is obtained from the matrix $(P_{jk}(t))$ by “row reducing” the first $i$ rows;
- the “pivot string” $\pi$ is an element of $\{0, 1, \ldots, N\}^i$, which “records the columns used in the row reductions”;
- the stack $S = \{(c, A, B, \pi, Q(t))\}$ is an (ordered) subset of $\mathbb{N} \times \mathbb{P}(\mathbb{Z}[t_1, \ldots, t_m])^2$.

Here, $\mathbb{P}$ denotes the power set.

The stack is required to be ordered as the algorithm takes elements from the “top” of the stack. The element at the top is the one that has been most recently added and is denoted by $\text{top}(S)$. The initial configuration in the algorithm is as follows:

- $c := (R_n)$;
- $A := \emptyset$;
- $B := \emptyset$;
- $\pi := (0)$;
- $Q(t)$ is the $1 \times N$ matrix with all entries equal to 0;
- $S := \{(R_0, \emptyset, \emptyset, (0), Q(t))\}$; and
- $O := \emptyset$.

Now we explain the next step in the algorithm; we have numbered the steps in the algorithm, so that we can refer back to it in the explanation given afterwards.

1. If the length of $c$ is $N$, then we are finished with this string. We set:
   
   (a) $O := O \cup \{(c, A, B)\}$.
   
   If $S = \emptyset$, then we finish.
   
   Else we make the following changes to the variables:
   
   (b) $(c, A, B, \pi, Q) := \text{top}(S)$; and
   
   (c) $S := S \setminus \{\text{top}(S)\}$.

2. If the length of $c$ is $i - 1 < N$, then we proceed by making the $i$th row reduction for the matrix $(P_{jk}^c(t))$ as defined above. Note that $Q(t)$ is the matrix resulting from the first $i - 1$ row reductions. We first append the $i$th row of $(P_{jk}^c(t))$ to $Q(t)$ and then make the row reduction as follows:
   
   (a) for $j = 1, \ldots, i - 1$, if $\pi_j \neq 0$ we set $Q_i(t) := Q_{i,\pi_j}(t)Q_j(t) - Q_{i,\pi_j}(t)Q_j(t)$, where $Q'_{j,\pi_j}(t)$ is $Q_{j,\pi_j}(t)$ divided by the highest common factor of $Q_{j,\pi_j}(t)$ and $Q_{i,\pi_j}(t)$, and $Q'_{i,\pi_j}(t)$ is defined analogously.
Let $R_i$ be the set of non-zero polynomials in the $i$th row of $Q(t)$ that are not divisible by any element of $A$. We next consider three cases:

(b) $R_i = \emptyset$. We update the variables as follows:

(i) $\pi := (\pi, 0)$;

(ii) $c := (c, R_n)$; and

(iii) $S := S \cup \{(c, R_0), A, B, \pi, Q\}$.

(c) $R_i \neq \emptyset$ and there is some non-zero element of $R_i$ that is a monomial or divides some element of $B$. We let $Q_{l,i}(t)$ be the least such polynomial with respect to our chosen order on the set of all polynomials. We update the variables as follows:

(iv) $c := (c, I)$; and

(v) $\pi := (\pi, l)$.

(d) Otherwise, we pick a least element $Q_{l,i}(t)$ of $R_i$. We update the variables as follows:

(vi) $S := S \cup \{(c, A \cup \{Q_{l,i}(t)\}, B, \pi, Q\}$;

(vii) $c := (c, I)$;

(viii) $B := B \cup \{Q_{l,i}(t)\}$; and

(ix) $\pi := (\pi, l)$.

The output of the algorithm is a collection of triples $(c, A, B)$. Each of these triples determines a subvariety

$$X_{c,A,B} = \left\{ \sum_{j : c_j = R_n} \tau_j e_{\beta_{c,j}} \in X_c \mid f(\tau) = 0 \text{ for all } f \in A \text{ and } g(\tau) \neq 0 \text{ for all } g \in B \right\}$$

(3.1)

of $X_c$. For fixed $c$, we can write $A^l_c, \ldots, A^{l_c}_c$ and $B^l_c, \ldots, B^{l_c}_c$ for the sets $A$ and $B$ occurring in a triple $(c, A, B)$. These are the determined choice of the sets $A^l_c$ and $B^l_c$ such that $X_c$ is the disjoint union of the varieties $X^i_c$ as defined earlier; each $X^i_c$ is equal to the corresponding $X_{c,A,B}$.

We now explain why the output of our algorithm can be used to determine all the minimal representatives of the $U(q)$-orbits in $u(q)$, for almost all primes $p$. As explained above there may be “implicit divisions” in step (2)(a) of the algorithm leading to a finite number of primes for which we cannot determine the minimal representatives of the $U(q)$-orbits in $u(q)$. We choose not to give a formal proof of the correctness of the algorithm as this would be very technical and just give an outline. We argue by induction on $i$ to show that the algorithm determines varieties $X_{c,A,B,i}$ (as defined below) from which one can calculate all the minimal representatives of the $U(q)$-orbits in $u_i(q)$ for each $i$ (and valid $p$). We do not discuss the part of the algorithm dealing with the row reductions in (2)(a), as this is elementary.

In the discussion below we often speak of all relevant $\tau \in k^m$. When considering the triple $(c, A, B)$, this means all $\tau \in k^m$ such that $f(\tau) = 0$ for all $f \in A$ and $g(\tau) \neq 0$ for all $g \in B$, i.e. all $\tau \in k^m$ for which $x_c(\tau) \in X_{c,A,B}$.

We need to explain our inductive hypothesis as this is not compatible with the depth-first search used in the algorithm. To do this we must consider all triples $(c, A, B)$ with $c$ of length $i$ that occur during the running of the algorithm. For such $(c, A, B)$ varieties, $X_{c,A,B}$ can be defined as in (3.1). Then the inductive hypothesis says that the varieties $X_{c,A,B,i} = \{x + m_i \mid x \in X_{c,A,B}\}$ give all minimal representatives of the $U$-orbits in $u_i$.

So assume the inductive hypothesis for $i - 1$. Then we have to show that for each $(c, A, B)$ with $c$ of length $i$ and all minimal representatives of $U$-orbits in $u_i$ of the form $x_c(\tau) + \lambda e_{\beta_i} + m_i$ with $x_c(\tau) \in X_{c,A,B}$ and $\lambda \in k$, we have that $x_c(\tau) + \lambda e_{\beta_i}$ lies in some variety of the form
of the torus $T$ where $c' = (c, Z)$ and $Z \in \{I, R_0, R_n\}$. To do this we have to consider the steps in (2) in the algorithm.

After the row reduction made in (2)(a), we have the row reduced matrix $Q(t)$ and the set $R_i$. As explained earlier we have that $i$ is an inert point of $x_c(\tau) \in X_{c,A,B}$ if and only if $\dim T_{u}(x_c(\tau) + m_i) = \dim c_{u}(x_c(\tau) + m_i) - 1$. Also the dimension of $c_{u}(x_c(\tau) + m_i)$ is the rank of the row reduced matrix $Q(\tau)$. Therefore, $i$ is an inert point of $x_c(\tau)$ if and only if $f(\tau) \neq 0$ for some $f \in R_i$. Note that we only have to consider the polynomials in $R_i$, because for any non-zero entry in the $i$th row of $Q(t)$ that is divisible by some polynomial $f(y) \in B$, we automatically have $f(\tau) = 0$ for all relevant $\tau$. At this stage one ideally wants to determine for which values of $\tau$ there is some $f \in R_i$ with $f(\tau) \neq 0$. However, this is a difficult task if $R_i$ contains several polynomials. Also we require a fixed value for the next entry of $\tau$, so that we can perform the row reductions later in the algorithm. So we proceed by considering the three cases (2)(b)–(d) in the algorithm:

(b) In this case it is clear that $i$ is a ramification point of $x_c(\tau)$ for all relevant $\tau$. Therefore, $x_c(\tau) + \lambda e_{\beta_i} + m_i$ is the minimal representative of its $U$-orbit in $u_i$ for all $\lambda \in k$. We see that the strings $c'$ corresponding to all such minimal representatives are passed on in the program: the non-zero values of $\lambda$ correspond to the updated string $c' = (c, R_n)$ in (ii), and the case $\lambda = 0$ corresponds to the string $c' = (c, R_0)$ added to the stack.

(c) For this case, it is clear that $f(\tau) \neq 0$ for all relevant $\tau$, where $f(t) = Q_{i,t}(t)$ is the least element of $R_i$ that is either a monomial or divides some element of $B$. Therefore, $i$ is an inert point of $x_c(\tau)$ for all relevant $\tau$. Thus the only minimal representative of its $U$-orbit in $u_i$ of the form $x_c(\tau) + \lambda e_{\beta_i} + m_i$ is when $\lambda = 0$. This minimal representative corresponds to the updated string $c' = (c, I)$ in (iv).

(d) This case is more complicated. We consider the least element $f(t) = Q_{i,t}(t)$ of $R_i$ and the following cases for $\tau$:

(I) If $f(\tau) \neq 0$, then $i$ is an inert point for $x_c(\tau)$. For such $\tau$, the only minimal representative of its $U$-orbit in $u_i$ of the form $x_c(\tau) + \lambda e_{\beta_i} + m_i$ when $\lambda = 0$. This minimal representative corresponds to the updated string $c' = (c, I)$ in (vii), along with $f$ being added to $B$ in (viii).

(II) If $f(\tau) = 0$, then we cannot say whether $i$ is an inert or ramification point of $x_c(\tau)$. The element $(c, A \cup \{f\}, B, \pi, Q(t))$ added to the stack in (vi) will be considered later in the algorithm. For this case $R_i$ will have fewer elements, and will be considered again either in case (a) or (c). This process will finish, and it is straightforward to see that when this happens all the required triples $(c', A', B')$ with $c' = (c, Z)$, will have occurred.

Putting together the above case analysis, verifies the inductive step. Therefore, for all minimal representatives $x_c(\tau)$ of the $U$-orbits in $u$, a corresponding triple $(c, A, B)$ occurs at some point of the program. This triple is added to the output in (1)(a), which means that the output of the algorithm determines all minimal representatives, as claimed.

Next we discuss two modifications to the algorithm that we make for its implementation in GAP. These changes are made in order to speed up the computations; we chose not to include them in the above description of the algorithm for simplicity.

Our first modification allows us to reduce the complexity by using the action of the maximal torus $T$ to “normalize” certain coefficients to be equal to 1. Let $c \in \{I, R_0, R_n\}^N$ and let $x_c(t) = \sum_{i=1}^{m_c} t^i e_{\beta_{c,i}}$ be defined as above. Suppose that $\{\beta_{c,i} \mid i \in J\}$ is linearly independent for some $J \subseteq \{1, \ldots, m_c\}$. Then for every $\tau \in (k^x)^{m_c}$ there is some $\sigma \in (k^x)^{m_c}$ with $\sigma_i = 1$ for all $i \in J$
and $x_c(\tau)$ is conjugate to $x_c(\sigma)$ via $T$, i.e. the action of $T$ can be used to “normalize $\tau_i = 1$” for all $i \in J$. In the computer program we replace $x_c(t)$ by $\sum_{i \in J} e_{p_{c,i}} + \sum_{i \notin J} t_i e_{p_{c,i}}$ thereby reducing the number of indeterminates required. This speeds up the row reduction of the matrix $Q(t)$ significantly. Some care is needed when using this modification, as the centralizer $C_T(x_c(\tau))$ can be disconnected. If this is the case then it can become difficult to determine the elements of $X_c(q)$ from the “normalized” elements of $X_c$. This problem can be resolved by not allowing certain normalizations; we omit the technical details here.

The second adaptation deals with “easy” elements of the set $A$. If there is a linear polynomial in $A$, then we may simplify future checks by “making a substitution.” If $t_i - a(t) \in A$, where $i \in \{1, \ldots, m_c\}$ and $a(t)$ is a linear polynomial not involving $t_i$, then we can make the substitution $t_i = a(t)$ in $x_c(t)$, in the polynomials in $A$ and $B$, and in the matrix $Q(t)$; we then remove $t_i - a(t)$ from $A$. This modification reduces the number of indeterminates, and also the number of polynomials in $A$. This helps to speed up the program.

We next explain a check that has to be included in the program to see which primes the output is valid for. When making the row reductions in (2) of our algorithm, there are implicit divisions by certain integers. Essentially, we need to be able to divide by the polynomials in $A$ is valid for. When making the row reductions in (2) of our algorithm, there are implicit divisions $p$ by the primes output by the program, then it would be straightforward to adapt the program to insist that there are no implicit divisions by $p$, and running this modification would give an output valid for $p$.

The algorithm is implemented in GAP with the two modifications and the check for primes. This is achieved using the functions for Lie algebras and polynomial rings in GAP. This allows us to define $U(q)$ within GAP and therefore allows us to obtain the matrices $(P_{jk}^c(t))$ that we row reduce using the method given in (2)(a) of the algorithm. The implementation is based on the algorithm given and the two modifications discussed above. We choose not to include any of the technical details.

In the next section we present the values of $k(U(q))$ that we have calculated from the output of our program. Each of the varieties $X_{c,A,B}$ is defined by polynomials with integer coefficients, so is defined over $\mathbb{F}_p$. We have

$$k(U(q)) = \sum_{(c,A,B)} |X_{c,A,B}(q)|.$$  

We can, therefore, calculate $k(U(q))$ by calculating $|X_{c,A,B}(q)|$ for all triples $(c, A, B)$. If the polynomials in $A$ and $B$ are not too complicated, then this can be achieved quite easily. We discuss this below.

It is most commonly the case that $c$ occurs in just one triple $(c, A, B)$ for which both $A$ and $B$ are empty. In which case it is easily seen that $X_c = X_{c,A,B}$ and $|X_c(q)| = (q - 1)^{mc}$. The next simplest case is when $A \cup B$ has one element that is linear. For example, consider the polynomial $t_1 - 1$: if $A = \{t_1 - 1\}$ and $B = \emptyset$, then $|X_{c,A,B}(q)| = (q - 1)^{mc-1}$; and if $A = \emptyset$ and $B = \{t_1 - 1\}$, then $|X_{c,A,B}(q)| = (q - 1)^{mc-1}(q - 2)$. More complicated sets $A$ and $B$ that we need to consider require a little thought to calculate $|X_{c,A,B}(q)|$.

As the rank of $G$ increases the polynomials become more complicated. For the $F_4$, $B_5$ and $C_5$ cases we get a number of quadratic polynomials. For the rank 6 cases, the polynomials become more complicated still and the number of triples $(c, A, B)$ with $A$ or $B$ non-empty gets large.
explained by the action of the split maximal torus $A$.

From the output of the program we can view all the polynomials occurring in the sets $A$ and $B$. One can check that for each of these polynomials it is possible to “solve for one indeterminate in terms of the others.” With a little further consideration one can see that this means each of the sets $X_{c,A,B}(q)$ has size a polynomial in $q$. However, the number of such $X_{c,A,B}(q)$ is so large that it would be rather time consuming to calculate $k(U(q))$ explicitly.

### 4. Results

In this final section we present some explicit results of our computations and go on to discuss some interesting features of the output.

In Table 1 we present the polynomials $k(U(q))$ for $G(q)$ of rank at most 5; in this table we let $v = q - 1$ to save space. We include the values for $G$ of type $A_r$ for completeness, though these polynomials have been known for some time, thanks to the work of Vera-López and Arregi referred to in the introduction. Also, as discussed below, the value of $k(U(q))$ is the same for $G$ of type $B_r$ and $C_r$, so we only include this polynomial once.

We make some comments about the polynomials in Table 1. We start by making the observation that $k(U(q))$ considered as a polynomial in $v = q - 1$ has non-negative coefficients. For the case $G$ is of type $A_r$ ($r \leq 12$), this was observed by Vera-López and Arregi in [19]. It would be interesting to have a geometric explanation of these positivity phenomena.

We give a heuristic idea why this occurs for the cases that we have calculated by considering the partition of the conjugacy classes used in our algorithm. As discussed at the end of the previous section, the number of $\mathbb{F}_q$-rational points of the varieties $X_c^i$ is most commonly $|X_c^i(q)| = v^{m_c}$. Although there are some values of $c$ and $i$ for which $|X_c^i(q)|$ is a polynomial in $v$ with negative coefficients, these negative coefficients are few enough so that they are canceled by the families of size $v^{m_c}$.

We observe that the constant coefficient in $k(U(q))$ as a polynomial in $v$ is always 1. This is explained by the action of the split maximal torus $T$ of $G$ on each $X_c$ for all $c \in \{I, R_n, R_0\}^N$. This action is non-trivial unless $c_i = R_0$ for all $i$, so that $X_c = \{0\}$. It is easy to see that if $X_c \neq \{0\}$,

<table>
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<td>$k(U(q))$, as polynomials in $v = q - 1$.</td>
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<tr>
<td>$G$</td>
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<td>$B_5/C_5$</td>
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then the orbits of $T(q)$ on $X_c(q)$ are all of size $v^a/b$ for some $a, b \in \mathbb{Z}_{\geq 1}$, so $|X_c(q)|$ is divisible by $v^a/b$. This implies that the constant coefficient in $k(U(q))$ as a polynomial in $v$ must be 1 (corresponding to the zero orbit).

We now comment on the fact that the value of $k(U(q))$ is the same for $G$ of type $B_r$ and $C_r$, for $r = 3, 4, 5$. One can see that the groups $U(q)$ are not isomorphic for $G$ of type $B_r$ and $C_r$; thanks to a result of A. Mal'cev [16], the maximal size of an abelian subgroup of $U(q)$ is different for $G$ of type $B_r$ and $C_r$. Using the variation of our program discussed below, one can also show that the number of $U(q)$-conjugacy classes in the derived subgroup $U^{(1)}(q)$ of $U(q)$ are different for $G$ of types $B_r$ and $C_r$, for $r = 3, 4, 5$. It would be interesting to have a reason for the coincidences in the numbers $k(U(q))$; we expect it should be explained by the duality of the root systems of type $B_r$ and $C_r$, see for example [3, Chapter 4] for similar phenomena.

As mentioned in the introduction, we have adapted our program to consider the action of $U$ on certain subquotients $M/N$. The adaptation is valid when $M \supseteq N$ are normal subgroups of $B$ contained in $U$. The algorithm runs in essentially the same way: one has to replace the filtration of $u$ by an analogous filtration of $m/n$, then change the initial configuration and the point at which variables are added to the output set $O$ accordingly.

In Table 2 we give some values of $k(U(q), U^{(l)}(q))$ for $G$ of exceptional type. We recall that the descending central series of $U$ is defined by $U^{(0)} = U$ and $U^{(l)} = [U^{(l-1)}, U]$ for $l \geq 1$. The cases that we have included are those for which we are able to compute $k(U(q), U^{(l)}(q))$ in a reasonable amount of time and for which there is an infinite number of $B$-orbits in $u^{(l)} = \text{Lie } U^{(l)}$; we refer the reader to [10] for a classification of all cases when there is only a finite number of $B$-orbits in $u^{(l)}$ for $G$ of exceptional type.

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