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Well-posedness results for triply nonlinear degenerate parabolic equations

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We study well-posedness of triply nonlinear degenerate elliptic–parabolic–hyperbolic problems of the kind

$$b(u)_t - \operatorname{div} \tilde{\mathfrak{a}}\big(u, \nabla\phi(u)\big) + \psi(u) = f, \qquad u|_{t=0} = u_0$$

in a bounded domain with homogeneous Dirichlet boundary conditions. The nonlinearities b, ϕ and ψ are supposed to be continuous non-decreasing, and the nonlinearity $\tilde{\mathfrak{a}}$ falls within the Leray–Lions framework. Some restrictions are imposed on the dependence of $\tilde{\mathfrak{a}}(u, \nabla\phi(u))$ on u and also on the set where ϕ degenerates. A model case is $\tilde{\mathfrak{a}}(u, \nabla\phi(u)) = \tilde{\mathfrak{f}}(b(u), \psi(u), \phi(u)) + k(u)\mathfrak{a}_0(\nabla\phi(u))$, with a nonlinearity ϕ which is strictly increasing except on a locally finite number of segments, and the nonlinearity $\mathfrak{a}_0$ which is of the Leray–Lions kind. We are interested in existence, uniqueness and stability of L^∞ entropy solutions. For the parabolic–hyperbolic equation ($b = \mathrm{Id}$), we obtain a general continuous dependence result on data u_0, f and nonlinearities $b, \psi, \phi, \tilde{\mathfrak{a}}$. Similar result is shown for the degenerate elliptic problem, which corresponds to the case of $b \equiv 0$ and general non-decreasing surjective ψ. Existence, uniqueness and continuous dependence on data u_0, f are shown in more generality. For instance, the assumptions $[b+\psi](\mathbf{R}) = \mathbf{R}$ and the continuity of $\phi \circ [b+\psi]^{-1}$ permit to

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achieve the well-posedness result for bounded entropy solutions of this triply nonlinear evolution problem.

1. Introduction

1.1. Problem and assumptions

In this paper we consider problems under the general form

$$\text{(P)}\qquad \begin{cases} \partial_t b(u) + \operatorname{div}\tilde{\mathfrak{f}}\big(b(u), \psi(u), \phi(u)\big) - \operatorname{div}\mathfrak{a}\big(u, \nabla\phi(u)\big) + \psi(u) = f & \text{in } Q_T = (0,T)\times\Omega, \\ u|_{t=0} = u_0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Sigma = (0,T)\times\partial\Omega, \end{cases}$$

where $u : (t,x) \in Q_T \to \mathbf{R}$ is the unknown function, $T > 0$ is a fixed time, $\Omega \subset \mathbf{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$.

We assume

(H₁) the functions $b, \psi, \phi : \mathbf{R} \mapsto \mathbf{R}$ are continuous non-decreasing, normalized by the value zero at the point zero.

We require the following technical assumption on ϕ:

(H₂) there exists a closed set $E \subset \mathbf{R}$ such that ϕ is strictly increasing on $\mathbf{R} \setminus E$, and the Lebesgue measure $\operatorname{meas}\phi(E)$ is zero;

and, moreover,

$$\text{(H}_3\text{)}\qquad \liminf_{\varepsilon\downarrow 0} \frac{\operatorname{meas}(G^\varepsilon)}{\varepsilon} < +\infty, \quad \text{where } G^\varepsilon := \big\{ z \in \mathbf{R} \,\big|\, \operatorname{dist}\big(z, \phi(E)\big) < \varepsilon \big\}.$$

Notice that since $\phi(\cdot)$ is continuous and strictly monotone on $\mathbf{R} \setminus E$, the set $G := \phi(E)$ is also closed.

Remark 1.1. (i) Hypotheses (H_2), (H_3) are trivially satisfied if ϕ is a strictly increasing function. In case ϕ has a finite number of segments on which it keeps constant values, E is just the union of all these "flatness segments", and (H_2), (H_3) are satisfied.

(ii) Property (H_2) is still true if ϕ is locally absolutely continuous. In general, the set of discontinuity points of ϕ^{-1} is not closed, and its closure can be large (this is the case, e.g., if ϕ is the "Cantor stairs function"). Thus (H_2) is a restriction, although it is fulfilled in most of the practical cases. Property (H_3) is a further restriction. Indeed, consider the following example. It is easy to construct a Lipschitz continuous non-decreasing function ϕ such that $G = \phi(E)$ is equal to $\{0\} \cup \{1/\sqrt{i} \mid i \in \mathbb{N}\}$. A straightforward calculation shows that for $\varepsilon = 1/n$, G^ε contains the whole interval $[0, 1/(2n^{2/3})]$; in this case $\operatorname{meas}(G^\varepsilon)/\varepsilon$ is of order $\varepsilon^{-1/3}$ and gets unbounded as $\varepsilon \to 0$.

The initial function $u_0 : \Omega \to \mathbf{R}$ and the source $f : Q \to \mathbf{R}$ are assumed to fulfill

(H₄) $u_0 \in L^\infty(\Omega)$; f is measurable such that $f(t,\cdot) \in L^\infty(\Omega)$ for a.e. $t \in (0,T)$ and $\displaystyle\int_0^T \big\| f(t,\cdot) \big\|_{L^\infty(\Omega)}\, dt < +\infty$.[1]

[1] In the sequel, we will abusively denote the latter quantity by $\|f\|_{L^1(0,T;L^\infty(\Omega))}$.

Furthermore, the following condition (automatically satisfied in the case $b(\mathbf{R})=\mathbf{R}$) is needed:

$$(\mathrm{H}_5)\qquad \left|\begin{array}{l}\text{in the case } b(+\infty)\neq+\infty\ \big(\text{resp., } b(-\infty)\neq-\infty\big)\\ \text{one has } \psi(+\infty)=+\infty \text{ and } f^+\in L^\infty(Q_T)\ \big(\text{resp., } \psi(-\infty)=-\infty \text{ and } f^-\in L^\infty(Q_T)\big).\end{array}\right.$$

Assumptions (H_4), (H_5) are imposed in order to limit our study to bounded solutions of (P).

Note that in view of (H_1) and (H_5), we are assuming at least that $(b+\psi)(\mathbf{R})=\mathbf{R}$. An important part of the paper is devoted to the case $b(\mathbf{R})=\mathbf{R}$. If, in addition, b is bijective, then performing a change of the unknown one puts the problem into the doubly nonlinear framework with $b=\mathrm{Id}$.

Our continuous dependence result for problem (P) (in which we perturb both the data and the nonlinearities) concerns the case where the structure condition

$$(\mathrm{H}_{\mathrm{str}})\qquad b(r)=b(s)\quad\Rightarrow\quad \phi(r)=\phi(s)$$

is satisfied. This result implies the existence of solutions for (P), by reduction to non-degenerate problems. Assumption $(\mathrm{H}_{\mathrm{str}})$ is trivially satisfied in the case $b=\mathrm{Id}$. If $(\mathrm{H}_{\mathrm{str}})$ fails, the convergence of approximate solutions to (P) is known only for a particular monotone approximation method developed by Ammar and Wittbold [4]. This approach leads to an existence result which bypasses $(\mathrm{H}_{\mathrm{str}})$; but interesting issues (such as proving convergence of numerical methods for (P) without requiring the structure condition $(\mathrm{H}_{\mathrm{str}})$) remain open. See Bénilan and Wittbold [14] for a thorough discussion of the role of the structure condition for a simple model one-dimensional case $\partial_t b(u)+(\mathfrak{f}(u))_x=u_{xx}$.

Furthermore,

$$(\mathrm{H}_6)\qquad \text{the function } \tilde{\mathfrak{f}}:\mathbf{R}\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}^N \text{ is assumed merely continuous.}$$

Notice that under the structure condition $(\mathrm{H}_{\mathrm{str}})$, the dependency of $\tilde{\mathfrak{f}}$ on $\phi(u)$ can be dropped. Whenever it is convenient (and in particular, in the case where b is bijective), we use the notation $\mathfrak{f}(\cdot):=\tilde{\mathfrak{f}}(b(\cdot),\psi(\cdot),\phi(\cdot))$.

The function $\mathfrak{a}:\mathbf{R}\times\mathbf{R}^N\to\mathbf{R}^N$ is assumed to satisfy the following conditions:

$$(\mathrm{H}_7)\qquad \mathfrak{a} \text{ is continuous}^2 \text{ on } \mathbf{R}\times\mathbf{R}^N,\ \text{ and } \mathfrak{a}(r,0)\equiv 0;$$

$$(\mathrm{H}_8)\qquad \left|\begin{array}{l}\mathfrak{a}(r,\cdot) \text{ is monotone, i.e.,}\\ \big(\mathfrak{a}(r,\xi)-\mathfrak{a}(r,\eta)\big)\cdot(\xi-\eta)\geqslant 0 \text{ for all } \xi,\eta\in\mathbf{R}^N \text{ and all } r\in\mathbf{R};\end{array}\right.$$

$$(\mathrm{H}_9)\qquad \left|\begin{array}{l}\mathfrak{a}(r,\cdot) \text{ is coercive at zero, i.e.,}\\ \text{there exist } p\in(1,+\infty) \text{ and } C\in C(\mathbf{R};\mathbf{R}^+) \text{ such that}\\ \mathfrak{a}(r,\xi)\cdot\xi\geqslant\dfrac{1}{C(r)}|\xi|^p \text{ for all } \xi\in\mathbf{R}^N \text{ and all } r\in\mathbf{R};\end{array}\right.$$

$$(\mathrm{H}_{10})\qquad \left|\begin{array}{l}\text{the growth of } \mathfrak{a}(r,\xi) \text{ is not greater than } |\xi|^{p-1}, \text{ i.e.,}\\ \text{there exists } C\in C(\mathbf{R};\mathbf{R}^+) \text{ such that}\\ \big|\mathfrak{a}(r,\xi)\big|\leqslant C(r)\big(1+|\xi|^{p-1}\big) \text{ for all } \xi\in\mathbf{R}^N \text{ and all } r\in\mathbf{R}.\end{array}\right.$$

It follows from (H_7)–(H_{10}) that for all r, the operator $w\mapsto -\operatorname{div}\mathfrak{a}(r,\nabla w)$ is an operator acting from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$, where $p'=\frac{p}{p-1}$. Since the work of Leray and Lions [29], this assumption became classical. It can be generalized to the framework of Orlicz spaces (see the works of Kačur [27] and those of Benkirane and collaborators [15,16]), and even to more general coercivity and growth assumptions. We refer to Bendahmane and Karlsen [10,11] for the case of the anisotropic p-Laplacian. In the case of dimension $N=1$, very general coercivity assumption $\lim_{\xi\to\pm\infty}\mathfrak{a}(r,\xi)/\xi=+\infty$ (uniformly for r bounded) can be considered. In this framework, the well-posedness for (P) was already established by Ouaro and Touré [37–39] and Ouaro [36] (see also Bénilan and Touré [13]); notice that some essential arguments of these works are specific to the case $N=1$. Notwithstanding the above

[2] The assumption of continuity of $\mathfrak{a}$ in the first variable can be relaxed: see Remark 1.3.

generalizations, the classical Leray–Lions assumptions are sufficient for us to illustrate the arguments of the existence proof for (P).

The relevant technical assumption in order to have uniqueness is

$$(\mathrm{H}_{11})\qquad \left|\begin{array}{l}\text{there exists } C \in C(\mathbf{R}^2;\mathbf{R}^+) \text{ such that}\\ \big(\mathfrak{a}(r,\xi)-\mathfrak{a}(s,\eta)\big)\cdot(\xi-\eta)+C(r,s)\big(1+|\xi|^p+|\eta|^p\big)\big|\phi(r)-\phi(s)\big|\geqslant 0\\ \text{for all } \xi,\eta\in\mathbf{R}^N \text{ and all } r,s\in\mathbf{R} \text{ such that}\\ \text{the segment which lies between } r \text{ and } s\\ \text{does not intersect the exceptional set } E.\end{array}\right.$$

This assumption goes along the lines of Carrillo and Wittbold [21] and combines the monotonicity condition (H_8) with a kind of Lipschitz continuity assumption on $\mathfrak{a}(\cdot,\xi)\circ\phi^{-1}$ on the connected components of $\mathbf{R}\setminus E$.

Remark 1.2. Notice that E is a set of the values of u that lead to $\mathfrak{a}(u,\nabla\phi(u))$ being zero. Indeed, since $\operatorname{meas}\phi(E)=0$, we have $\nabla\phi(u)=0$ a.e. on the set where $u\in E$; then we have $\mathfrak{a}(u,\nabla\phi(u))=\mathfrak{a}(u,0)=0$, regardless of the exact value of $u\in E$.

Remark 1.3. Notice that we do not assume the structure condition

$$\phi(r)=\phi(s)\quad\Rightarrow\quad \mathfrak{a}(r,\xi)=\mathfrak{a}(s,\xi)\quad \forall\xi\in\mathbf{R}^N.$$

This means that $\mathfrak{a}(r,\xi)$ can be discontinuous with respect to $\phi(r)$; the set of discontinuities is included in $\phi(E)$ which, by (H_2), is a closed set of measure zero. This technical assumption is needed to be able to "cut off" the discontinuity set.

One can also consider $\mathfrak{a}(r,\xi)$ which is discontinuous in r. E.g., take the case of $\mathfrak{a}(r,\xi)=\mathfrak{a}_1(k(r),\xi)$ with $k(\cdot)$ piecewise continuous. Thanks to Remark 1.2, it is reduced to our setting by a change of unknown function u into v such that $u=\rho(v)$ with ρ non-strictly increasing, chosen so that $k(\rho(z))\equiv\tilde{k}(z)$ with $\tilde{k}(\cdot)$ continuous. Indeed, such change of the unknown preserves assumption (H_{str}).

Let us mention that the assumptions (H_4), (H_5) and (H_9), (H_{10}) can also be generalized. Within the framework of "variational" solutions, one usually works with "bounded energy initial data", i.e., with u_0 measurable and such that $B(u_0)<+\infty$, where

$$B(z):=\int_0^z \phi(s)\,db(s) \tag{1}$$

is the function depending on b and ϕ which we introduce following Alt and Luckhaus [1], and with relaxed growth and coercivity assumptions allowing for additional terms which depend on $B(r)+|\phi(r)|^p+r\psi(r)$, these terms being controlled by means of *a priori* estimates (see, e.g., [1,5,27]).

A more general framework is the one of renormalized solutions (see [4,10,17–19,21] and the references therein). Nonetheless, the assumptions (H_4), (H_5), (H_9), and (H_{10}) are sufficiently weak to provide the starting point for the well-posedness theory for renormalized solutions of (P). Indeed, the uniqueness proof for renormalized solutions of (P) remains essentially the same as the one of Carrillo and Wittbold in [21], and the existence result is most easily obtained using bi-monotone sequences of bounded entropy solutions, following Ammar and Wittbold [4] and Ammar and Redwane [3].

1.2. The notion of solution and known results

Problem (P) is of mixed elliptic–parabolic–hyperbolic type, and thus combines the difficulties related to nonlinear conservation laws with those related to nonlinear degenerate diffusion equations.

We refer to Kruzhkov [28] and to Leray and Lions [29], Lions [30], Alt and Luckhaus [1], Otto [35] for the fundamental works on these classes of equations, respectively.

One consequence is that the notion of weak solution (sometimes called "variational solution") generally leads to non-uniqueness, unless $\phi(\cdot)$ is strictly increasing. The notion of entropy solution we use is adapted form the founding paper of Carrillo [20], which extends the classical framework of entropy solutions to scalar conservation laws to the case of problem (P) with the linear diffusion $\mathfrak{a}(u, \nabla\phi(u)) = \nabla\phi(u)$. The uniqueness arguments of [20] were adapted by Carrillo and Wittbold [21] to the case of a nonlinear Leray–Lions diffusion operator of the form $\mathfrak{a}(u, \nabla u)$ corresponding to $\phi = \mathrm{Id}$. The case of $b = \mathrm{Id}$ and of a diffusion operator of the form $\mathfrak{a}(\phi(u), \nabla\phi(u))$ is similar; one particular case is considered in Andreianov, Bendahmane and Karlsen [6]. Both frameworks are sometimes referred to as "doubly nonlinear". Notice that a new definition of an entropy solution, suitable also for the case of doubly nonlinear anisotropic diffusion operators, was used in a series of works by Bendahmane and Karlsen (see [10,11] and references therein); the issue of existence in this general anisotropic framework is still open. The case of triply nonlinear problems of the form (P) has been first addressed by Ouaro and Touré (see [39] and the references therein) and Ouaro [36]. Well-posedness results are obtained in dimension $N = 1$, under very general coercivity conditions; see also the works of Bénilan and Touré ([13] and the references therein). The multi-dimensional elliptic analogue of (P) was recently addressed in the work of Ammar and Redwane [3], in the framework of renormalized solutions; their approach is quite similar to the ours, but the proofs of [3] require a special structure of the nonlinearity ϕ. We bypass the difficulties of [3] using two observations in Section 3 (see Lemmas 3.2, 3.3).

In most of the works cited hereabove, the homogeneous Dirichlet boundary conditions were chosen. One should bear in mind that, unless ϕ is strictly increasing, the boundary condition is also understood in an entropy sense (see, e.g., Bardos, LeRoux and Nédélec [9], Otto [34], Carrillo [20]). Focusing on the homogeneous boundary condition is a simplification which seems to be not merely technical. A partial extension of the Carrillo's techniques to inhomogeneous boundary data can be found in Ammar, Wittbold and Carrillo [2]. Different techniques for the inhomogeneous problem, based on the weak trace framework introduced by Otto [34] and developed by Chen and Frid [22], were used by Mascia, Porretta and Terracina [32], by Michel and Vovelle [33], and by Vallet [41]. A related, though somewhat more straightforward approach was attempted by Andreianov and Igbida [7]. Notice that in all cases, a technique of "going up to the boundary" is preceded by obtaining the fundamental weak formulations and entropy inequalities "inside the domain". In the present paper, we also make the simplest choice of the homogeneous Dirichlet data, and focus on deriving the entropy inequalities.

1.3. Main techniques and the outline of the paper

Our main concern is the existence for (P) (and more generally, the continuous dependence result with respect to perturbations of the data and the nonlinearities) in the case $b = \mathrm{Id}$ (or, more generally, under the structure condition ($\mathrm{H_{str}}$)). We extend the arguments of Andreianov, Bendahmane and Karlsen [6] developed for the case of $\mathfrak{a}(u, \nabla\phi(u)) \equiv \mathfrak{a}(\nabla\phi(u))$; the main difficulty stems from the fact that we do not assume the structure condition of Remark 1.3, so that $\mathfrak{a}(\phi^{-1}(\cdot), \xi)$ can be discontinuous.

In order to use the weak convergence while passing to the limit in the nonlinear diffusion term in (P), we use a version of the classical Minty–Browder monotonicity argument. We use cut-off functions to focalize on the intervals of the strict monotonicity of ϕ. We then show that the complementary of this set can be neglected, thanks to a particular *a priori* estimate with the cut-off function; this estimate focalizes on a neigbourhood of the "exceptional set" E of the values of u introduced in ($\mathrm{H_2}$) (see Remark 1.2 and Lemma 3.2).

In order to deal with the convection term in (P), we use the technique of nonlinear weak-$\star$ convergence to a measure-valued solution (as considered by Tartar, DiPerna, Szepessy, Panov), or more exactly, to an entropy process solution as developed by Gallouët and collaborators (see [23–25] and references therein). Considering entropy process solutions is a purely technical issue, since the uniqueness proof also contains their identification to entropy solutions.

The chain rule arguments of Lemmas 3.3, 3.4 permit to separate the two aforementioned weak convergence arguments, the one for the diffusion term and the one for the convection term.

The uniqueness of an entropy solution is shown under the assumption (H_{11}), with the help of (H_3) and the estimate of Lemma 3.2; we follow Carrillo [20], Carrillo and Wittbold [21], Eymard, Gallouët, Herbin and Michel [24] and Andreianov, Bendahmane and Karlsen [6].

While the full continuous dependence result strongly relies upon the structure condition $(\mathrm{H}_{\mathrm{str}})$, the stability of entropy solutions to (P) with respect to the perturbation of the data (u_0, f) is shown under the weaker structure assumption

$$(\mathrm{H}'_{\mathrm{str}}) \qquad (b+\psi)(r) = (b+\psi)(s) \quad \Rightarrow \quad \phi(r) = \phi(s).$$

Let us stress the fact that our argument using $(\mathrm{H}'_{\mathrm{str}})$ is not "robust", in the sense that it cannot be directly adapted to the proof of convergence of various kinds of approximate solutions. In order to address the question of convergence of numerical approximations of (P), the stronger structure assumption $(\mathrm{H}_{\mathrm{str}})$ still seems essential. For finite volume approximations of the doubly nonlinear problem (P) with $b = \mathrm{Id}$, a convergence proof using $(\mathrm{H}_{\mathrm{str}})$ is given in [6].

Let us give an outline of the paper. We start with definitions and the formulation of the main results in Section 2. In Section 3, we give the key ingredients of our techniques. Section 4 concerns the adaptation of the standard uniqueness, L^1 contraction and comparison result for entropy and entropy process solutions of (P). Section 5 contains the *a priori* estimates for solutions. In Section 6, we assume that b is bijective (or, more generally, conditions $(\mathrm{H}_{\mathrm{str}})$, (H_5) are satisfied); we deal with the convergence of solutions to problems (P_n) with perturbed coefficients. Section 7 is devoted to the proof of the well-posedness result for (P). In Section 8 we give existence, uniqueness and continuous dependence results for the related elliptic equation $\psi(u) + \operatorname{div}\tilde{\mathfrak{f}}(\psi(u), \phi(u)) - \operatorname{div}\mathfrak{a}(u, \nabla\phi(u)) = s$, $s \in L^\infty(\Omega)$.

2. Entropy solutions and well-posedness results

2.1. Entropies and related notation

As it was explained in the introduction, we need the notion of weak solution for (P) with additional "entropy" conditions. In order to use entropy conditions in the interior of Q_T and, moreover, take into account the homogeneous Dirichlet boundary condition, following Carrillo [20] we will work with the so-called "semi-Kruzhkov" entropy–entropy flux pairs $(\eta_c^\pm, \mathfrak{q}_c^\pm)$ for each $c \in \mathbf{R}$:

$$\eta_c^+(z) = (z-c)^+, \qquad \eta_c^-(z) = (z-c)^-,$$
$$\mathfrak{q}_c^+(z) = \operatorname{sign}^+(z-c)\big(\mathfrak{f}(z) - \mathfrak{f}(c)\big), \qquad \mathfrak{q}_c^-(z) = \operatorname{sign}^-(z-c)\big(\mathfrak{f}(z) - \mathfrak{f}(c)\big).$$

By convention, we assign $(\eta_c^\pm)'(c)$ to be zero. Here $(z-c)^\pm$ stand for the non-negative quantities such that $z-c = (z-c)^+ - (z-c)^-$, but we denote

$$\operatorname{sign}^+(z-c) = \big(\eta_c^+\big)'(z) = \begin{cases} 1, & z > c, \\ 0, & z \leqslant c, \end{cases} \qquad \operatorname{sign}^-(z-c) = \big(\eta_c^-\big)'(z) = \begin{cases} 0, & z \geqslant c, \\ -1, & z < c. \end{cases}$$

At certain points, we will also need smooth regularizations of the semi-Kruzhkov entropy–entropy flux pairs; it is sufficient to consider regular "boundary" entropy pairs $(\eta_{c,\varepsilon}^\pm, \mathfrak{q}_{c,\varepsilon}^\pm)$ (cf. Otto [34] and the book [31]), which are $W^{2,\infty}$ pairs with the same support as $(\eta_c^\pm, \mathfrak{q}_c^\pm)$, converging pointwise to $(\eta_c^\pm, \mathfrak{q}_c^\pm)$ as $\varepsilon \to 0$. In particular, the functions

$$\operatorname{sign}_\varepsilon^+(z) = \frac{1}{\varepsilon}\min\{z^+, \varepsilon\}, \qquad \operatorname{sign}_\varepsilon^-(z) = \frac{1}{\varepsilon}\max\{-z^-, -\varepsilon\}$$

will be used to approximate $\operatorname{sign}^\pm(\cdot) = (\eta_0^\pm)'(\cdot)$.

Definition 2.1. For a function φ which is monotone on $\mathbf{R}$, for all locally bounded piecewise continuous function θ on $\mathbf{R}$ we can define (using, e.g., the Stiltjes integral)

$$\varphi_\theta : z \in \mathbf{R} \mapsto \int_0^z \theta(s)\, d\varphi(s).$$

Moreover (see Lemma 3.1 below), there exists a continuous function $\tilde{\varphi}_\theta$ such that $\varphi_\theta(u) = \tilde{\varphi}_\theta(\varphi(u))$.

In the sequel, we denote by $b_c^\pm(\cdot)$ the function $z \mapsto \int_0^z (\eta_c^\pm)'(s)\, db(s)$.

2.2. Entropy and entropy process solutions

For the sake of simplicity, we will in this paper work with bounded entropy solutions, i.e., we require that $u \in L^\infty(Q_T)$ and put corresponding hypotheses on the data u_0, f and functions b, ψ. Note that the boundedness assumption can be bypassed in the framework of renormalized solutions (see in particular Ammar and Redwane [3]); or in the framework of variational solutions in the spirit of Alt and Luckhaus [1] (in this case, one has to replace the functions $C(\cdot)$ in assumptions (H_9), (H_{10}), (H_{11}) with a constant C; further changes are indicated in Remark 6.1).

Definition 2.2 *(Entropy solution).* An entropy solution of (P) is a measurable function $u : Q_T \to \mathbf{R}$ satisfying the following conditions:

(D.1) (Regularity) $u \in L^\infty(Q_T)$ and $w = \phi(u) \in L^p(0, T; W_0^{1,p}(\Omega))$.
(D.2) For all $\xi \in \mathcal{D}([0, T) \times \Omega)$,

$$\iint_{Q_T} \big(b(u)\partial_t \xi + \mathfrak{f}(u) \cdot \nabla \xi - \mathfrak{a}(u, \nabla w) \cdot \nabla \xi - \psi(u)\xi\big)\, dx\, dt + \int_\Omega b(u_0)\xi(0, \cdot)\, dx + \iint_{Q_T} f \xi\, dx\, dt = 0. \tag{2}$$

(D.3) For all $(c, \xi) \in \mathbf{R}^\pm \times \mathcal{D}([0, T) \times \overline{\Omega})$, $\xi \geqslant 0$, and also for all $(c, \xi) \in \mathbf{R} \times \mathcal{D}([0, T) \times \Omega)$, $\xi \geqslant 0$,

$$\iint_{Q_T} \big(b_c^\pm(u)\partial_t \xi + \mathfrak{q}_c^\pm(u) \cdot \nabla \xi - \big(\eta_c^\pm\big)'(u)\mathfrak{a}(u, \nabla w) \cdot \nabla \xi - \big(\eta_c^\pm\big)'(u)\psi(u)\xi\big)\, dx\, dt + \int_\Omega b_c^\pm(u_0)\xi(0, \cdot)\, dx + \iint_{Q_T} \big(\eta_c^\pm\big)'(u) f \xi\, dx\, dt \geqslant 0.$$

Remark 2.1. If in the above definition,

- u satisfies $u \in L^\infty(Q_T)$, $w = \phi(u) \in L^p(0, T; W^{1,p}(\Omega))$ and w^+ (resp., w^-) belongs to $L^p(0, T; W_0^{1,p}(\Omega))$,
- the equality sign in the weak formulation (2) is replaced by the inequality "$\geqslant$" (resp., with the inequality "$\leqslant$"),
- entropy inequalities in (D.3) hold with the entropies η_c^+ for $c \in \mathbf{R}^+$ (resp., with the entropies η_c^- for $c \in \mathbf{R}^-$),

then u is called entropy subsolution (resp., entropy supersolution) of (P).

Remark 2.2. Following Alt and Luckhaus [1], we can rewrite the weak formulation (D.2) of (P) as follows:

– $b(u)|_{t=0} = b(u_0)$ and the distributional derivative $\partial_t b(u)$ satisfies

$$\partial_t b(u) \in L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big) + L^1(Q_T)$$

in the sense

$$\int_0^T \big\langle \partial_t b(u), \zeta \big\rangle = -\iint_{Q_T} b(u)\partial_t\zeta - \int_\Omega b(u_0)\zeta(0,\cdot)$$

for all $\zeta \in L^p(0,T,W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ such that $\partial_t\zeta \in L^\infty(Q_T)$ and $\zeta(T,\cdot)=0$;

– equation (P) is satisfied in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T)$.

We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T)$ and $L^p(0,T, W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$.

For technical reasons, it is convenient to introduce the notion of entropy process solution adapted from Eymard, Gallouët and Herbin [23], Gallouët and Hubert [25] and Eymard, Gallouët, Herbin and Michel [24]. This definition is based upon an L^∞ representation of Young measures via their distribution functions, see Panov [40] and Gallouët et al. [23,25]. This allows for the so-called "nonlinear weak-$\star$ convergence" property (see, e.g., Ball [8] and Hungerbühler [26] for the formulation in terms of Young measures):

$$\left|\begin{array}{l}\text{each sequence } (u_n)_n \text{ of measurable functions}\\ \text{admits a subsequence such that for all } F \in C(\mathbf{R},\mathbf{R}),\\ F\big(u_n(\cdot)\big) \to \displaystyle\int_0^1 F\big(\mu(\cdot,\alpha)\big)\,d\alpha \text{ weakly in } L^1(Q_T)\\ \text{whenever the set } \big(F(u_n)\big)_n \text{ is weakly relatively compact in } L^1(Q_T),\end{array}\right. \tag{3}$$

where the function $\mu \in L^\infty(Q_T \times (0,1))$ is referred to as the "process function". Notice that in the above statement, one also concludes that $F(\mu(\cdot,\alpha))$ is independent of α whenever $F(u_n)$ converges to $F(u)$ in measure (see, e.g., [26]).

Definition 2.3 *(Entropy process solution).* An entropy process solution of (P) is a couple (μ, w) of measurable functions $\mu : Q_T \times (0,1) \to \mathbf{R}$ and $w : Q_T \to \mathbf{R}$ satisfying the following conditions:

(D′.1) (Regularity and consistency) $\mu \in L^\infty(Q_T \times (0,1))$, $w \in L^p(0,T;W_0^{1,p}(\Omega))$, and $\phi(\mu(t,x,\alpha)) \equiv w(t,x)$ for a.e. $(t,x,\alpha) \in Q_T \times (0,1)$.

(D′.2) For all $\xi \in \mathcal{D}([0,T) \times \Omega)$,

$$\int_0^1 \iint_{Q_T} z\big(b(\mu)\partial_t\xi + \mathfrak{f}(\mu)\cdot\nabla\xi - \mathfrak{a}(\mu,\nabla w)\cdot\nabla\xi - \psi(\mu)\xi\big)\,dx\,dt\,d\alpha$$
$$+ \int_\Omega u_0\xi(0,\cdot)\,dx + \iint_{Q_T} f\xi\,dx\,dt = 0.$$

(D′.3) For all $(c,\xi)\in\mathbf{R}^{\pm}\times\mathcal{D}([0,T)\times\overline{\Omega})$, $\xi\geqslant 0$, and also for all $(c,\xi)\in\mathbf{R}\times\mathcal{D}([0,T)\times\Omega)$, $\xi\geqslant 0$,

$$\int_0^1\iint_{Q_T}\big(b_c^{\pm}(\mu)\partial_t\xi+\mathfrak{q}_c^{\pm}(\mu)\cdot\nabla\xi-\big(\eta_c^{\pm}\big)'(\mu)\mathfrak{a}(\mu,\nabla w)\cdot\nabla\xi-\big(\eta_c^{\pm}\big)'(\mu)\psi(\mu)\big)\,dx\,dt\,d\alpha$$
$$+\int_\Omega b_c^{\pm}(u_0)\xi(0,\cdot)\,dx+\int_0^1\iint_{Q_T}\big(\eta_c^{\pm}\big)'(\mu)f\xi\,dx\,dt\,d\alpha\geqslant 0.$$

Remark 2.3. In (D′.3), setting $u:=\int_0^1\mu(\alpha)\,d\alpha$ one can rewrite the third term under the form

$$\int_0^1\iint_{Q_T}\big(\eta_c^{\pm}\big)'(\mu)\mathfrak{a}(\mu,\nabla w)=\iint_{Q_T}\big(\eta_c^{\pm}\big)'(u)\mathfrak{a}(u,\nabla w).$$

Indeed, we have $w\equiv\phi(\mu)$, and ϕ is invertible on $\mathbf{R}\setminus E$, so that $\mu(t,x,\alpha)\equiv\phi^{-1}(w(t,x))=u(t,x)$ whenever $w(t,x)\in\mathbf{R}\setminus\phi(E)$; furthermore, $\nabla w=0$ a.e. on $[w\in\phi(E)\cup\{\phi(c)\}]$, and the exact value of $(\eta_c^{\pm})'(\mu)$ on $[w\in\phi(E)\cup\{\phi(c)\}]$ does not matter because $\mathfrak{a}(r,0)\equiv 0$. For the same reasons, $\mathfrak{a}(\mu,\nabla w)$ can be replaced by $\mathfrak{a}(u,\nabla w)$ in (D′.2).

Remark 2.4. If u is an entropy solution of (P), then the couple (μ,w) defined by $\mu(t,x,\alpha)=u(t,x)$ a.e. on $Q_T\times(0,1)$ (resp., by $w(t,x)=\phi(u(t,x))$ a.e. on Q_T), is an entropy process solution of (P). In turn, if (μ,w) is an entropy process solution of (P) such that $\mu(t,x,\alpha)=u(t,x)$ a.e. on $Q_T\times(0,1)$ for some $u:Q_T\to\mathbf{R}$, then the function u is an entropy solution of (P).

2.3. Well-posedness of problem (P) in the framework of entropy solutions

First note the uniqueness result, which requires no range condition nor structure condition on the nonlinearities b and ϕ.

Theorem 2.1. *Assume that* (H_1)–(H_5) *and* (H_6)–(H_{11}) *hold.*

(i) *Assume that* (μ,w) *is an entropy process solution of* (P). *Then*

$$u(t,x)=\int_0^1\mu(t,x,\alpha)\,d\alpha$$

is an entropy solution of (P). *Moreover, we have* $b(\mu)(t,x,\alpha)\equiv b(u)(t,x)$ *and* $\psi(\mu)(t,x,\alpha)\equiv\psi(u)(t,x)$ *a.e. on* $Q_T\times(0,1)$. *If* $(b+\phi+\psi)$ *is strictly increasing, then* $\mu(t,x,\alpha)=u(t,x)$ *a.e. on* $Q_T\times(0,1)$.

(ii) *Assume that* u *and* $\hat u$ *are two entropy solutions of* (P) *corresponding to the data* u_0, f *and* $\hat u_0,\hat f$, *respectively. Then for a.e.* $t\in(0,T)$,

$$\int_\Omega\big(b(u)-b(\hat u)\big)^+(t)+\int_0^t\int_\Omega\big(\psi(u)-\psi(\hat u)\big)^+$$
$$\leqslant\int_\Omega\big(b(u_0)-b(\hat u_0)\big)^+ +\int_0^t\int_\Omega \operatorname{sign}^+(u-\hat u)(f-\hat f).$$

(iii) *In particular, if* $u,\hat u$ *are two entropy solutions of* (P), *then* $b(u)\equiv b(\hat u)$ *and* $\psi(u)\equiv\psi(\hat u)$.

Remark 2.5. In Theorem 2.1(ii), one can replace u, resp. $\hat{u}$, with an entropy subsolution, resp. with an entropy supersolution. The same proof applies.

The following continuous dependence property is the central result of this paper.

Theorem 2.2. *Let $(b_n, \psi_n, \phi_n, \mathfrak{a}_n, \tilde{\mathfrak{f}}_n; u_0^n, f_n)$, $n \in \mathbb{N}$, be a sequence converging to $(b, \psi, \phi, \mathfrak{a}, \tilde{\mathfrak{f}}; u_0, f)$ in the following sense*:

$$\begin{aligned} &\bullet\ b_n, \psi_n, \phi_n \text{ converge pointwise to } b, \psi, \phi \text{ respectively};\\ &\bullet\ \tilde{\mathfrak{f}}_n, \mathfrak{a}_n \text{ converge to } \tilde{\mathfrak{f}}, \mathfrak{a}, \text{ respectively, uniformly on compacts};\\ &\bullet\ b_n\big(u_0^n\big) \to b(u_0) \text{ in } L^1(\Omega), \text{ and } f_n \to f \text{ in } L^1(Q_T). \end{aligned} \tag{4}$$

Assume that $(b, \psi, \phi, \mathfrak{a}, \tilde{\mathfrak{f}}; u_0, f)$ and $(b_n, \psi_n, \phi_n, \mathfrak{a}_n, \tilde{\mathfrak{f}}_n; u_0^n, f_n)$ (for each n) satisfy the hypotheses (H_1), (H_4), (H_5), *and* (H_6)–(H_{11}), *and, moreover, that the functions $C(\cdot)$ in* (H_9), (H_{10}), *and* (H_{11}) *as well as the $L^\infty(\Omega)$ and $L^1(0,T,L^\infty(\Omega))$ bounds in* (H_4) *are independent of n. We denote by* (P_n) *the analogue of problem* (P) *corresponding to the data and coefficients $(b_n, \psi_n, \phi_n, \mathfrak{a}_n, \tilde{\mathfrak{f}}_n; u_0^n, f_n)$.*

Assume that either $b(\mathbf{R}) = \mathbf{R}$, or the $L^\infty(Q_T)$ bounds on $f_n^\pm$ in (H_5) *are independent of n. Assume that the structure condition* $(\mathrm{H}_{\mathrm{str}})$ *holds, and ϕ satisfies the technical hypotheses* (H_2), (H_3).

Let u_n be an entropy solution of problem (P_n). *Then the functions u_n converge to an entropy solution u of* (P) *in $L^\infty(Q_T)$ weakly-$\star$, up to a subsequence. Furthermore, the functions $\phi_n(u_n)$ converge to $\phi(u)$ in $L^1(Q_T)$ up to a subsequence, and the whole sequences $b_n(u_n)$, $\psi_n(u_n)$ converge in $L^1(Q_T)$ to $b(u)$, $\psi(u)$, respectively.*

Remark 2.6. The structure condition $(\mathrm{H}_{\mathrm{str}})$ seems to be essential for results like Theorem 2.2 to hold (see [14]).

The surjectivity assumption on b or assumption (H_5) were only required to ensure the L^∞ estimate on u. One could work with unbounded solutions, in which case these assumptions can be replaced by a growth condition on $\tilde{\mathfrak{f}}$ (see Remark 6.1 below).

Notice that assuming simultaneously $(\mathrm{H}_{\mathrm{str}})$, $b(\mathbf{R}) = \mathbf{R}$ and $\psi \equiv 0$, by a change of the unknown u into $v = b(u)$ we can always reduce the triply nonlinear problem (P) to the doubly nonlinear problem with $b \equiv \mathrm{Id}$.

Finally, we state the well-posedness result for (P). Note that when only the data (u_0, f) are perturbed, the continuous dependence result analogous to Theorem 2.2 holds under the structure assumption $(\mathrm{H}'_{\mathrm{str}})$ which is weaker than $(\mathrm{H}_{\mathrm{str}})$.

Theorem 2.3. (i) *Assume that* (H_1)–(H_5) *and* (H_6)–(H_{11}) *hold. Then there exists an entropy solution to* (P). *Moreover, it is unique, in the sense of Theorem* 2.1(iii).

(ii) *Assume in addition that the structure condition* $(\mathrm{H}'_{\mathrm{str}})$ *holds. Then the entropy solution of* (P) *depends continuously on the data (u_0, f). More precisely, let u_n be an entropy solution of* (P) *with data (u_0^n, f_n). Assume $b(u_0^n) \to b(u_0)$ in $L^1(\Omega)$ and $f_n \to f$ in $L^1(Q_T)$, as $n \to \infty$. Assume that the bounds in* (H_4), (H_5) *are uniform in the sense*

- $\|u_0^n\|_{L^\infty(\Omega)} \leqslant \mathrm{Const}$;
- *either* $b(+\infty) = +\infty$ *and* $\int_0^T \|f_n^+(t,\cdot)\|_{L^\infty(\Omega)}\,dt \leqslant \mathrm{Const}$, *or* $\psi(+\infty) = +\infty$ *and* $\|f_n^+\|_{L^\infty(Q_T)} \leqslant \mathrm{Const}$;
- *either* $b(-\infty) = -\infty$ *and* $\int_0^T \|f_n^-(t,\cdot)\|_{L^\infty(\Omega)}\,dt \leqslant \mathrm{Const}$, *or* $\psi(-\infty) = -\infty$ *and* $\|f_n^-\|_{L^\infty(Q_T)} \leqslant \mathrm{Const}$.

Then $b(u_n)$, $\psi(u_n)$ and $\phi(u_n)$ converge, respectively, to $b(u)$, $\psi(u)$ and $\phi(u)$ in $L^1(Q_T)$ as $n \to \infty$, where u is an entropy solution of (P) *with data (u_0, f).*

Moreover, if we reinforce hypothesis (H_8) *by requiring the uniform monotonicity of* $\mathfrak{a}(r,\cdot)$ *in the sense*

$$(H_8') \qquad \left|\begin{array}{l} \text{there exists } C \in C\big(\mathbf{R}^2;\mathbf{R}^+\big) \text{ such that} \\ \big(\mathfrak{a}(r,\xi)-\mathfrak{a}(r,\eta)\big)\cdot(\xi-\eta) \geqslant 1/C\left(r,\dfrac{1}{|\xi-\eta|}\right), \end{array}\right.$$

then also $\nabla\phi(u_n)$ *converge to* $\nabla\phi(u)$ *in* $(L^p(Q_T))^N$ *and* $\mathfrak{a}(u_n,\nabla\phi(u_n))$ *converge to* $\mathfrak{a}(u,\nabla\phi(u))$ *in* $(L^{p'}(Q_T))^N$*, as* $n\to\infty$.

3. Notation and preliminary lemmas

Let us give some notation which will be used throughout the remaining sections.

We will use the notation like $[u\in F]$ for the sets like $\{(t,x)\in Q_T \mid u(t,x)\in F\}$. For a measurable set H, we denote by $\mathbb{1}_H$ the characteristic function of H. For $H\subset\mathbf{R}$, we set

$$T_H(z) := \int\limits_0^z \mathbb{1}_H(s)\,ds;$$

clearly, $T_H(\cdot)$ is a Lipschitz function with $T_H(0)=0$.

We denote by G the image $\phi(E)$ by $\phi(\cdot)$ of the "exceptional set" E introduced in (H_2); recall that E is closed, G is closed and $\operatorname{meas}(G)=0$. We denote by I a generic open interval in $\mathbf{R}\setminus E$, and by J its image $\phi(I)$ which is a generic open interval in $\mathbf{R}\setminus G$. For all $\varepsilon>0$, we choose an open set $G_\varepsilon\supset G$ such that $\operatorname{meas}(G_\varepsilon)<\text{Const}\times\varepsilon$. We denote by E_ε the open set $\phi^{-1}(G_\varepsilon)$ which contains E. When (H_3) holds, we can simply take $G_\varepsilon = G^\varepsilon := \{z\in\mathbf{R} \mid \operatorname{dist}(z,G)<\varepsilon\}$.

Now let us prove the representation property used in Definition 2.1.

Lemma 3.1. *Let* $\varphi_\theta(\cdot)$ *be the function defined by*

$$\varphi_\theta : z\in\mathbf{R}\mapsto \int\limits_0^z \theta(s)\,d\varphi(s),$$

for a continuous non-decreasing function $\varphi:\mathbf{R}\to\mathbf{R}$ *and a bounded piecewise continuous function* $\theta:\mathbf{R}\to\mathbf{R}^n$*. Then there exists a Lipschitz continuous function* $\tilde\varphi_\theta:\varphi(\mathbf{R})\to\mathbf{R}^n$ *such that for all* $z\in\mathbf{R}$,

$$\varphi_\theta(z)=\tilde\varphi_\theta\big(\varphi(z)\big).$$

Proof. If $\varphi(z)=\varphi(\hat z)$, then the measure $d\varphi(s)$ vanishes between z and $\hat z$; thus $\varphi_\theta(z)-\varphi_\theta(\hat z)=\int_{\hat z}^z\theta(s)\,d\varphi(s)$ is zero. Therefore $\tilde\varphi_\theta$ is well defined. For all $r,\hat r\in\varphi(\mathbf{R})$, $\tilde\varphi_\theta(r)-\tilde\varphi_\theta(\hat r)=\varphi_\theta(z)-\varphi_\theta(\hat z)=\int_{\hat z}^z\theta(s)\,d\varphi(s)$, where $z\in\varphi^{-1}(r)$, $\hat z\in\varphi^{-1}(\hat r)$. Thus

$$\big|\tilde\varphi_\theta(r)-\tilde\varphi_\theta(\hat r)\big|\leqslant\|\theta\|_{L^\infty}\big|\varphi(z)-\varphi(\hat z)\big|=\|\theta\|_{L^\infty}|r-\hat r|. \qquad\square$$

Now, let us give a localized estimate of the gradient of $w=\phi(u)$.

Lemma 3.2. *Let* u *be a bounded weak solution of* (P)*. Then there exists a constant* C *depending on* $C(\cdot)$ *in* (H_9)*, on* $\|u\|_{L^\infty(Q_T)}$ *and on* $\|b(u_0)\|_{L^1(\Omega)}$*,* $\|f\|_{L^1(Q)}$ *such that for all Borel measurable set* $F\subset\mathbf{R}$,

$$I_F(u) := \iint\limits_{[u\in F]} \big|\nabla\phi(u)\big|^p = \iint\limits_{[w\in\phi(F)]} |\nabla w|^p \leqslant C\,\mathrm{Var}_F\,\phi(\cdot) = C\,\mathrm{meas}\big(\phi(F)\big). \tag{5}$$

Proof. Without loss of restriction, one can assume that F is bounded; indeed, otherwise we can replace F with $F^M := F \cap [-M, M]$ and then pass to the limit as $M \to +\infty$ in inequality (5) written for F^M.

Set $H = \phi(F)$, and note that $T_H(w) = T_H(\phi(u)) \in L^p(0, T; W_0^{1,p}(\Omega))$ can be approximated by admissible test functions in (2) of Definition 2.2; one has

$$\nabla T_H(w) = \nabla w \mathbb{1}_H(w) = \nabla \phi(u) \mathbb{1}_F(u),$$

and

$$\left\| T_H(\cdot) \right\|_\infty \leqslant \int_F d\phi(s) = \mathrm{Var}_F\, \phi(\cdot) = \mathrm{meas}(H). \tag{6}$$

Using this test function, with Remark 2.2 and the standard chain rule argument known as the Mignot–Bamberger and Alt–Luckhaus formula (see, e.g., Alt and Luckhaus [1], Otto [35], Carrillo and Wittbold [21]) we get

$$\begin{aligned} &\int_\Omega B_F(u)(T, \cdot) + \iint_{Q_T} \psi(u) T_H\big(\phi(u)\big) + \iint_{Q_T} \mathfrak{a}(u, \nabla w) \cdot \nabla T_H(w) \\ &\quad = \int_\Omega B_F(u_0) + \iint_{Q_T} f T_H(w) - \iint_{Q_T} \mathfrak{f}(u) \cdot \nabla T_H\big(\phi(u)\big), \end{aligned} \tag{7}$$

where $B_F(z) = \int_0^z T_H(\phi(s))\, db(s)$. The last term is zero thanks to the boundary condition $w|_\Sigma = 0$. Indeed, because F is assumed bounded, $\mathfrak{f}(\cdot)$ is bounded on the support of $T_H'(\phi(\cdot))$; thus by Lemma 3.1 there exists a Lipschitz continuous vector-valued function $\mathfrak{g}(\cdot)$ such that

$$\int_0^z \mathfrak{f}(s)\, dT_H\big(\phi(s)\big) = \mathfrak{g}\big(\phi(z)\big).$$

Hence $\mathfrak{g} \circ w \in L^p(0, T; W_0^{1,p}(\Omega; \mathbf{R}^N))$, so that one can apply the Green–Gauss formula to get

$$\iint_Q \mathrm{div}\left(\int_0^w \mathfrak{f}(s)\, dT_H\big(\phi(s)\big) \right) = \int_0^T \int_{\partial\Omega} \mathfrak{g}(w) \cdot \nu = 0,$$

where ν is the exterior unit normal vector to $\partial\Omega$. By definition of $T_H(\cdot)$ and because $\phi(\cdot)$ is nondecreasing, dropping positive terms in the left-hand side of (7), by (H_9) we infer

$$\begin{aligned} \frac{1}{C(\|u\|_{L^\infty(Q_T)})} \iint_{[u \in F]} \left| \nabla \phi(u) \right|^p &\leqslant \iint_{[w \in \phi(F)]} \mathfrak{a}(u, \nabla w) \cdot \nabla w \\ &\leqslant \left(\left\| b(u_0) \right\|_{L^1(\Omega)} + \| f \|_{L^1(Q)} \right) \left\| T_H(\cdot) \right\|_\infty. \end{aligned}$$

Hence the claim follows by (6). $\square$

In the above proof, we have used two chain rule lemmas. Now we notice that both apply for $u(\cdot)$ replaced with a "process function" $\mu(\cdot, \alpha)$, as in (D′.1), provided that for a.e. $\alpha \in (0, 1)$ one can substitute

$$u(t,x) := \int_0^1 \mu(t,x,\alpha)\,d\alpha$$

by $\mu(t,x,\alpha)$ in the expression of the test function.

Lemma 3.3. *Let (μ, w) satisfy (D′.1), and $S : \mathbf{R} \to \mathbf{R}$ be a Lipschitz continuous function such that $S(0) = 0$. Let $\zeta \in L^\infty(0,T)$. Then*

$$\iint_{Q_T} \int_0^1 \mathfrak{f}\big(\mu(t,x,\alpha)\big) \cdot \nabla S\big(w(t,x)\big)\zeta(t)\,dt\,dx\,d\alpha = 0.$$

Proof. By Lemma 3.1, there exists a Lipschitz vector-valued function $\mathfrak{g}$ such that $\int_0^z \mathfrak{f}(s)\,dS(\phi(s)) = \mathfrak{g}(\phi(z))$ for $|z| \leqslant \|\mu\|_{L^\infty(Q_T\times(0,1))}$. We have

$$\begin{aligned}
\iint_{Q_T} \int_0^1 \mathfrak{f}\big(\mu(\alpha)\big)\,d\alpha \cdot \nabla S\big(\phi(u)\big)\zeta &= \iint_{Q_T} \int_0^1 \mathfrak{f}\big(\mu(\alpha)\big) \cdot \nabla S\big(\phi\big(\mu(\alpha)\big)\big)\,d\alpha\,\zeta \\
&= \iint_{Q_T} \int_0^1 \operatorname{div}\left(\int_0^{\mu(\alpha)} \mathfrak{f}(s)\,dS\big(\phi(s)\big)\right) d\alpha\,\zeta \\
&= \int_0^T \int_\Omega \operatorname{div}\mathfrak{g}(w)\zeta = \int_0^T \int_{\partial\Omega} \mathfrak{g}(w)\cdot\nu\zeta = 0
\end{aligned}$$

because for a.e. $\alpha \in (0,1)$, $\phi(\mu(\alpha)) \equiv \phi(u) = w \in L^p(0,T;W_0^{1,p}(\Omega))$. $\square$

Lemma 3.4. *Let Ω be a bounded domain of $\mathbf{R}^n$, $T > 0$, $Q_T := (0,T)\times\Omega$, and $1 < p < +\infty$. Let $g \in C(\mathbf{R};\mathbf{R})$. Let $b \in C(\mathbf{R};\mathbf{R})$ be non-decreasing. Set*

$$B_g(z) := \int_0^z g(s)\,db(s).$$

Let $\mu \in L^\infty(Q_T \times (0,1))$; set $u = \int_0^1 \mu(\alpha)\,d\alpha$. Assume that

$$g(u) \in L^p\big(0,T;W_0^{1,p}(\Omega)\big) \cap L^\infty(Q_T)$$

and, moreover,

$$g\big(\mu(\alpha)\big) \equiv g(u).$$

Assume that

$$\partial_t\left(\int_0^1 b\big(\mu(\alpha)\big)\,d\alpha\right) \in L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big) + L^1(Q_T)$$

and

$$\int_0^1 b\bigl(\mu(\alpha)\bigr)\,d\alpha|_{t=0} = b(u_0)$$

in the following sense:

$$\forall \xi \in L^p\bigl(0,T;W_0^{1,p}(\Omega)\bigr) \quad \text{such that} \quad \partial_t\xi \in L^\infty(Q_T) \text{ and } \xi(T,\cdot)=0,$$

$$\int_0^T \left\langle \partial_t\left(\int_0^1 b\bigl(\mu(\alpha)\bigr)\,d\alpha\right), \xi \right\rangle = -\iint_{Q_T}\int_0^1 b\bigl(\mu(\alpha)\bigr)\,d\alpha\,\partial_t\xi - \int_\Omega b(u_0)\xi(0,\cdot).$$

Then for all $\zeta \in \mathcal{D}([0,T))$,

$$\int_0^T \left\langle \partial_t\left(\int_0^1 b\bigl(\mu(\alpha)\bigr)\,d\alpha\right), g(u)\zeta \right\rangle = -\iint_{Q_T}\int_0^1 B_g\bigl(\mu(\alpha)\bigr)\,d\alpha\,\zeta_t - \int_\Omega B_g(u_0)\zeta(0).$$

Proof (sketched)**.** Note that the claim of Lemma 3.4 cannot be deduced directly from the usual Mignot–Bamberger and Alt–Luckhaus chain rule lemma; the reason is that we cannot expect $\partial_t b(\mu(\alpha))$ to belong to $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T)$ for a.e. α. But it suffices to reproduce the proof (see, e.g., [21]) which is by discretization of $\partial_t(\int_0^1 b(\mu(\alpha))\,d\alpha)$. Indeed, we have for a.e. $t, t-h \in (0,T)$,

$$\begin{aligned}
&\frac{1}{h}\left(\int_0^1 b\bigl(\mu(t,\alpha)\bigr)\,d\alpha - \int_0^1 b\bigl(\mu(t-h,\alpha)\bigr)\,d\alpha\right) g\bigl(u(t)\bigr)\\
&\quad = \int_0^1 \frac{1}{h}\bigl(b\bigl(\mu(t,\alpha)\bigr) - b\bigl(\mu(t-h,\alpha)\bigr)\bigr) g\bigl(\mu(t,\alpha)\bigr)\,d\alpha,
\end{aligned}$$

and now we can reason separately for each α. Thus the arguments of [21] apply. □

4. Proof of L^1 contraction and comparison principles

Now we turn to the proof of Theorem 2.1 and Remark 2.1. Most of the statements are standard. We only notice that while proving Theorem 2.1(i), one obtains that $b(\mu)$ and $\psi(\mu)$ are independent of α; since $\phi(\mu) = w$ is independent of α by definition, one concludes that $\mathfrak{f}(\mu) \equiv \tilde{\mathfrak{f}}(b(\mu), \psi(\mu), \phi(\mu))$ is also independent of α, thus the entropy process solution μ gives rise to the entropy solution $u = \int_0^1 \mu(\alpha)\,d\alpha$. This is the only point where the special structure of the dependency of $\mathfrak{f}$ on u is used.

The proof of Theorem 2.1 is essentially the same as in Carrillo and Wittbold [21]; it is based on the techniques of Carrillo [20] and on hypothesis (H_{11}) (notice that in the case $\phi = \mathrm{Id}$, one has $E = \emptyset$; therefore (H_{11}) reduces to the Carrillo–Wittbold hypothesis in this case). For Theorem 2.1(i), we also need the adaptation of the Carrillo arguments to the framework of entropy process solutions. This has been done by Eymard, Gallouët, Herbin and Michel [24], Michel and Vovelle [33] and Andreianov, Bendahmane and Karlsen [6]. Therefore we only point out why hypothesis (H_{11}) is sufficient for the uniqueness of an entropy solution in the case of problem (P) with ϕ that can be not strictly increasing.

The role of hypothesis (H_{11}) is to ensure that

$$\limsup_{\varepsilon\to 0} \iint_{Q_T}\iint_{Q_T} \frac{1}{\varepsilon}\bigl(\mathfrak{a}(u,\nabla w) - \mathfrak{a}(\hat{u},\nabla\hat{w})\bigr)\cdot(\nabla w - \nabla\hat{w})\mathbb{1}_{[0<w-\hat{w}<\varepsilon]} \geqslant 0, \tag{8}$$

where

$$u=\int_0^1 \mu(\alpha)\,d\alpha, \qquad w=\phi(u), \qquad \hat u=\int_0^1 \hat\mu(\alpha)\,d\alpha, \qquad \hat w=\phi(\hat u),$$

and $(\mu(t,x,\alpha), w(t,x))$ and $(\hat\mu(s,y,\alpha), \hat w(t,x))$ are two entropy process solutions of (P). Here, following Kruzhkov [28], we have taken two independent sets of the variables (t,x) and (s,y).

We split the integration domain $Q_T \times Q_T$ into several pieces.

First, notice that a.e. on $[w \in G] \times Q_T$, we have $\nabla w = 0$; thus the integrand in (8) is reduced to $\mathfrak{a}(\hat u, \nabla \hat w)\nabla \hat w \mathbb{1}_{[0<w-\hat w<\varepsilon]}$, which is non-negative. The same argument applies on $Q_T \times [\hat w \in G]$.

To conclude the proof of (8), it remains to investigate the integrand in (8) on the set $[w \notin G] \times [\hat w \notin G]$. Let us introduce $G^\varepsilon := \{z \in \mathbf{R} \mid \operatorname{dist}(z,G) < \varepsilon\}$. For a.e. $(t,x,s,y) \in [w \notin G] \times [\hat w \notin G]$ we have:

(a) either $w(t,x)$ and $w(s,y)$ belong to the same connected component of $\mathbf{R} \setminus G$;
(b) or (a) fails, but $w(t,x) \in G^\varepsilon \setminus G$ and $\hat w(s,y) \in G^\varepsilon \setminus G$;
(c) or both (a) and (b) fail, but then $|w(t,x) - \hat w(s,y)| \geqslant 2\varepsilon$.

We split $[w \notin G] \times [\hat w \notin G]$ into disjoint union of sets $S_a \cup S_b \cup S_c$, according to which of the above cases (a), (b), (c) takes place at $(t,x,s,y) \in [w \notin G] \times [\hat w \notin G]$. On S_a, we use assumption (H_{11}) and infer that the integrand in (8) is lower bounded by

$$-\max\bigl\{C(r,s) \bigm| |r|,|s| \leqslant \|u\|_\infty\bigr\}\bigl(1+|\nabla w|^p+|\nabla \hat w|^p\bigr)\mathbb{1}_{[0<w-\hat w<\varepsilon]}.$$

Because the $2(N+1)$-dimensional Lebesgue measure of the set $[0 < w-\hat w < \varepsilon]$ goes to zero as $\varepsilon \to 0$, the limit of the corresponding part of the integral in (8) is lower bounded by zero.

On S_b, we bound the integrand in (8) from below by $-\frac{p}{\varepsilon}(|\nabla w|^p + |\nabla \hat w|^p)$. Using Lemma 3.2 we have, e.g.,

$$\frac{1}{\varepsilon}\iint\limits_{[w\in G^\varepsilon\setminus G]}\iint\limits_{[\hat w\in G^\varepsilon\setminus G]} |\nabla w|^p \leqslant C\frac{\operatorname{meas}(G^\varepsilon)}{\varepsilon}\iint \mathbb{1}_{[\hat w \in G^\varepsilon\setminus G]}.$$

By the continuity of the Lebesgue measure and because $\bigcup_{\varepsilon>0} G^\varepsilon \setminus G = \emptyset$, the measure of the set $[\hat w \in G^\varepsilon \setminus G]$ tends to zero as $\varepsilon \to 0$. Therefore, using assumption (H_3), we deduce that the corresponding part of the limit in (8) is non-negative.

Finally, on S_c the integrand in (8) is zero.

This ends the proof of (8).

5. *A priori* estimates

The following estimates are rather standard.

Lemma 5.1. *Let $(b_n, \psi_n, \phi_n, \mathfrak{a}_n, \tilde{\mathfrak{f}}_n; u_0^n, f_n)$, $n \in \mathbb{N}$, be a sequence of data satisfying the assumptions of Theorem 2.2. Assume that the corresponding limiting data $(b, \psi, \phi, \mathfrak{a}, \tilde{\mathfrak{f}}; u_0, f)$ are such that (H_5) and $(\mathrm{H}_{\mathrm{str}})$ hold.*

Let u_n be an entropy solution of problem (P_n). Then there exists a constant M and a modulus of continuity $\omega : \mathbf{R}^+ \to \mathbf{R}^+$, such that for all $n \in \mathbb{N}$,

(i) $\|u_n\|_{L^\infty(Q_T)} \leqslant M$;
(ii) *the following quantities are all upper bounded by M:*

$$\bigl\|\phi_n(u_n)\bigr\|_{L^p(0,T;W_0^{1,p}(\Omega))}, \ \|\phi_n(u_n)\|_{L^p(Q_T)}, \ \bigl\|\mathfrak{a}_n\bigl(u_n, \nabla\phi_n(u_n)\bigr)\bigr\|_{L^{p'}(Q_T)},$$
$$\bigl\|\psi_n(u_n)\phi_n(u_n)\bigr\|_{L^1(Q_T)}, \ \bigl\|B_n(u_n)\bigr\|_{L^\infty(0,T;L^1(\Omega))},$$

where B_n is defined in (1) with b, ϕ replaced by b_n, ϕ_n;

(iii) $$\text{for all } \Delta > 0, \quad \iint\limits_{Q_{T-\Delta}} \left|\phi_n\big(u_n(t+\Delta, x)\big) - \phi_n\big(u_n(t,x)\big)\right| \leqslant \omega(\Delta).$$

Proof. (i) First assume $b(\mathbf{R}) = \mathbf{R}$. Consider the function

$$M(t) := \sup_{n\in\mathbb{N}} \left(\left\|b_n\big(u_0^n\big)\right\|_{L^\infty(\Omega)} + \int_0^t \left\|f_n(\tau,\cdot)\right\|_{L^\infty(\Omega)} d\tau \right) < +\infty.$$

Then for any measurable choice of $\overline{u}(t,x) \in b_n^{-1}(M(t))$, $\overline{u}$ is an entropy supersolution of (P_n). Similarly, $\underline{u}(t,x) \in b_n^{-1}(-M(t))$ is an entropy subsolution of (P_n). The comparison principle of Remark 2.5 ensures that a.e. on Q,

$$-M(T) \leqslant -M(t) \leqslant b_n(u_n)(t,x) \leqslant M(t) \leqslant M(T).$$

Now the assumption $b(\mathbf{R}) = \mathbf{R}$ and the pointwise convergence of b_n to b ensure the uniform $L^\infty(Q_T)$ bound on u_n.

If $b(+\infty) < +\infty$, then (H_5) ensures that any constant $\underline{u} \in \psi_n^{-1}(\|f^+\|_{L^\infty(Q_T)})$ is an entropy supersolution of (P_n). As hereabove, the comparison principle and the pointwise convergence of ψ_n to ψ (recall that $\psi(+\infty) = +\infty$) yield a uniform majoration of u_n. The case $b(-\infty) > -\infty$ is analogous.

(ii) We use the test function $\phi_n(u_n)$ in the weak formulation of (P_n). The duality product between

$$\phi_n(u_n) \in L^p\big(0,T; W_0^{1,p}(\Omega)\big) \cap L^\infty(Q_T)$$

and

$$\partial_t b_n(u_n) \in L^{p'}\big(0,T; W^{-1,p'}(\Omega)\big) + L^1(Q_T)$$

is treated via the standard chain rule argument [1,21,35]. Using in addition the chain rule of Lemma 3.3, the L^∞ bound on u_n shown in (i), the uniform coercivity assumption (H_9), and the obvious inequality $B_n(z) \leqslant b_n(z)\phi_n(z)$, we obtain the inequality

$$\int_\Omega B_n\big(u_n(t,\cdot)\big) + \int_0^t \int_\Omega \big(\psi_n(u_n)\phi_n(u_n) + c\left|\nabla\phi_n(u_n)\right|^p\big) \leqslant \int_\Omega b_n\big(u_0^n\big)\phi_n\big(u_0^n\big) + \int_0^t \int_\Omega f_n\phi_n(u_n)$$

with some $c > 0$ independent of n. Notice that the functions b_n, ϕ_n are locally uniformly bounded because they are monotone and converge pointwise to b, ϕ, respectively. Therefore the right-hand side of the above inequality is bounded uniformly in n, thanks to (i) and to the uniform bounds on the data u_0^n and f_n in $L^\infty(\Omega)$ and in $L^1(Q_T)$, respectively. The uniform estimate of the left-hand side follows. We then estimate $\|\phi_n(u_n)\|_{L^p(Q_T)}$ by the Poincaré inequality; the $(L^{p'}(Q_T))^N$ bound on $\mathfrak{a}_n(u_n, \nabla\phi_n(u_n))$ follows from the growth assumption (H_{10}).

(iii) Let $\Delta > 0$. The weak formulation of (P_n) yields, for a.e. $t, t+\Delta \in (0,T)$,

$$\int_\Omega \big(b_n(u_n)(t+\Delta) - b_n(u_n)(t)\big)\xi = \int_t^{t+\Delta} \int_\Omega \big[\big(-\mathfrak{f}_n(u_n) + \mathfrak{a}\big(u_n, \nabla\phi_n(u_n)\big)\big) \cdot \nabla\xi - \psi_n(u_n)\xi + f\xi\big]$$

for all $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (here and in the sequel, we drop the dependence on x in the notation). We take $\xi = \phi_n(u_n(t+\Delta)) - \phi_n(u_n(t))$ and integrate in t, then use the Fubini theorem which makes appear the factor Δ in the right-hand side; with the estimates of (i), (ii) and the uniform bounds on the data, we deduce that

$$\iint\limits_{Q_{T-\Delta}} \big|b_n(u_n)(t+\Delta) - b_n(u_n)(t)\big|\big|\phi_n(u_n)(t+\Delta) - \phi_n(u_n)(t)\big| \leqslant d|\Delta|. \tag{9}$$

Here d is a generic constant independent of n. In the sequel, denote by $(r_n)_n$ a generic sequence vanishing as $n \to \infty$. Notice that by the Dini theorem, b_n, ϕ_n converge to b, ϕ, respectively, uniformly on compact subsets of $\mathbf{R}$. By (i), it follows that we can replace b_n, ϕ_n in (9) by b, ϕ, provided that a term r_n is added to the right-hand side of (9).

Now notice that assumption $(\mathrm{H_{str}})$ implies that $\tilde{\phi} := \phi \circ b^{-1}$ is a continuous function. Let M be the L^∞ bound for u_n in (i). Let π be a concave modulus of continuity of $\tilde{\phi}$ on $[-M, M]$, $\varPi$ be its inverse, and $\overline{\varPi}(r) = r\varPi(r)$. Let $\overline{\pi}$ be the inverse of $\overline{\varPi}$. Note that $\overline{\pi}$ is concave, continuous, $\overline{\pi}(0) = 0$. Denote $v = b(u_n)(t+\Delta)$ and $y = b(u_n)(t)$. We have

$$\iint\limits_{Q_{T-\Delta}} \big|\tilde{\phi}(v) - \tilde{\phi}(y)\big| = \iint\limits_{Q_{T-\Delta}} \overline{\pi}\big(\overline{\varPi}\big(\big|\tilde{\phi}(v) - \tilde{\phi}(y)\big|\big)\big) \leqslant |Q_{T-\Delta}|\overline{\pi}\left(\frac{1}{|Q_{T-\Delta}|} \iint\limits_{Q_{T-\Delta}} \overline{\varPi}\big(\big|\tilde{\phi}(v) - \tilde{\phi}(y)\big|\big)\right).$$

Since $|\tilde{\phi}(v) - \tilde{\phi}(y)| \leqslant \pi(|v - y|)$, we have $\varPi(|\tilde{\phi}(v) - \tilde{\phi}(y)|) \leqslant |v - y|$ and

$$\begin{aligned} \overline{\varPi}\big(\big|\tilde{\phi}(v) - \tilde{\phi}(y)\big|\big) &= \varPi\big(\big|\tilde{\phi}(v) - \tilde{\phi}(y)\big|\big)\big|\tilde{\phi}(v) - \tilde{\phi}(y)\big| \leqslant |v - y|\big|\tilde{\phi}(v) - \tilde{\phi}(y)\big| \\ &\equiv \big|b(u_n)(t+\Delta) - b(u_n)(t)\big|\big|\phi(u_n)(t+\Delta) - \phi(u_n)(t)\big|. \end{aligned}$$

Therefore the estimate (9) (with b_n, ϕ_n and $d\Delta$ replaced by b, ϕ and $d\Delta + r_n$, respectively) implies

$$\begin{aligned} &\iint\limits_{Q_{T-\Delta}} \big|\phi(u_n)(t+\Delta) - \phi(u_n)(t)\big| \\ &\quad \leqslant |Q_{T-\Delta}|\overline{\pi}\left(\frac{1}{|Q_{T-\Delta}|} \iint\limits_{Q_{T-\Delta}} \big|u_n(t+\Delta) - u_n(t)\big|\big|\phi(u_n)(t+\Delta) - \phi(u_n)(t)\big|\right) \\ &\quad = |Q_{T-\Delta}|\overline{\pi}\left(\frac{1}{|Q_{T-\Delta}|}\Delta\right) \leqslant d\overline{\pi}(d\Delta + r_n), \end{aligned}$$

and finally, replacing ϕ with ϕ_n we get

$$\iint\limits_{Q_{T-\Delta}} \big|\phi_n\big(u_n(t+\Delta, x)\big) - \phi_n\big(u_n(t, x)\big)\big| \leqslant d\overline{\pi}(d\Delta + r_n) + r_n. \tag{10}$$

Now using the fact that $r_n \to 0$ as $n \to \infty$ and the fact that for all fixed $n \in \mathbb{N}$, the left-hand side of (10) tends to zero as $\Delta \to 0$, we deduce the claim of (iii). $\square$

6. Proof of the general continuous dependence property

In this section, we prove Theorem 2.2. First notice that the uniform estimates of Lemmas 5.1 and 3.2 apply. It follows that there exists a (not relabelled) subsequence $(u_n)_n$ such that

- $w_n := \phi_n(u_n)$ converges strongly in $L^1(Q_T)$ and a.e. on Q_T to a function w;
- w_n converges weakly in $L^p(0, T; W_0^{1,p}(\varOmega))$;

- $\chi_n := \mathfrak{a}(u_n, \nabla w_n)$ converges weakly in $L^{p'}(Q_T)$ to some limit χ;
- u_n converges to $\mu : Q_T \times (0,1) \to \mathbf{R}$ in the sense of the nonlinear weak-$\star$ convergence (3).

Denote $u(\cdot) = \int_0^1 \mu(\cdot, \alpha)\, d\alpha$. Thanks to the uniform L^∞ bound on u_n and to the uniform convergence of ϕ_n to ϕ on compact subsets of $\mathbf{R}$, we can identify the limit of $w_n(\cdot)$ with $\int_0^1 \phi(\mu(\alpha, \cdot))\, d\alpha$; moreover, $\phi(\mu(\alpha, \cdot))$ is independent of $\alpha \in (0,1)$, because the convergence of $\phi(u_n)$ to w is actually strong. Thus $w, \phi(u)$ and $\phi(\mu(\alpha))$ coincide. We also identify the limit of ∇w_n with ∇w, because the two functions coincide as elements of $\mathcal{D}'$.

The following lemma permits to deduce strong convergence of u_n to u on the set $[u \notin E]$ (recall that E is assumed to be closed).

Lemma 6.1. *Let $\phi_n(\cdot)$ be a sequence of continuous non-decreasing functions converging pointwise to a continuous function $\phi(\cdot)$. Assume $\phi(\cdot)$ is strictly increasing on $\mathbf{R} \setminus E$. Let I be an open interval contained in $\mathbf{R} \setminus E$, and $\phi_n(u_n) \to \phi(u)$ a.e. on Q. Then $u_n \to u$ a.e. on $[u \in I]$.*

Proof. Let $I = (a, b)$ and choose $I' = (a', b')$ with $a < a' < b' < b$. Introduce $\delta > 0$ by

$$\delta := \min\{\phi(a') - \phi(a), \phi(b) - \phi(b')\}.$$

Notice that by the Dini theorem, the convergence of $\phi_n(\cdot)$ to $\phi(\cdot)$ is uniform on all compact subset of $\mathbf{R}$. Thus for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\|\phi_n - \phi\|_{C([a,b])} < \varepsilon/2$. With $\varepsilon = \delta$, it follows that for all n sufficiently large, $\phi_n(b) - \phi(b') = \phi_n(b) - \phi(b) + \phi(b) - \phi(b') > -\delta/2 + \delta = \delta/2$, and similarly, $\phi(a') - \phi_n(a) > \delta/2$. By the monotonicity of $\phi_n(\cdot), \phi(\cdot)$,

$$\max_{u \in I',\, z \notin I} \left|\phi_n(z) - \phi(u)\right| = \max\{\phi_n(b) - \phi(b'), \phi(a') - \phi_n(a)\} > \delta/2.$$

Hence if $u(t,x) \in I'$ and if $|\phi_n(u_n(t,x)) - \phi(u(t,x))| < \delta/2$, we have $u_n(t,x) \in I$. Next, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|\phi_n(u_n(t,x)) - \phi(u(t,x))| \leqslant \varepsilon/2$ for all $n > N$. We conclude in particular that for a.e. $(t,x) \in [u \in I']$, one has $u_n \in I$ for all n large enough.

Thus for all $\varepsilon < \delta$, for a.e. point of $[u \in I']$ there exists $N \in \mathbb{N}$ such that at this point one has, for all $n > N$,

$$\begin{aligned} \left|\phi(u_n) - \phi(u)\right| &\leqslant \left|\phi(u_n) - \phi_n(u_n)\right| + \left|\phi_n(u_n) - \phi(u)\right| \\ &\leqslant \|\phi_n - \phi\|_{C([a,b])} + \varepsilon/2 \leqslant \varepsilon. \end{aligned}$$

Therefore $\phi(u_n)$ converges to $\phi(u)$ a.e. on $[u \in I']$. Since $\phi(\cdot)$ is continuously invertible on I, one also has $u_n \to u$ a.e. on $[u \in I']$. Since I is open and $I' \Subset I$ is arbitrary, the claim of the lemma follows. $\square$

Now we start to identify χ with $\mathfrak{a}(u, \nabla w)$. First, according to Remark 1.2, $\mathfrak{a}(u, \nabla w) = 0$ a.e. on the set $[u \in E] = [w \in G]$. Using Lemma 3.2, we now deduce that also $\chi = 0$ on this set.

Lemma 6.2. *Let χ be the weak $L^{p'}(Q_T)$ limit of the sequence*

$$\chi_n = \mathfrak{a}_n(u_n, \nabla\phi_n(u_n)),$$

and let $w_n = \phi_n(u_n)$ converge to $w = \phi(u)$ a.e. Then $\chi = 0$ a.e. on $[u \in E]$. Moreover, χ_n converges strongly to zero in $L^1([u \in E])$.

Proof. Since $\mathbb{1}_{[u \in E]} \in L^\infty(Q_T) \subset L^p(Q_T)$, by the definition of the weak convergence, the function $\chi \mathbb{1}_{[u \in E]} \equiv \chi \mathbb{1}_{[w \in G]}$ is the weak $(L^{p'}(Q_T))^N$ limit of $\chi_n \mathbb{1}_{[w \in G]}$. For all $\varepsilon > 0$, choose an open set

$G_\varepsilon \supset G$ of measure less than ε. Because $w_n \to w$ a.e. and G_ε is an open neighbourhood of G, we have

$$[u \in E] = [w \in G] = R \cup \left(\bigcup_{N \in \mathbb{N}} [w \in G, w_n \in G_\varepsilon \ \forall n \geqslant N] \right),$$

where $\mathrm{meas}(R) = 0$. Since

$$\big([w \in G,\ \ w_n \in G_\varepsilon \ \forall n \geqslant N] \big)_{N \in \mathbb{N}}$$

is an increasing collection of sets, the corresponding sequence of measures converges to $\mathrm{meas}([w \in G])$ as $N \to \infty$, by the continuity of the Lebesgue measure. Because

$$\mathrm{meas}\big([w \in G]\big) \geqslant \mathrm{meas}\big([w \in G,\ \ w_N \in G_\varepsilon]\big) \geqslant \mathrm{meas}\big([w \in G,\ \ w_n \in G_\varepsilon \ \forall n \geqslant N]\big),$$

we conclude that $\mathrm{meas}([w \in G,\ \ w_n \notin G_\varepsilon])$ tends to zero as $n \to \infty$. Now by the Hölder inequality,

$$\begin{aligned}
\|\chi_n \mathbb{1}_{[w \in G]}\|_{L^1(Q)} &= \iint_{[w \in G]} \big|\mathfrak{a}_n(u_n, \nabla w_n)\big| \\
&\leqslant \iint_{[w_n \in G_\varepsilon]} \big|a_n(u_n, \nabla w_n)\big| + \left(\iint_{Q_T} \big|a_n(u_n, \nabla w_n)\big|^{p'} \right)^{\frac{1}{p'}} \times \big(\mathrm{meas}\big([w \in G,\ \ w_n \notin G_\varepsilon]\big)\big)^{\frac{1}{p}}.
\end{aligned}$$

For all fixed $\varepsilon > 0$, the last term tends to zero as $n \to \infty$, thanks to the boundedness of the sequence $\mathfrak{a}_n(u_n, \nabla w_n)$ in $L^{p'}(Q_T)$. The first term in the right-hand side converges to zero as $\varepsilon \to 0$ uniformly in n; indeed, by (H$_9$) and by Lemma 3.2, it is majorated by $\frac{1}{c}\,\mathrm{meas}(G_\varepsilon) \leqslant \varepsilon / c$.

It follows that $\chi \mathbb{1}_{[u \in E]} = 0$ a.e. on Q_T, which ends the proof. $\square$

Now we use the Minty–Browder argument to identify χ with $\mathfrak{a}(u, \nabla w)$ on the sets $[u \in I]$, for all open $I \subset \mathbf{R} \setminus E$. A crucial role is played by Lemma 3.3, which permits us to pass to the limit in the product

$$\mathfrak{f}_n(u_n) \cdot \nabla w_n$$

of two weakly converging sequences.

We proceed in the classical way, but use the test functions $T_J(w)$ (at the limit) and $T_J(w_n)$ (before the passage to the limit), where $J = \phi(I)$ and

$$T_J(z) := \int_0^z \mathbb{1}_J(s)\, ds$$

is the truncation function that localizes the values of the solution to the interval I. Notice that we can assume that $\nabla T_J(w_n)$ converges to $\nabla T_J(w)$ weakly in $(L^p(Q_T))^N$. Indeed, the sequence $(T_J(w_n))_n$ is bounded in $L^p(0, T; W_0^{1,p}(\Omega))$, hence (up to a subsequence) it converges to a limit; this limit is identified with $T_J(w)$ since w_n converges to w in $L^1(Q_T)$, and T_J is Lipschitz continuous. Now let us give the details.

We first pass to the limit into the weak formulation of (P_n) (see Remark 2.2), getting

$$\begin{cases} \partial_t\left(\int_0^1 b(\mu(\alpha))\,d\alpha\right) + \operatorname{div}\left(\int_0^1 \mathfrak{f}(\mu(\alpha))\,d\alpha\right) + \int_0^1 \psi(\mu(\alpha))\,d\alpha = \operatorname{div}\chi + f \\ \quad \text{in } L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T), \\ \int_0^1 b(\mu(\alpha))\,d\alpha|_{t=0} = b(u_0) \end{cases} \tag{11}$$

in the same sense as in Remark 2.2. Take $\zeta \in \mathcal{D}([0,T))$. Because $w = \phi(u) \equiv \phi(\mu(\alpha))$ for a.e. $\alpha \in (0,1)$, using Lemma 3.4 we have

$$\int_0^T \left\langle \partial_t\left(\int_0^1 b(\mu(\alpha))\,d\alpha\right), T_J(\phi(u))\zeta \right\rangle = -\iint_{Q_T}\int_0^1 B_J(\mu(\alpha))\,d\alpha\,\partial_t\zeta - \int_\Omega B_J(u_0)\zeta(0) \tag{12}$$

with $B_J(z) := \int_0^z T_J(\phi(s))\,db(s)$. Define similarly $B_J^n(z) := \int_0^z T_J(\phi_n(s))\,db_n(s)$. By the standard chain rule lemma of [1,21,35] we get

$$\int_0^T \langle \partial_t b_n(u_n), T_J(\phi_n(u_n))\zeta\rangle = -\iint_{Q_T} B_J^n(u_n)\partial_t\zeta - \int_\Omega B_J^n(u_0^n)\zeta(0). \tag{13}$$

One shows easily that B_J^n converges to B_J uniformly on compact subsets of $\mathbf{R}$, because of (4). In particular, $B_J^n(u_0)$ converge to $B_J(u_0)$ in $L^1(\Omega)$. Moreover, the nonlinear weak-$\star$ convergence of u_n to μ yields

$$\lim_{n\to+\infty}\iint_{Q_T} B_J^n(u_n)\partial_t\zeta = \iint_{Q_T}\int_0^1 B_J(\mu(\alpha))\,d\alpha\,\partial_t\zeta.$$

Thus the left-hand side of (12) coincides with the "$n \to +\infty$ limit" of the left-hand side of (13).

Similarly, the nonlinear weak-$\star$ convergence of u_n to μ permits to conclude that

$$\begin{aligned} \iint_{Q_T}\int_0^1 \psi(\mu(\alpha))\,d\alpha\,T_J(\phi(u))\zeta &= \iint_{Q_T}\int_0^1 \psi(\mu(\alpha))T_J(\phi(\mu(\alpha)))\,d\alpha\,\zeta \\ &= \lim_{n\to+\infty}\iint_{Q_T} \psi_n(u_n)T_J(\phi_n(u_n))\zeta. \end{aligned}$$

Now let us take the test functions $T_J(w_n)\zeta$ and $T_J(w)\zeta$ in the weak formulation of (P_n) and in (11), respectively. Without loss of restriction, we can assume that $t = T$ is a Lebesgue point of $\int_0^1 B_J(\mu(\alpha))\,d\alpha$. Using the above calculations and Lemma 3.3, then letting $\zeta(t)$ converge to $\mathbb{1}_{[0,T)}(t)$ we deduce the equality

$$\iint_{Q_T} \chi\cdot\nabla T_J(w) = \lim_{n\to\infty}\iint_{Q_T} \mathfrak{a}_n(u_n,\nabla w_n)\cdot\nabla T_J(w_n). \tag{14}$$

Lemma 6.3. *Assume that for all $\xi \in \mathbf{R}^N$, $\mathfrak{a}_n(\cdot,\xi)$ converge to $\mathfrak{a}(\cdot,\xi)$ uniformly on compact subsets of an open interval $I \subset \mathbf{R}\setminus E$. With the notation and assumptions above, one has $\chi = \mathfrak{a}(u,\nabla w)$ a.e. on $[u\in I]=[w\in J]$.*

Proof. Take an arbitrary function $\tilde{\zeta} \in (L^p(Q_T))^N$ such that $\tilde{\zeta}=0$ a.e. on $[w\notin J]$. Take $\lambda\in\mathbf{R}$. Set $\zeta = \nabla T_J(w)+\lambda\tilde{\zeta}$; we have $\zeta=0$ a.e. on $[w\notin J]$. By the classical Minty–Browder argument, considering $\pm\lambda\downarrow 0$ one concludes that $\chi=\mathfrak{a}(u,\nabla w)$ a.e. on $[w\in J]$, provided the following relations can be justified:

$$\begin{aligned}
\iint\limits_{Q_T} \chi\cdot\big(\nabla T_J(w)-\zeta\big) &\geqslant \liminf_{n\to\infty}\iint\limits_{Q_T}\mathfrak{a}_n(u_n,\nabla w_n)\cdot\big(\nabla T_J(w_n)-\zeta\big)\\
&\geqslant \liminf_{n\to\infty}\iint\limits_{Q_T}\mathfrak{a}_n(u_n,\zeta)\cdot\big(\nabla T_J(w_n)-\zeta\big)\\
&= \liminf_{n\to\infty}\iint\limits_{Q_T}\mathfrak{a}(u,\zeta)\cdot\big(\nabla T_J(w_n)-\zeta\big)\\
&= \iint\limits_{Q_T}\mathfrak{a}(u,\zeta)\cdot\big(\nabla T_J(w)-\zeta\big).
\end{aligned}\tag{15}$$

Now we justify (15). Because (14) holds and $\nabla T_J(w_n)$ converges to $\nabla T_J(w)$ weakly in $(L^p(Q_T))^N$, the first inequality and the last equality in (15) are clear.

The second inequality comes from the monotonicity of $\mathfrak{a}_n(u_n,\cdot)$. Indeed, by the choice of ζ,

$$\begin{aligned}
\iint\limits_{[w\notin J]}\mathfrak{a}_n(u_n,\nabla w_n)\cdot\big(\nabla T_J(w_n)-\zeta\big) &= \iint\limits_{[w\notin J,\ w_n\in J]}\mathfrak{a}_n(u_n,\nabla w_n)\cdot\nabla w_n\\
&\geqslant 0 = \iint\limits_{[w\notin J]}\mathfrak{a}_n(u_n,\zeta)\cdot\big(\nabla T_J(w_n)-\zeta\big),
\end{aligned}$$

because $\mathfrak{a}_n(\cdot,0)\equiv 0$. Further, since J is open and $w_n\to w$ a.e. on Q_T, as in the proof of Lemma 6.2 we have $\mathrm{meas}([w\in J, w_n\notin J])\to 0$ as $n\to\infty$. On $[w\in J, w_n\in J]$ we have $\mathfrak{a}_n(u_n,\nabla w_n)=\mathfrak{a}_n(u_n,\nabla T_J(w_n))$, and ($\mathrm{H}_8$) applies. On $[w\in J, w_n\notin J]$ the terms with $\nabla T_J(w_n)$ are zero; finally, as in the proof of Lemma 6.2, the integrals

$$\iint\limits_{[w\in J,\ w_n\notin J]}\mathfrak{a}_n(u_n,\nabla w_n)\cdot\zeta \quad\text{and}\quad \iint\limits_{[w\in J,\ w_n\notin J]}\mathfrak{a}_n(u_n,\zeta)\cdot\zeta$$

both tend to zero, by the equi-integrability argument.

It remains to justify the last but one equality in (15). Thanks to Lemma 6.1, we have $u_n\to u$ a.e. on $[w\in J]=[u\in I]$. In particular, a.e. on $[u\in I]$ one has $u_n\in I$ for sufficiently large n. Using the locally uniform on I convergence of $\mathfrak{a}_n(\cdot,\xi)$ to $\mathfrak{a}(\cdot,\xi)$, we conclude that $\mathfrak{a}_n(u_n,\zeta)$ converges to $\mathfrak{a}(u,\zeta)$ a.e. on $[u\in I]$. By (H_{10}) and the Lebesgue dominated convergence theorem, (15) follows. $\square$

Finally, notice that thanks to (D′.1) and Remark 1.2, we have $\mathfrak{a}(\mu,\nabla w)\equiv\mathfrak{a}(u,\nabla w)$. The identification of χ thus being complete, from (11) we readily conclude that (μ,w) verifies (D′.2).

Now we can pass to the limit in the entropy inequalities corresponding to (P_n) and deduce (D′.3). Because regular boundary entropy pairs $(\eta^{\pm}_{c,\varepsilon},\mathfrak{q}^{\pm}_{c,\varepsilon})$ can be approximated by convex combinations of semi-Kruzhkov pairs, we have the analogue of (D′.3) for u_n with $(\eta^{\pm}_c,\mathfrak{q}^{\pm}_c)$ replaced by $(\eta^{\pm}_{c,\varepsilon},\mathfrak{q}^{n,\pm}_{c,\varepsilon})$ (with $\mathfrak{q}^{n,\pm}_{c,\varepsilon}$ converging to $\mathfrak{q}^{\pm}_{c,\varepsilon}$ uniformly on compact subsets of $\mathbf{R}$).

Consider the third term in (D′.3). We have

$$\lim_{n\to\infty}\iint\limits_{Q_T}\left(\eta^{\pm}_{c,\varepsilon}\right)'(u_n)\mathfrak{a}_n(u_n,\nabla w_n)\cdot\nabla\xi$$
$$=\lim_{n\to\infty}\left(\iint\limits_{[w\in G]}+\iint\limits_{[w\notin G]}\right)\left(\eta^{\pm}_{c,\varepsilon}\right)'(u_n)\nabla\xi\cdot\mathfrak{a}_n(u_n,\nabla w_n). \tag{16}$$

By Lemma 6.2 and because $(\eta^{\pm}_{c,\varepsilon})'(u_n)\nabla\xi$ are bounded uniformly in n, the first term converges to zero; also notice that

$$0=\iint\limits_{[w\in G]}\left(\eta^{\pm}_{c,\varepsilon}\right)'(u)\nabla\xi\cdot\mathfrak{a}(u,\nabla w)$$

because $\mathfrak{a}(u,\nabla w)=\mathfrak{a}(u,0)=0$ a.e. on $[w\in G]$. By Lemma 6.1, we have $u_n\to u$ a.e. on $[w\notin G]$; by the dominated convergence theorem, we infer that $(\eta^{\pm}_{c,\varepsilon})'(u_n)\nabla\xi$ converges to $(\eta^{\pm}_{c,\varepsilon})'(u)\nabla\xi$ strongly in $L^p([w\notin G])$. Because also $\chi_n=a_n(u_n,\nabla w_n)$ converges to $\chi=a(u,\nabla w)$ weakly in $L^{p'}([w\notin G])$, the second term in the right-hand side of (16) converges to

$$\int\limits_{[w\notin G]}\left(\eta^{\pm}_{c,\varepsilon}\right)'(u)\nabla\xi\cdot\mathfrak{a}(u,\nabla w).$$

The passage to the limit as $\varepsilon\to 0$ in the other terms in (D.3) is straightforward, using the uniform boundedness of $(u_n)_n$ and the nonlinear weak-$\star$ convergence property (3). At the limit, we conclude that (D′.3) also holds. Thus (μ,w) is an entropy process solution of (P).

Now by the result of Theorem 2.1(i), (μ,w) gives rise to an entropy solution

$$u=\int_0^1\mu(\alpha)\,d\alpha$$

of (P). By (3) (with $F=\mathrm{Id}$), we conclude that $(u_n)_n$ possesses a subsequence that converges in $L^\infty(Q_T)$ weak-$\star$ to u; we have already shown that the corresponding subsequence $\phi_n(u_n)$ converges to $\phi(u)$ in $L^1(Q_T)$.

Moreover, $b(\mu(\alpha))$ and $\psi(\mu(\alpha))$ are in fact independent of α. By Theorem 2.1(iii), we also have the uniqueness of $b(u)$ and $\psi(u)$ such that u is an entropy solution of (P). By the well-known result of the nonlinear weak-$\star$ convergence (see, e.g., Hungerbühler [26]), it follows that the whole sequences $(b_n(u_n))_n$ and $(\psi_n(u_n))_n$ converge to $b(u)$ and $\psi(u)$, respectively, in measure on Q_T and in $L^1(Q_T)$.

This ends the proof of Theorem 2.2.

Remark 6.1. In the case assumption (H_5) is dropped, in order to deduce that u is an entropy process solution, along with the assumption (H_{str}) one needs

– a growth restriction on $\tilde{\mathfrak{f}}$ of the following kind: there exists a function $M\in C(\mathbf{R}^+;\mathbf{R}^+)$ and a function $\mathcal{L}$ with $\lim\limits_{z\to+\infty}\mathcal{L}(z)/z=0$ such that for all $r\in\mathbf{R}$,

$$\left|\tilde{\mathfrak{f}}\big(b(r),\phi(r),\psi(r)\big)\right|\leqslant M\big(|b(r)|\big)\mathcal{L}\left(\left|\phi(r)\right|^p+\int_0^r\phi(s)\,db(s)+\psi(r)\phi(r)\right);$$

– the same growth restriction with $|\tilde{\mathfrak{f}}(b(r),\phi(r),\psi(r))|$ replaced with $|b(r)|$ and with $|\psi(r)|$.

Indeed, these inequalities make it possible to use the nonlinear weak-$\star$ convergence framework without the uniform L^∞ bound on $(u_n)_n$ (see Ball [8] and Hungerbühler [26]).

7. Proof of Theorem 2.3

In this section, we prove Theorem 2.3. The uniqueness claim was shown in Theorem 2.1; also notice that the continuous dependence result under the structure assumption $(\mathrm{H}_{\mathrm{str}})$ was shown in Theorem 2.2. Let us first prove the existence claim.

(i) First, consider the case where assumption $(\tilde{\mathrm{H}}_{\mathrm{str}})$ is fulfilled. Consider an approximation of (P) by regular problems (P_n) with data $(b_n, \psi, \phi_n, \mathfrak{a}, \tilde{\mathfrak{f}}; u_0, f)$ such that the assumptions of Theorem 2.2 are fulfilled, and $b_n, [b_n]^{-1}, \mathfrak{f}_n, \phi_n, [\phi_n]^{-1}$ are Lipschitz continuous on $\mathbf{R}$. Using classical methods (cf. Alt and Luckhaus [1], Lions [30]), one shows that there exists a weak solution $u_n \in L^p(0,T;W_0^{1,p}(\Omega))$ to the problem (P_n) in the sense

$$\partial_t b_n(u_n) + \operatorname{div} \tilde{\mathfrak{f}}_n(u_n) + \psi(u_n) = \operatorname{div} \mathfrak{a}\big(u_n, \nabla\phi_n(u_n)\big) + f$$

in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T)$, with initial data

$$b_n(u_n)|_{t=0} = b_n(u_0).$$

In addition, u_n verifies the entropy formulation (D′.3), obtained along the lines of Carrillo [20]. By Theorem 2.2, we conclude that there exists an entropy solution of (P).

To prove existence without the structure condition $(\mathrm{H}_{\mathrm{str}})$, we use the particular multi-step approximation approach of Ammar and Wittbold [4] (see also Ammar and Redwane [3]). We replace b by $b_k := b + \frac{1}{k}\mathrm{Id}$ and ψ, by $\psi_{m,n} := \psi + \frac{1}{n}\mathrm{Id}^+ - \frac{1}{m}\mathrm{Id}^-$. The result of (i) for the corresponding problem $(P^k_{m,n})$ is already proved.

There exists a function $u^k_{m,n}$, constructed by means of the nonlinear semigroup theory (see, e.g., [12]), such that $b_k(u^k_{m,n})$ is the unique integral solution to the abstract evolution problem associated with $(P^k_{m,n})$ (here and below, we refer to Ammar and Wittbold [4], Ammar and Redwane [3] for details). One then shows that $u^k_{m,n}$ coincides with the unique entropy solution of $(P^k_{m,n})$, the existence of this entropy solution being already shown. Further, the whole set $(u^k_{m,n})_{k,m,n}$ verifies the uniform *a priori* estimates of Lemmas 5.1, 3.2.

We then pass to the limit in $u^k_{m,n}$ in the following order: first $k \to +\infty$, then $n \to +\infty$, $m \to +\infty$.

While letting $k \to +\infty$, we use the fact that $\psi^{-1}_{m,n}$ is Lipschitz continuous. The fundamental estimates for the semigroup solutions permit to show that $\psi_{m,n}(u_{m,n})$ are uniformly continuous on $(0,T)$ with values in $L^1(\Omega)$; thus we get the strong precompactness of $(u^k_{m,n})_k$ in $L^1(Q_T)$. Thus, up to a subsequence, $u^k_{m,n}$ converge to $u_{m,n}$ which is an entropy solution of problem $(P_{m,n})$ corresponding to the data $(b, \psi_{m,n}, \phi, \mathfrak{a}, \tilde{\mathfrak{f}}; u_0, f)$.

Finally, we use the inequalities $u_{m+1,n} \leqslant u_{m,n} \leqslant u_{m,n+1}$ which follow readily form the comparison principle of Theorem 2.1(ii). The monotonicity argument yields the strong convergence of $u_{m,n}$. Now the whole scheme of the proof of Theorem 2.2 applies with considerable simplifications, because no nonlinear weak-$\star$ convergence arguments are not needed. Passing to the limit in $u_{m,n}$, we conclude that the limit u is an entropy solution of the original problem (P). This ends the existence proof.

(ii) Existence for the limit data (u_0, f) is now shown and we can apply Theorem 2.1(ii). Then we deduce the $L^1(Q_T)$ convergence of $b(u_n), \psi(u_n)$ to $b(u), \psi(u)$, respectively. The convergence of $\phi(u_n)$ to $\phi(u)$ in $L^1(Q_T)$ follows by hypothesis $(\mathrm{H}'_{\mathrm{str}})$ and because our assumptions imply the uniform $L^\infty(Q_T)$ bound on u_n.

The remaining claim of the strong $(L^p(Q_T))^N$ convergence of $\nabla\phi(u_n)$ is a rather standard part of the Minty–Browder trick. The argument is based upon the proof of Theorem 2.2. Because we already have the strong compactness of $(\phi(u_n))_n$, we can bypass the hypothesis $(\mathrm{H}_{\mathrm{str}})$ and deduce that the L^∞ weak-$\star$ limit $\hat{u}$ of u_n is an entropy solution of (P) with the data (u_0, f). In particular, the $(L^{p'}(Q_T))^N$ weak limit χ of $\mathfrak{a}(u_n, \nabla\phi(u_n))$ is equal to $\mathfrak{a}(\hat{u}, \nabla\phi(\hat{u}))$. Now notice that by Theorem 2.1(iii), under the structure condition (H'_{str}) we also have the uniqueness of $\phi(u)$ such that u is an entropy solution of (P); moreover, by Remark 1.2, we also have the uniqueness of $\mathfrak{a}(u, \nabla\phi(u))$ such that u is an entropy solution of (P). Thus χ coincides with $\mathfrak{a}(u, \nabla\phi(u))$, so that (14) now reads as

$$\iint_{Q_T} \mathfrak{a}\big(u, \nabla\phi(u)\big) \cdot \nabla\phi(u) = \lim_{n\to\infty} \iint_{Q_T} \mathfrak{a}_n(u_n, \nabla w_n) \cdot \nabla\phi(u_n). \tag{17}$$

It follows by the weak convergences of $\nabla\phi(u_n)$ and of $\mathfrak{a}(u_n, \nabla\phi(u_n))$ that

$$\lim_{n\to\infty} \iint_{Q_T} \big(\mathfrak{a}\big(u, \nabla\phi(u)\big) - \mathfrak{a}\big(u_n, \nabla\phi(u_n)\big)\big) \cdot \big(\nabla\phi(u) - \nabla\phi(u_n)\big) = 0. \tag{18}$$

Notice that a.e. on the set $[u \in E]$, $\nabla\phi(u_n)$ converges to $0 = \nabla\phi(u)$, by Lemma 6.2 and by the coercivity assumption (H_9). On the set $[u \notin E]$, we can use Lemma 6.1 to replace $\mathfrak{a}(u, \nabla\phi(u))$ with $\mathfrak{a}(u_n, \nabla\phi(u))$ in the above formula (18). Using the uniform monotonicity assumption (H'_8), we can now conclude that the convergence of $\nabla\phi(u_n)$ to $\nabla\phi(u)$ holds a.e. on Q_T.

Separating again the sets $[u \in E]$ and $[u \notin E]$, we deduce that the sequence of non-negative functions $\mathfrak{a}(u_n, \nabla\phi(u_n)) \cdot \nabla\phi(u_n)$ converges to $\mathfrak{a}(u, \nabla\phi(u)) \cdot \nabla\phi(u)$ a.e. on Q_T. Together with (17), this implies that $\mathfrak{a}(u_n, \nabla\phi(u_n)) \cdot \nabla\phi(u_n)$ also converge to $\mathfrak{a}(u, \nabla\phi(u)) \cdot \nabla\phi(u)$ in $L^1(Q_T)$; in particular, they are equi-integrable on Q_T. The coercivity assumption (H_9) now implies the equi-integrability on Q_T of the functions $|\nabla\phi(u_n)|^p$. Combining this argument with the a.e. convergence of $\nabla\phi(u_n)$, we deduce our claim from the Vitali theorem. The $(L^{p'}(Q_T))^N$ convergence of $\mathfrak{a}(u_n, \nabla\phi(u_n))$ to $\mathfrak{a}(u, \nabla\phi(u))$ follows in the same way.

8. Well-posedness for the doubly nonlinear elliptic problem

We first notice that the well-posedness result for the degenerate elliptic problem

$$\text{(S)} \qquad \begin{cases} \psi(u) + \operatorname{div}\tilde{\mathfrak{f}}\big(\psi(u), \phi(u)\big) - \operatorname{div}\mathfrak{a}\big(u, \nabla\phi(u)\big) = s & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$(H'_5) \qquad s \in L^\infty(\Omega) \quad \text{and} \quad \psi(\mathbf{R}) = \mathbf{R}$$

follows from Theorem 2.3, upon setting $b \equiv 0$, $f(t, \cdot) \equiv s(\cdot)$ and arbitrarily prescribing u_0. The definition of an entropy solution of (S) can also be formally obtained from Definition 2.2.

Let us notice that the analogue of the general continuous dependence property of Theorem 2.2 holds without any additional structure condition:

Theorem 8.1. *Let $(\psi_n, \phi_n, \mathfrak{a}_n, \tilde{\mathfrak{f}}_n; s)$, $n \in \mathbb{N}$, be a sequence which converges to $(\psi, \phi, \mathfrak{a}, \tilde{\mathfrak{f}}; s)$ in the following sense*:

- *ψ_n, ϕ_n converge pointwise to ψ, ϕ respectively*;
- *$\tilde{\mathfrak{f}}_n, \mathfrak{a}_n$ converge to $\tilde{\mathfrak{f}}, \mathfrak{a}$, respectively, uniformly on compacts*;
- *s_n converges to s in $L^1(\Omega)$.*

Assume that $(\psi, \phi, \mathfrak{a}, \tilde{\mathfrak{f}}; s)$ and $(\psi_n, \phi_n, \mathfrak{a}_n, \tilde{\mathfrak{f}}_n; s_n)$ (for each n) satisfy the hypotheses (H_1), (H_6)–(H_{11}), *and* (H'_5), *and, moreover, that the functions C in* (H_9), (H_{10}), *and* (H_{11}) *as well as the $L^\infty(\Omega)$ bound in* (H'_5) *are independent of n. We denote by* (S_n) *the analogue of problem* (S) *corresponding to the data and coefficients $(\psi_n, \phi_n, \mathfrak{a}_n, \tilde{\mathfrak{f}}_n; s_n)$.*

Assume that ϕ satisfies the technical hypotheses (H_2), (H_3).

Let u_n be an entropy solution of problem (S_n). Then the functions u_n converge to an entropy solution u of (S) in $L^\infty(\Omega)$ weakly-$\star$, up to a subsequence. Furthermore, the functions $\phi_n(u_n)$ converge to $\phi(u)$ in $L^p(\Omega)$ up to a subsequence, and the whole sequence $\psi_n(u_n)$ converges to $\psi(u)$ in $L^1(\Omega)$.

The proof of Theorem 8.1 is contained in the one of Theorem 2.2, because the $L^p(\Omega)$ bound on ∇w_n is sufficient for the strong precompactness of w_n.

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