Periodic Solutions for Delay Differential Equations Model of Plankton Allelopathy

JIN ZHEN AND ZHIEN MA
Department of Applied Mathematics
Xi'an Jiaotong University
Xi'an, Shaan'xi 710049, P.R. China

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Abstract—In this paper, we propose a modified delay differential equation model of the growth of two species of plankton having competitive and allelopathic effects on each other. By using the continuation theorem of coincidence degree theory, a set of easily verifiable sufficient conditions are obtained for the existence of positive periodic solutions for this model. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The study of tremendous fluctuations in abundance of many phytoplankton communities is an important subject in aquatic ecology. These changes of size and density of phytoplankton depend on several factors, such as physical factors, seasonal, variation of necessary nutrients, effects of time delays, or a combination of these, etc. Several workers have noted that the increased population of one species of phytoplankton might affect the growth of one or several other species by the production of allelopathic toxins or stimulators, influencing bloom, pulses, and seasonal succession. For detailed literature studies on allelopathic interactions in the phytoplanktonic world, see the review of Hellebust [1] and the book by Rice [2].

Maynard-Smith [3] incorporated the effect of toxic substances in a two species Lotka-Volterra competitive system by considering that each species produces a substance toxic to the other but only when the other is present.

Chattopadhyay [4] investigated the stability properties of the above system. Mukhopadhy et al. [5] modified the model of Maynard-Smith [3]; they propose a discrete delay differential equation model of the growth of two species of plankton having competitive and allelopathic effects on each other. It may be more realistic to assume delay is not discrete but a distributed delay.
Environmental fluctuation is important in an ecosystem. Naturally, more realistic models require the inclusion of the effect of environmental changing, especially the environmental parameters which are time-dependent periodic changing (e.g., seasonal changes, food supplies, etc.). This motivates us to modify the model of Mukhopadhyay et al. [5].

In this paper, we study a two-dimensional system that arises in plankton allelopathy involving distributed time delays and environmental periodical changing. We also assume that parameters are periodic functions. By using coincidence degree theory, sufficient conditions are obtained for the existence of positive periodic solutions for this model.

The organization of this paper is as follows. In the next section, the mathematical model will be introduced. Section 3 presents the main results. An example is given in Section 4.

2. THE MATHEMATICAL MODEL

Lotka-Volterra two species competition model can be written as

\[
\frac{dN_1}{dt} = N_1[r_1 - a_{11}N_1(t) - a_{12}N_2(t)],
\]

\[
\frac{dN_2}{dt} = N_2[r_2 - a_{21}N_1(t) - a_{22}N_2(t)],
\]

where \(N_1(t), N_2(t)\) are the population densities of two competing species; \(r_1, r_2\) are the rates of cell proliferation per hour; \(a_{11}, a_{22}\) are the rates of intraspecific competition of the first and second species, respectively; \(a_{12}, a_{21}\) are the rates of interspecific competition of the first and second species, respectively; \(r_i/a_{ii} \ (i = 1, 2)\) are environmental carrying capacities.

Maynard-Smith [3] and finally Chattopadhyay [4] modified system (2.1) by considering that each species produces a substance toxic to the other, but only when the other is present. Then system (2.1) can be written as

\[
\frac{dN_1}{dt} = N_1[r_1 - a_{11}N_1(t) - a_{12}N_2(t) - b_1N_1(t)N_2(t)],
\]

\[
\frac{dN_2}{dt} = N_2[r_2 - a_{21}N_1(t) - a_{22}N_2(t) - b_2N_1(t)N_2(t)],
\]

where \(b_1\) and \(b_2\) are, respectively, the rates of toxic inhibition of the first species by the second and vice versa.

Notice that the production of the toxic substance allelopathic to the competing species, simply termed “allelochemic”, will not be instantaneous, but delayed by different discrete time lags required for the maturity of both species. Hence, Mukhopadhyay et al. [5] have modified system (2.2), and the modified system can be written as

\[
\frac{dN_1}{dt} = N_1[r_1 - a_{11}N_1(t) - a_{12}N_2(t) - b_1N_1(t)N_2(t - \tau_2)],
\]

\[
\frac{dN_2}{dt} = N_2[r_2 - a_{21}N_1(t) - a_{22}N_2(t) - b_2N_1(t - \tau_1)N_2(t)],
\]

where \(\tau_i > 0, i = 1, 2\) are the times required for the maturity of first species and second species, respectively. Considering the periodic changing of environment and distributed delay, we modify system (2.3) to the form

\[
\frac{dN_1}{dt} = N_1 \left[ r_1(t) - \sum_{j=1}^{2} a_{1j}(t) \int_{-T_{1j}}^{0} K_{1j}(s)N_2(t + s) \, ds - b_1(t)N_1(t) \int_{-\tau_2}^{0} f_2(s)N_2(t + s) \, ds \right],
\]

\[(2.4)\]
\[
\frac{dN_2}{dt} = N_2 \left[ r_2(t) - \sum_{j=1}^{2} a_{2j}(t) \int_{-\tau_j}^{0} K_{2j}(s) N_j(t + s) \, ds \right. \\
\left. - b_2(t) N_2(t) \int_{-\tau_i}^{0} f_1(s) N_1(t + s) \, ds \right],
\]

where \( r_i(t), a_{ij}(t) > 0, \) \( b_i(t) > 0 \) \((i, j = 1, 2)\) are continuous \( \omega \)-periodic functions, \( T_{ij}, \) \( \tau_i \) are positive constants, \( K_{ij} \in C([T_{ij}, 0], (0, \infty)) \), and \( \int_{0}^{T_{ij}} K_{ij}(s) \, ds = 1, \) \( f_1 \in C([-\tau_i, 0], (0, \infty)) \) and \( \int_{-\tau_i}^{0} f_1(s) \, ds = 1 \) \((i, j = 1, 2)\). For the work concerning the existence of periodic solutions of (2.4) which was done using coincidence degree theory, we shall study system (2.4) in the following sections.

3. EXISTENCE OF A POSITIVE PERIODIC \SOLUTION

In this section, by using the Mawhin’s continuation theorem, we shall show the existence of at least one positive periodic solution of system (2.4). To do so, we need to make some preparations.

Let \( X, Y \) be real Banach spaces, \( L : \text{dom} \ L \subset X \to Y \) be a Fredholm mapping of index zero (index \( L = \text{dim} \text{ker} L - \text{codim} \text{Im} L \)), and let \( P : X \to X, Q : Y \to Y \) be continuous projectors such that \( \text{Im} P = \text{Ker} L, \) \( \text{Ker} Q = \text{Im} L \subset \text{Im} Q \). Denote by \( L_P \) the restriction of \( L \) to \( \text{Dom} L \cap \text{Ker} P, \) \( K_P : \text{Im} L \to \text{Ker} P, \) \( K_P : \text{im} L \to \text{Ker} P \cap \text{Dom} L \) the inverse (to \( L_P \)), and \( J : \text{Im} Q \to \text{Ker} L \) an isomorphism of \( \text{Im} Q \) onto \( \text{Ker} L \).

For convenience, we introduce Mawhin’s continuation theorem [6] as follows.

**Lemma 3.1.** Let \( \Omega \subset X \) be an open bounded set and let \( N : X \to Y \) be a continuous operator which is \( L \)-compact on \( \Omega \) (i.e., \( QN : \Omega \to Y \) and \( \text{Ker} L \cap \Omega \to Y \) and \( \text{Ker} L \subset \text{Im} L \cap \Omega \) are compact), \( L : \text{dom} L \subset Y \to Z \) be a Fredholm mapping of index zero. Assume

(a) for each \( \lambda \in (0, 1), x \in \partial \Omega \cap \text{Ker} L, Lx \neq \lambda Nx; \)
(b) for each \( x \in \text{Ker} L \cap \partial \Omega, QNx \neq 0, \) and

\[ \text{deg} \{ JQN, \Omega \cap \text{Ker} L, 0 \} \neq 0. \]

Then \( Lx = Nx \) has at least one solution in \( \text{dom} L \cap \Omega \).

In the following, we shall use the notation

\[ \bar{z} = \frac{1}{\omega} \int_0^\omega z(t) \, dt, \quad |z|_0 = \max_{t \in [0, \omega]} |z(t)|, \]

\[ \alpha_{ij} = -a_{ji}, \quad \beta_{ij} = -\bar{r}_i \bar{a}_{ij} - \bar{a}_{ii} \bar{b}_j + \bar{a}_{ij} \bar{a}_{ij}, \]

\[ \gamma_{ij} = \bar{r}_i \bar{a}_{jj} - \bar{r}_j \bar{a}_{ij}, \quad i \neq j, \quad i, j = 1, 2. \]

We make the following assumptions.

\( (H_1) \) \( \bar{R}_i = \frac{1}{\omega} \int_0^\omega |r_i(t)| \, dt \geq (1/\omega) \int_0^\omega r_i(t) \, dt > 0. \)

\( (H_2) \) \( \bar{r}_i \bar{a}_{jj} - \bar{r}_j \bar{a}_{ij} \epsilon (\bar{r}_i + \bar{r}_j) \omega > 0, \quad i, j = 1, 2, \quad i \neq j. \)

Next, we introduce Lemma 3.2.

**Lemma 3.2.** Consider the following algebraic equations:

\[ \bar{a}_{11} N_1 + \bar{a}_{12} N_2 + \bar{b}_1 N_1 N_2 = \bar{r}_1, \]

\[ \bar{a}_{21} N_1 + \bar{a}_{22} N_2 + \bar{b}_2 N_1 N_2 = \bar{r}_2. \]

Assuming that \( (H_1) \) and \( (H_2) \) hold, then the following conclusions hold.

(i) If \( \alpha_{12} \leq 0 \) and \( \alpha_{21} \leq 0, \) then equations (3.1) have a unique positive solution \((N_1^*, N_2^*)\).
(ii) If $\alpha_{12} > 0$, then equations (3.1) have two positive solutions $(N_{11}^*, N_{22}^*)$ and $(N_{12}^*, N_{21}^*)$.

(iii) If $\alpha_{21} > 0$, then equations (3.1) have two positive solutions $(N_{11}^*, N_{22}^*)$ and $(N_{11}^*, N_{21}^*)$, where

\[
N_{11}^* = \frac{-\beta_{12} - \sqrt{\beta_{12}^2 - 4\alpha_{12}\gamma_{12}^2}}{2\alpha_{12}}, \quad N_{12}^* = \frac{-\beta_{12} + \sqrt{\beta_{12}^2 - 4\alpha_{12}\gamma_{12}^2}}{2\alpha_{12}},
\]

\[
N_{22}^* = \frac{-\beta_{21} - \sqrt{\beta_{21}^2 - 4\alpha_{21}\gamma_{21}^2}}{2\alpha_{21}}, \quad N_{21}^* = \frac{-\beta_{21} + \sqrt{\beta_{21}^2 - 4\alpha_{21}\gamma_{21}^2}}{2\alpha_{21}}.
\]

**Proof.** The nonzero solutions of algebraic equations (3.1) can be determined by (see [5, p. 170])

\[
\alpha_{12}N_2^2 + \beta_{12}N_1 + \gamma_{12} = 0, \quad (3.2)
\]

\[
\alpha_{21}N_2^2 + \beta_{21}N_1 + \gamma_{21} = 0. \quad (3.3)
\]

From Assumption (H2), we obtain

\[
\gamma_{12} > 0, \quad \gamma_{21} > 0, \quad \tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}\tilde{a}_{21} > 0. \quad (3.4)
\]

(i) If $\alpha_{12} < 0$, $\alpha_{21} < 0$, from equations (3.2) and (3.3) we see that equations (3.1) have a unique positive solution $(N_{11}^*, N_{21}^*)$.

(ii) If $\alpha_{12} > 0$, it follows from (3.4) that $\alpha_{21} < 0$. Hence, equation (3.3) has a unique positive solution $N_{21}^*$. Notice that

\[
\beta_{12} = -\left(\frac{\tilde{b}_1}{\tilde{a}_{11}} + \frac{\tilde{a}_{12}}{\tilde{r}_1}\right)\gamma_{21} - \left(\frac{\tilde{r}_{11}\alpha_{12}}{\tilde{a}_{11}} + \frac{\tilde{a}_{11}\gamma_{12}}{\tilde{r}_1}\right) < 0, \quad (3.5)
\]

and

\[
\beta_{12}^2 - 4\alpha_{12}\gamma_{12} = \left(\frac{\tilde{b}_1}{\tilde{a}_{11}} + \frac{\tilde{a}_{12}}{\tilde{r}_1}\right)^2 \gamma_{21}^2 + \left(\frac{\tilde{r}_{11}\alpha_{12}}{\tilde{a}_{11}} - \frac{\tilde{a}_{11}\gamma_{12}}{\tilde{r}_1}\right)^2 + 4\gamma_{21}\left(\frac{\tilde{b}_1}{\tilde{a}_{11}} + \frac{\tilde{a}_{12}}{\tilde{a}_{11}}\tilde{r}_1\right)^2 \left(\frac{-\tilde{r}_{11}\alpha_{12}}{\tilde{a}_{11}} + \frac{\tilde{a}_{11}\gamma_{12}}{\tilde{r}_1}\right)^2 > 0. \quad (3.6)
\]

From (3.5) and (3.6), we obtain that equation (3.2) has two positive solutions $N_{11}^*$ and $N_{12}^*$, which implies that (3.1) have two positive solutions $(N_{11}^*, N_{22}^*)$ and $(N_{12}^*, N_{21}^*)$.

In a similar way as in the proof of (ii), we can prove that conclusion (iii) holds.

**Lemma 3.3.** (See [7].) Let $D \subset \mathbb{R}^n$ be open bounded symmetric and $0 \in D$. Let $F \in C(\bar{D})$ be such that $0 \notin F(\partial D)$ and $F(x)/|F(x)| \neq F(-x)/|F(-x)|$ on $\partial D$. Then $\deg(F, D, 0)$ is odd.

**Lemma 3.4.** The domain $R^*_+ = \{(N_1, N_2)^T \mid N_i > 0, \ i = 1, 2\}$ is positive invariant with respect to equations (2.4).

**Proof.** Since

\[
N_i(t) = N_i(0) \exp \left\{ \int_0^t \left[ r_i(t) - \sum_{j=1}^2 a_{ij}(t) \int_{-T_{ij}}^0 K_{ij}(s)e^{x_j(t+s)} ds - b_i(t)e^{x_i(t)} \int_{-\tau_k}^0 f_k(s)e^{x_k(t+s)} ds \right] dt \right\}, \quad i \neq k, \ i, k, 1, 2,
\]

the assertion of the lemma follows immediately for all $t \in [0, +\infty)$ since considering the biological significance of equations (2.4), we specify $N_i(0) > 0$; that is, $(N_1(0), N_2(0)) \in R^*_+$. The proof is complete.
THEOREM 3.5. Under Hypotheses (H1) and (H2), equations (2.4) have at least one positive \( \omega \)-periodic solution.

PROOF. Making the change of variable

\[
N_i(t) = e^{x_i(t)}, \quad i = 1, 2,
\]

then equations (2.4) are reformulated as

\[
\frac{dx_i}{dt} = r_i(t) - \sum_{j=1}^{2} a_{ij}(t) \int_{-T_{ij}}^{0} K_{ij}(s) e^{x_j(t+s)} ds - b_i(t) e^{x_i(t)} \int_{-\tau_i}^{0} f_k(s) e^{x_k(t+s)} ds,
\]

where \( k \neq i, i, k = 1, 2 \). It is obvious that if system (3.8) has an \( \omega \)-periodic solution \((x_1^*(t), x_2^*(t))^T\), then \((N_1(t), N_2(t))^T = (\exp[x_1^*(t)], \exp[x_2^*(t)])\) is a positive \( \omega \)-periodic solution of system (2.4).

So, to complete the proof, it suffices to show that system (3.8) has an \( \omega \)-periodic solution.

In order to use the continuation theorem of coincidence degree theory to establish the existence of an \( \omega \)-periodic solution of system (3.8), we take

\[
X = Y = \{ x(t) = (x_1(t), x_2(t))^T \in C(R, R^2) : x(t + \omega) = x(t), \} ,
\]

\[
\| x \| = \left( \sum_{i=1}^{2} \| x_i(t) \|_0 \right)^{1/2}, \quad \text{for any } x \in X \quad (\text{or } x \in Y).
\]

Then \( X \) and \( Y \) are both Banach spaces when they are endowed with the norm \( \| \cdot \| \).

Let \( N : X \rightarrow X \), and

\[
N x = \begin{pmatrix}
\left( r_1(t) - \frac{1}{\omega} \int_{0}^{\omega} \sum_{j=1}^{2} a_{1j}(t) \int_{-T_{1j}}^{0} K_{1j}(s) e^{x_j(t+s)} ds - b_1(t) e^{x_1(t)} \int_{-\tau_1}^{0} f_1(s) e^{x_1(t+s)} ds \right)

\left( r_2(t) - \frac{1}{\omega} \int_{0}^{\omega} \sum_{j=1}^{2} a_{2j}(t) \int_{-T_{2j}}^{0} K_{2j}(s) e^{x_j(t+s)} ds - b_2(t) e^{x_2(t)} \int_{-\tau_2}^{0} f_2(s) e^{x_2(t+s)} ds \right)
\end{pmatrix};
\]

\[
L x = \dot{x}, \quad Px = \frac{1}{\omega} \int_{0}^{\omega} x(t) dt = Q x, \quad x(t) \in X.
\]

Obviously, \( \text{Ker} L = R^2, \text{Im} L = \{ x \in X : \int_{0}^{\omega} x(t) dt = 0 \} \), and \( \text{dim Ker} L = \text{codim Im} L = 2 \).

Since \( \text{Im} L \) is closed in \( Y \), \( L \) is a Fredholm mapping of index zero. It is easy to show that \( P \) and \( Q \) are continuous projectors such that

\[
\text{Im} P = \text{Ker} L, \quad \text{Ker} Q = \text{Im} L = \text{Im}(I - Q).
\]

Furthermore, through an easy computation, we can find that the inverse \( K_P \) of \( L_n \) has the form

\[
K_P : \text{Im} L \rightarrow \text{Dom} L \subset \text{Ker} P,
\]

\[
K_P(x) = \int_{0}^{t} x(s) ds - \frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{\eta} x(s) ds d\eta.
\]

Notice that

\[
QN : X \rightarrow X,
\]

\[
QN x = \begin{pmatrix}
\int_{0}^{\omega} \sum_{j=1}^{2} a_{1j}(t) \int_{-T_{1j}}^{0} K_{1j}(s) e^{x_j(t+s)} ds dt

- \frac{1}{\omega} \int_{0}^{\omega} b_1(t) e^{x_1(t)} \int_{-\tau_1}^{0} f_1(s) e^{x_1(t+s)} ds dt

- \frac{1}{\omega} \int_{0}^{\omega} b_2(t) e^{x_2(t)} \int_{-\tau_2}^{0} f_2(s) e^{x_2(t+s)} ds dt
\end{pmatrix},
\]

\[
QN x = \begin{pmatrix}
\int_{0}^{\omega} \sum_{j=1}^{2} a_{2j}(t) \int_{-T_{2j}}^{0} K_{2j}(s) e^{x_j(t+s)} ds dt
\end{pmatrix},
\]

\[
QN x = \begin{pmatrix}
\int_{0}^{\omega} \sum_{j=1}^{2} a_{1j}(t) \int_{-T_{1j}}^{0} K_{1j}(s) e^{x_j(t+s)} ds dt
\end{pmatrix},
\]

\[
QN x = \begin{pmatrix}
\int_{0}^{\omega} \sum_{j=1}^{2} a_{2j}(t) \int_{-T_{2j}}^{0} K_{2j}(s) e^{x_j(t+s)} ds dt
\end{pmatrix}.
\]
Thus,

\[ K_P(I - Q)N : X \rightarrow X, \]
\[ K_P(I - Q)Nx \]

\[ = \left( \int_0^t \left[ r_1(\mu) - \sum_{j=1}^2 a_{1j}(\mu) \int_{-\tau_{1j}}^{0} K_{1j}(s)e^{\xi_j(\mu+s)} ds - b_1(\mu)e^{\xi_1(t)} \int_{-\tau_1}^{0} f_2(s)e^{\xi_2(t+s)} ds \right] d\mu \right) \]

\[ - \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega \left[ r_1(t) - \sum_{j=1}^2 a_{1j}(t) \int_{-\tau_{1j}}^{0} K_{1j}(s)e^{\xi_j(t+s)} ds \right] dt \]

\[ = \left( \int_0^t \left[ r_2(\mu) - \sum_{j=1}^2 a_{2j}(\mu) \int_{-\tau_{2j}}^{0} K_{2j}(s)e^{\xi_j(\mu+s)} ds - b_2(\mu)e^{\xi_2(t)} \int_{-\tau_2}^{0} f_1(s)e^{\xi_1(t+s)} ds \right] d\mu \right) \]

\[ - \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega \left[ r_2(t) - \sum_{j=1}^2 a_{2j}(t) \int_{-\tau_{2j}}^{0} K_{2j}(s)e^{\xi_j(t+s)} ds \right] dt \]

\[ = \left( \int_0^t \left[ r_1(\mu) - \sum_{j=1}^2 a_{1j}(\mu) \int_{-\tau_{1j}}^{0} K_{1j}(s)e^{\xi_j(\mu+s)} ds - b_1(\mu)e^{\xi_1(t)} \int_{-\tau_1}^{0} f_2(s)e^{\xi_2(t+s)} ds \right] d\mu \right) \]

\[ - \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega \left[ r_1(t) - \sum_{j=1}^2 a_{1j}(t) \int_{-\tau_{1j}}^{0} K_{1j}(s)e^{\xi_j(t+s)} ds \right] dt \]

\[ = \left( \int_0^t \left[ r_2(\mu) - \sum_{j=1}^2 a_{2j}(\mu) \int_{-\tau_{2j}}^{0} K_{2j}(s)e^{\xi_j(\mu+s)} ds - b_2(\mu)e^{\xi_2(t)} \int_{-\tau_2}^{0} f_1(s)e^{\xi_1(t+s)} ds \right] d\mu \right) \]

\[ - \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega \left[ r_2(t) - \sum_{j=1}^2 a_{2j}(t) \int_{-\tau_{2j}}^{0} K_{2j}(s)e^{\xi_j(t+s)} ds \right] dt \]

Obviously, \( QN \) and \( K_P(I - Q)N \) are continuous by the Lebesgue theorem and, moreover, \( QN(\tilde{\Omega}) \), \( K_P(I - Q)N(\tilde{\Omega}) \) are relatively compact for any open bounded set \( \Omega \subset X \). Therefore, \( N \) is \( L \)-compact on \( \tilde{\Omega} \) for any open bounded set \( \Omega \subset X \). Corresponding to the operator equation \( Lx = \lambda Nx, \lambda \in (0,1) \), one has

\[ \frac{dx_i}{dt} = \lambda \left[ r_i(t) - \sum_{j=1}^2 a_{ij}(t) \int_{-\tau_{ij}}^{0} K_{ij}(s)e^{\xi_j(t+s)} ds - b_i(t)e^{\xi_i(t)} \int_{-\tau_i}^{0} f_k(s)e^{\xi_k(t+s)} ds \right] ds, \quad (3.9) \]

where \( k \neq i, i, k = 1,2 \). Suppose that \( x(t) = (x_1(t), x_2(t)) \in X \) is a solution of system (3.9) for some \( \lambda \in (0,1) \). Integrating (3.9) over the interval \([0,\omega]\), we obtain

\[ \sum_{j=1}^2 \int_{-\tau_{ij}}^{0} a_{ij}(t) \int_{-\tau_{ij}}^{0} K_{ij}(s)e^{\xi_j(t+s)} ds dt + \int_0^\omega b_i(t)e^{\xi_i(t)} \int_{-\tau_i}^{0} f_k(s)e^{\xi_k(t+s)} ds dt = \bar{r}_i\omega, \quad (3.10) \]

where \( k \neq i, i, k = 1,2 \).

It follows from (3.9) and (3.10) that

\[ \int_0^\omega |\dot{x}_i(t)| dt = \lambda \int_0^\omega \left| r_i(t) - \sum_{j=1}^2 a_{ij}(t) \int_{-\tau_{ij}}^{0} K_{ij}(s)e^{\xi_j(t+s)} ds \right. \]

\[ - b_i(t)e^{\xi_i(t)} \int_{-\tau_i}^{0} f_k(s)e^{\xi_k(t+s)} ds \left| dt \right. \]
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\[ < \int_0^\omega |r_i(t)| \, dt + \sum_{j=1}^2 \int_0^\omega a_{ij}(t) \int_{-\tau_{ij}}^0 K_{ij}(s)e^{x_j(t+s)} \, ds \, dt \]
\[ + \int_0^\omega b_i(t)e^{x_i(t)} \int_{-\tau}^0 f_k(s)e^{x_k(t+s)} \, ds \, dt \]
\[ = (\bar{r}_i + \bar{R}_i) \omega, \quad i = 1, 2; \]

that is,
\[ \int_0^\omega |x_i(t)| \, dt < (\bar{r}_i + \bar{R}_i) \omega, \quad i = 1, 2. \quad (3.11) \]

Since \( x(t) \in X \), there exist \( \xi_i \in [0, \omega] \) such that
\[ x_i(\xi_i) = \min_{t \in [0, \omega]} x_i(t), \quad i = 1, 2. \quad (3.12) \]

From (3.10) and (3.12), we see that
\[ \bar{a}_{ii}e^{x_i(\xi_i)} \leq \int_0^\omega a_{ii}(t) \int_{-T_{ii}}^0 K_{ii}(s)e^{x_i(t+s)} \, ds \, dt < \bar{r}_i \omega, \quad i = 1, 2; \]

that is,
\[ x_i(\xi_i) < \ln \frac{\bar{r}_i}{\bar{a}_{ii}}, \quad i = 1, 2. \quad (3.13) \]

From (3.11) and (3.13), one obtains
\[ x_i(t) \leq x_i(\xi_i) + \int_0^\omega |x_i(t)| \, dt < \ln \frac{\bar{r}_i}{\bar{a}_{ii}} + (\bar{r}_i + \bar{R}_i) \omega. \quad (3.14) \]

On the other hand, there also exist \( \eta_i \in [0, \omega] \) such that
\[ x_i(\eta_i) = \max_{t \in [0, \omega]} x_i(t), \quad i = 1, 2. \quad (3.15) \]

From (3.10) and (3.15), it follows that
\[ r_i \omega = \sum_{j=1}^2 \int_0^\omega a_{ij}(t) \int_{-T_{ij}}^0 K_{ij}(s)e^{x_j(t+s)} \, ds \, dt + \int_0^\omega b_i(t)e^{x_i(t)} \int_{-\tau_k}^0 f_k(s)e^{x_k(t+s)} \, ds \, dt \]
\[ \leq \sum_{j=1}^2 \bar{a}_{ij}e^{x_j(\eta_j)} + \bar{b}_i e^{x_i(\eta_i)}e^{x_k(\eta_k)}, \quad k \neq i, \quad i, k = 1, 2. \quad (3.16) \]

From (3.16), we have
\[ e^{x_i(\eta_i)} \geq \frac{\bar{r}_i - \bar{a}_{ik}e^{x_k(\eta_k)}}{\bar{a}_{ii} + \bar{b}_i e^{x_k(\eta_k)}}, \quad i \neq k, \quad i, k = 1, 2. \quad (3.17) \]

From (3.14) and (3.17), we obtain
\[ e^{x_i(\eta_i)} \geq \frac{\bar{a}_{kk} \bar{r}_i - \bar{a}_{ik} \bar{r}_k e^{(\bar{r}_k + \bar{R}_k) \omega}}{\bar{a}_{ii} \bar{a}_{kk} + \bar{b}_k \bar{r}_k e^{(\bar{r}_k + \bar{R}_k) \omega}}, \quad k \neq i, \quad i, k = 1, 2, \]

which imply
\[ x_i(\eta_i) \geq \ln \left( \frac{\bar{a}_{kk} \bar{r}_i - \bar{a}_{ik} \bar{r}_k e^{(\bar{r}_k + \bar{R}_k) \omega}}{\bar{a}_{ii} \bar{a}_{kk} + \bar{b}_k \bar{r}_k e^{(\bar{r}_k + \bar{R}_k) \omega}} \right) \overset{\text{def}}{=} M_i, \quad k \neq i, \quad i, k = 1, 2. \quad (3.18) \]
From (3.11) and (3.18), one obtains
\[ x_i(t) \geq x_i(\eta_i) - \int_0^\omega |\dot{x}_i(t)| \, dt > M_i - (\bar{r}_i + \bar{R}_i) \omega, \quad i = 1, 2. \tag{3.19} \]

By (3.14) and (3.19), this yields
\[ \max_{t \in [0, \omega]} |x_i(t)| < \max\left\{ \left| \ln \frac{\bar{r}_i}{\alpha_{ii}} + (\bar{r}_i + \bar{R}_i) \omega \right|, \left| M_i - (\bar{r}_i + \bar{R}_i) \omega \right| \right\} \quad \text{def} \quad A_i. \]

Clearly, \( A_i \) are independent of \( \lambda \).

To complete the proof, we must consider several cases.

**CASE 1.** \( \alpha_{12} < 0 \) and \( \alpha_{21} < 0 \). In this case, we have already proven equations (3.1) have a unique positive solution \((N_{11}^*, N_{22}^*)\).

Set \( A = (\sum_{i=1}^2 A_i^2)^{1/2} + C \), where \( C \) is taken sufficiently large such that the unique positive solution \((N_{11}^*, N_{22}^*)\) of (3.1) satisfies \( \|\ln N_{11}^*, \ln N_{22}^*\| < C \); then \( \|x\| < A \).

Let \( \Omega = \{x = (x_1, x_2)^T \in X \mid \|x\| < A\} \); it is clear that \( \Omega \) satisfies Condition (a) in Lemma 3.1. When \( x = (x_1, x_2)^T \in \partial \Omega \cap \text{Ker} \, L = \partial \Omega \cap R^2 \), \( x \) is a constant vector in \( R^2 \) with \( \|x\| = A \).

Furthermore, take \( J = I : \text{Im} \, Q \to \text{Ker} \, L, (x_1, x_2)^T \to (x_1, x_2)^T \). Since (3.1) have a unique positive solution \((N_{11}^*, N_{22}^*)\), and we easily verify that \((x_1^*, x_2^*) = (\ln N_{11}^*, \ln N_{22}^*)\) is a regular point of operator \( Q \), by a straightforward computation, we find
\[ \deg [JQN, \text{Ker} \, L \cap \Omega, 0] = \text{sgn} \{N_{11}^* N_{22}^* (a_{11} a_{22} - a_{12} a_{21} - \alpha_{12} N_{11}^* - \alpha_{21} N_{22}^*)\} = 1 \neq 0. \]

According to Lemma 3.1, system (3.8) has at least one \( \omega \)-periodic solution.

**CASE 2.** \( \alpha_{12} > 0 \).

In this case, equations (3.1) have two positive solutions \((N_{11}^*, N_{22}^*)\) and \((N_{12}^*, N_{22}^*)\).

Taking \( J = I : \text{Im} \, Q \to \text{Ker} \, L, (x_1, x_2)^T \to (x_1, x_2)^T \), then
\[ JQN(x) = \begin{pmatrix} \bar{r}_1 - a_{11} e^{x_1} - a_{12} e^{x_2} - b_1 e^{x_1+x_2} \\ \bar{r}_2 - a_{21} e^{x_1} - a_{22} e^{x_2} - b_2 e^{x_1+x_2} \end{pmatrix}, \]
where \( x = (x_1, x_2) \in \text{Ker} \, L = R^2 \) is a constant vector.

Notice that
\[ \frac{JQN(x)}{|JQN(x)|} - \frac{JQN(-x)}{|JQN(-x)|} \]
is equivalent to
\[ \begin{pmatrix} \bar{r}_1 - a_{11} N_1 - a_{12} N_2 - b_1 N_1 N_2 \\ \bar{r}_2 - a_{21} N_1 - a_{22} N_2 - b_2 N_1 N_2 \end{pmatrix} = \pm \begin{pmatrix} \bar{r}_1 N_1 N_2 - a_{11} N_2 - a_{12} N_1 - b_1 N_1 N_2 \\ \bar{r}_2 N_1 N_2 - a_{21} N_1 - a_{22} N_2 - b_2 N_1 N_2 \end{pmatrix}, \]
\[ \begin{pmatrix} \bar{r}_1 - a_{11} N_1 - a_{22} N_2 - b_1 N_1 N_2 \\ \bar{r}_2 - a_{21} N_1 - a_{12} N_2 - b_2 N_1 N_2 \end{pmatrix} = \pm \begin{pmatrix} \bar{r}_1 N_1 N_2 - a_{11} N_1 - a_{12} N_2 - b_1 N_1 N_2 \\ \bar{r}_2 N_1 N_2 - a_{21} N_1 - a_{22} N_2 - b_2 N_1 N_2 \end{pmatrix}, \tag{3.20} \]

where \( N_i = e^{x_i} \, (i = 1, 2) \) is constant. Clearly, equations (3.20) only have finite solutions \((N_{11}^{(1)}, N_{22}^{(1)}), \ldots, (N_{11}^{(m)}, N_{22}^{(m)})\).

Take \( C_1 \) sufficiently large such that all solutions of (3.20) satisfy \( \|\ln N_{11}^{(i)}, \ln N_{22}^{(i)}\| < C_1 \) \((i = 1, 2, \ldots, m)\). Similarly, take \( C_2 \) sufficiently large such that all solutions of (3.1) satisfy \( \|\ln N_{11}^*, \ln N_{22}^*\| < C_1 \) \((i = 1, 2)\).
Set

\[ A = \left( \sum_{i=1}^{2} A_i^2 \right)^{1/2} + \sum_{i=1}^{2} C_i; \]

then \( \|x\| < A \).

Let

\[ \Omega = \{ x = (x_1, x_2)^\top \in X \mid \|x\| < A \} . \]

In a similar way as in the proof Case 1, we can obtain that \( \Omega \) satisfies Condition (a) in Lemma 3.1, and for \( x \in \partial \Omega \cap \text{Ker} L, QNx \neq 0 \).

Define \( D = \ker L \cap \Omega, F(x) = QNx \). Clearly, the conditions in Lemma 3.3 are satisfied. An application of Lemma 3.3 yields

\[ \text{deg}(JQN, \text{Ker} L \cap \Omega, 0) \neq 0. \]

Hence, system (3.8) has at least one \( \omega \)-periodic solution.

**CASE 3.** \( \alpha_{21} > 0 \).

In this case, we can similarly show that (3.8) has \( \omega \)-periodic solutions.

By combining Cases 1–3, we see that (3.8) has at least one \( \omega \)-periodic solution \( x^*(t) = (x_1^*(t), x_2^*(t))^\top \), and by \( N_1^*(t) = e^{\tau_1^*(t)} \) we know that \( (N_1^*(t), N_2^*(t)) \) is a positive \( \omega \)-periodic solution of (2.4). The proof is complete.

**REMARK 1.** In view of the proof theorem, one sees that in theorem, when some of the \( T_{ij}, \tau_i, \) or all of them are \( \infty \), the theorem is still true.

**REMARK 2.** Fan [8] discussed the existence of positive periodic solutions of periodic \( N \)-species Lotka-Volterra competition system with time delays. When \( N = 2 \), conditions of the main Theorem 2.1 in [8] are

\[ \tau_{12} > F_{2a12}e(\bar{\tau}_2 + R_2)^2 \]

Obviously, these conditions are the same as those of our Theorem 3.5. It implies that for a Lotka-Volterra competition system, the plankton allelopathy are “harmless” for the existence of positive periodic solutions.

### 4. APPLICATIONS

Finally, as an application of our main results, we consider the following systems.

\[ \dot{N}_1(t) = N_1(t) \left[ 3 \sin t - (3c^{12\pi} \cos t) \int_{-T_{11}}^{0} K_{11}(s) N_1(s + t) ds \right. \]

\[ \left. - (2 + \sin t) \int_{-T_{12}}^{0} K_{12}(s) N_2(s + t) ds - (2 + \cos t) N_3(t) \int_{-\tau_3}^{0} f_2(s) N_2(t + s) ds \right] , \]

\[ \dot{N}_2(t) = N_2(t) \left[ 4 - \cos t - (3 - \cos t) \int_{-T_{11}}^{0} K_{21}(s) N_1(s + t) ds \right. \]

\[ \left. - (3c^{16\pi} + \sin t) \int_{-T_{22}}^{0} K_{22}(s) N_2(s + t) ds \right. \]

\[ \left. - (5 + \sin t) N_3(t) \int_{-\tau_1}^{0} f_1(s) N_1(t + s) ds \right] , \]

where \( K_{ij} \in C([-T_{ij}, 0], (0, \infty)), \int_{-T_{ij}}^{0} K_{ij}(s) \, ds = 1, f_i \in C([\tau_i, 0], (0, \infty)), \int_{-\tau_1}^{0} f_i(s) \, ds = 1, \)

and \( T_{ij}, \tau_i \) are positive constants, \( j \neq i, i, j = 1,2 \).

It is easy to see that \( \bar{\tau}_1 = \bar{R}_1 = 3, \bar{\tau}_2 = \bar{R}_2 = 4, \bar{a}_{11} = 3c^{12\pi} \), \( \bar{a}_{12} = 2, \bar{a}_{21} = 3, \bar{a}_{22} = 3c^{16\pi} \).

Therefore, we have

\[ \bar{\tau}_1 \bar{a}_{22} - \bar{\tau}_2 \bar{a}_{12} e^{(\bar{r}_2 + \bar{R}_2)^2} = e^{16\pi} > 0, \quad \text{and} \quad \bar{\tau}_2 \bar{a}_{11} - \bar{\tau}_1 \bar{a}_{21} e^{(\bar{r}_1 + \bar{R}_1)^2} = e^{12\pi} > 0. \]

According to Theorem 3.5 and Remark 1, we see that system (4.1) has a positive 2\( \pi \)-periodic solution.
REFERENCES