Comp. & Maths. with Appls. Vol. 8, No. 4, pp. 305–311, 1982
Printed in Great Britain.

0097-4943/82/040305-07$03.00/0

APPROXIMATION FOR BESSEL FUNCTIONS AND THEIR APPLICATION IN THE COMPUTATION OF HANKEL TRANSFORMS

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(*Received August* 1981; *in revised form October* 1981)

Communicated by E. Y. Rodin

Abstract—We present rational approximations of the Bessel functions $J_\nu(x)$, $\nu = 0, 1, \dots, 10$, which can be used to simplify the computation of the Hankel transform to the computation of two Fourier transforms.

1. INTRODUCTION

The Hankel transform of integer order ν of the function f is defined as[1]

$$\mathcal{H}_\nu[f; p] = \int_0^\infty xf(x)\, J_\nu(px)\, dx,\ p > 0. \tag{1}$$

Some authors employ a different but related definition (see for example Erdélyi *et al.*[2]).

A number of partial differential equations and integral equations can be solved using this transform.

For the numerical computation of (1) only a few number of methods are known. Longman[3] reports the use of Gauss–Legendre quadrature formulas for the computation of the finite-interval integrals in the right-hand member of

$$\mathcal{H}_\nu[f; p] = \frac{1}{p^2} \sum_{s=0}^{\infty} \int_{j_{\nu,s}}^{j_{\nu+1,s}} xf(x/p)\, J_\nu(x)\, dx \tag{2}$$

where $j_{\nu,s}$, $s = 1, 2, \dots$ is the sth positive zero of $J_\nu(x)$ and $j_{\nu,0} = 0$. The summation in (2) can be carried out using the convergence accelerating algorithm of Euler[4]. Piessens[5] has constructed particular quadrature formulas of the Gaussian type for the computation of the integrals in (2).

Longman's method is not suitable for large values of p and the use of Piessens' quadrature formulas requires the storage of a large number of abscissae and weights. Linz[6] presents an ingenious but unefficient method, based on the Abel transformation. His method requires the solution of the Abel integral equation.

In this paper we present a new method which reduces the computation of the Hankel transform to a numerically simpler task, namely the computation of two Fourier transforms. For the numerical evaluation of the Fourier transform a number of methods and computer-programs are available[7, 8] (for a survey, see Davis and Rabinowitz[4]).

2. DESCRIPTION OF THE METHOD

The asymptotic behaviour of the Bessel function $J_\nu(x)$ suggests an approximation of the form

$$J_\nu(x) \simeq \sqrt{\left(\frac{2}{\pi x}\right)}\, \psi_\nu(x) \sin[\varphi_\nu(x)], \quad x \geq 0 \tag{3}$$

where

$$\varphi_\nu(x) = x + \left(\frac{1}{4} - \frac{\nu}{2}\right)\pi + Q(x) \tag{4}$$

and

$$\psi_\nu(x) = 1 + R(x) \tag{5}$$

where $Q(x)$ and $R(x)$ are functions which are to be determined and which must tend to zero if $x \to \infty$.

It is difficult to determine $Q(x)$ and $R(x)$ so that (3) is a good approximation over the whole interval $[0, \infty)$. For this reason, we limit the approximation interval to $[j_{\nu,5}, \infty)$. The construction of the functions $Q(x)$ and $R(x)$ will be discussed in the following section.

We now use approximation (3) for the computation of the Hankel transform. The Hankel transform can be written as

$$\mathcal{H}_\nu[f; p] = \int_0^A xf(x)\,J_\nu(px)\,dx + \int_A^\infty xf(x)\,J_\nu(px)\,dx \tag{6}$$

where $A = j_{\nu,5}/p$.

Substituting (3) into the second integral of (6) yields

$$\int_A^\infty xf(x)\,J_\nu(px)\,dx = \int_0^\infty S_{\nu,p}(x)\sin x\,dx + \int_0^\infty C_{\nu,p}(x)\cos x\,dx \tag{7}$$

where

$$S_{\nu,p}(x) = F_{\nu,p}(x)\cos[G_\nu(x)] \tag{8}$$

$$C_{\nu,p}(x) = F_{\nu,p}(x)\sin[G_\nu(x)] \tag{9}$$

$$F_{\nu,p}(x) = \sqrt{[2\pi(x + j_{\nu,5})]}\, f((x + j_{\nu,5})/p)\, \psi_\nu(x + j_{\nu,5})/(\pi p^2) \tag{10}$$

$$G_\nu(x) = j_{\nu,5} + \left(\frac{1}{4} - \frac{\nu}{2}\right)\pi + Q(x + j_{\nu,5}). \tag{11}$$

The final result is

$$\mathcal{H}_\nu[f; p] = \int_0^{j_{\nu,5}/p} xf(x)\,J_\nu(px)\,dx + \int_0^\infty S_{\nu,p}(x)\sin x\,dx + \int_0^\infty C_{\nu,p}(x)\cos x\,dx. \tag{12}$$

The evaluation of (12) requires the computation of an integral over a finite interval and the computation of a Fourier-sine and a Fourier-cosine transform. For both problems, a lot of methods are available.

3. CONSTRUCTION OF THE FUNCTIONS $R(x)$ AND $Q(x)$

$R(x)$ and $Q(x)$ are chosen as rational functions

$$Q(x) = \frac{\sum_{i=0}^{k} a_i x^{2i}}{x + \sum_{i=1}^{k} b_i x^{2i+1}} \tag{13}$$

and

$$R(x) = \frac{\sum_{i=0}^{m} c_i x^{2i}}{1 + \sum_{i=1}^{m+1} d_i x^{2i}} \tag{14}$$

The function $\cdot\, Q(x)$ is constructed so that

$$\max_{5 \le s \le q} |\varphi_\nu(j_{\nu,s}) - s\pi| \tag{15}$$

Table 1. Coefficients of the approximation for $J_0(x)$

i	a_i	b_i
0	-0.89832 48414 017943 E-01	
1	-0.89826 97321 899602 E-01	0.75760 99392 106824 E+00
2	-0.98956 77193 079057 E-02	0.79994 80847 179244 E-01
3	-0.19905 38234 058590 E-03	0.15924 30588 607080 E-02
4	-0.32646 07444 783483 E-12	0.26116 84988 704736 E-11
i	**c_i**	**d_i**
0	-0.16545 37111 290759 E+00	
1	-0.98268 66474 693050 E-01	0.46423 91184 482107 E+01
2	-0.78527 26716 572958 E-02	0.17686 02586 723141 E+01
3	-0.12403 80789 773397 E-03	0.12893 06365 578512 E+00
4		0.19846 09263 637456 E-02

Table 2. Coefficients of the approximation for $J_1(x)$

i	a_i	b_i
0	0.25062 20793 228356 E+00	
1	0.35214 37339 269938 E+00	0.98804 96383 279081 E+00
2	0.43704 14499 062160 E-01	0.11763 50362 597494 E+00
3	0.93484 25220 856040 E-03	0.24929 13387 288671 E-02
4	-0.42339 70583 554150 E-11	-0.11290 58771 833683 E-10
i	**c_i**	**d_i**
0	0.32218 72118 523913 E+00	
1	0.63284 73303 884832 E-01	0.20353 24019 097700 E+01
2	0.18291 71032 254493 E-02	0.34753 55592 995276 E+00
3	0.25215 62930 546599 E-05	0.97694 47434 814370 E-02
4		0.13448 33562 958140 E-04

Table 3. Coefficients of the approximation for $J_2(x)$

i	a_i	b_i
0	0.10444 03233 704498 E+02	
1	-0.24428 87745 565414 E+02	-0.13673 69471 541962 E+02
2	-0.57555 71445 411089 E+01	-0.30857 31537 219188 E+01
3	-0.16093 43282 006637 E+00	-0.85831 64174 871470 E-01
4	-0.41693 93506 654575 E-09	-0.22236 76571 634556 E-09
i	**c_i**	**d_i**
0	-0.81949 11008 221835 E+01	
1	-0.16639 15562 245167 E+01	-0.73096 50286 669034 E+01
2	-0.65156 80934 641177 E-02	-0.17632 51982 909614 E+01
3	0.13545 41489 823665 E-02	-0.81691 47004 437662 E-02
4		0.14448 44255 811907 E-02

Table 4. Coefficients of the approximation for $J_3(x)$

i	a_i	b_i
0	0.36850 55882 357572 E+01	
1	0.35374 82270 268847 E+00	0.75646 38172 999489 E-01
2	0.25800 71736 489745 E+00	0.58284 41216 038589 E-01
3	0.13147 07090 673293 E-01	0.30050 44779 272259 E-02
4	-0.11272 14999 274785 E-10	-0.25764 91374 782481 E-11
i	**c_i**	**d_i**
0	0.15108 89617 694366 E+01	
1	0.60037 20004 873568 E+00	-0.12811 05455 921623 E+00
2	0.45565 85496 163818 E-01	0.19414 20765 464895 E+00
3	0.40162 55081 111366 E-03	0.20101 44170 346349 E-01
4		0.18360 02322 793766 E-03

Table 5. Coefficients of the approximation for $J_4(x)$

i	a_i	b_i
0	0.72129 06331 327618 E+01	
1	-0.11658 12056 278308 E+00	-0.25222 20983 548825 E-01
2	0.11133 39994 513846 E+00	0.13919 69855 069034 E-01
3	0.21124 59344 929020 E-02	0.26824 88095 108118 E-03
4	-0.36884 45619 811673 E-10	-0.46837 40467 021193 E-11
i	c_i	d_i
0	-0.23437 19520 992144 E+01	
1	-0.25650 56350 845120 E+00	-0.35820 44170 611220 E+00
2	-0.39997 03948 655609 E-01	0.13037 79032 641513 E-01
3	-0.13854 24937 061899 E-02	-0.72221 96868 050648 E-02
4		-0.35185 39522 696883 E-03

Table 6. Coefficients of the approximation for $J_5(x)$

i	a_i	b_i
0	0.12280 77090 815733 E+02	
1	0.89943 46593 284196 E+00	0.75050 87285 321884 E-01
2	-0.27496 46687 767164 E-02	-0.12923 77389 533210 E-02
3	0.84758 54336 708531 E-02	0.68491 75694 807555 E-03
4	-0.37479 41188 549673 E-09	-0.30286 39343 864416 E-10
i	c_i	d_i
0	-0.48797 74240 074589 E+00	
1	0.25017 81609 647496 E-01	-0.13454 48687 563001 E+00
2	-0,11723 54699 525138 E-02	0.67416 18419 753168 E-02
3	0.21370 48842 021672 E-04	-0.23771 69631 742314 E-03
4		0.34538 16310 338014 E-05

Table 7. Coefficients of the approximation for $J_6(x)$

i	a_i	b_i
0	-0.29467 26920 155173 E+01	
1	-0.12215 13912 186148 E+02	-0.76121 10570 738810 E+00
2	0.58239 51857 979850 E+00	0.35765 28046 160190 E-01
3	-0.22954 97784 575715 E-01	-0.12841 95383 325744 E-02
4	0.59016 56639 187234 E-08	0.33016 26091 800279 E-09
i	c_i	d_i
0	-0.47984 70828 883366 E+00	
1	0.33428 49523 125688 E-01	-0.12487 85017 590801 E+00
2	-0.11368 38352 198206 E-02	0.63731 80643 556723 E-02
3	0.24191 63756 839612 E-04	-0.18361 76556 943689 E-03
4		0.27067 56651 009368 E-05

Table 8. Coefficients of the approximation for $J_7(x)$

i	a_i	b_i
0	0.12731 84976 093666 E+02	
1	-0.43564 73544 888421 E+01	-0.20007 48873 887149 E+00
2	0.15837 65711 876680 E+00	0.69522 39197 525618 E-02
3	-0.31113 82288 215723 E-02	-0.12764 68690 980973 E-03
4	0.28480 14134 985890 E-08	0.11684 16055 375338 E-09
i	c_i	d_i
0	-0.48366 57510 984620 E+00	
1	0.52041 19212 998210 E-03	-0.43997 54618 022570 E-01
2	0.82882 20783 222314 E-04	-0.11028 13395 302394 E-03
3	-0.94365 42419 313562 E-05	0.29230 52685 524707 E-04
4		-0.77428 04036 360101 E-06

Table 9. Coefficients of the approximation for $J_8(x)$

i	a_i	b_i
0	0.18349 31338 210380 E+02	
1	-0.37386 25787 185874 E+01	-0.13155 60071 648959 E+00
2	0.10462 12748 575944 E+00	0.34907 14601 954437 E-02
3	-0.13808 49117 045964 E-02	-0.43321 11271 225241 E-04
4	0.23585 78088 571698 E-08	0.73994 60670 020975 E-10
i	**c_i**	**d_i**
0	-0.47723 17709 515633 E+00	
1	0.16798 01791 806652 E-02	-0.35948 24854 132290 E-01
2	0.79289 31676 209592 E-04	-0.42784 76496 528333 E-04
3	-0.46022 42135 142107 E-05	0.16047 46908 837459 E-04
4		-0.28876 81339 695742 E-06

Table 10. Coefficients of the approximation for $J_9(x)$

i	a_i	b_i
0	0.23811 56206 599237 E+02	
1	-0.35981 91873 462367 E+01	-0.10040 64583 312075 E+00
2	0.82324 76093 270257 E-01	0.21663 22581 032705 E-02
3	-0.82524 17594 349535 E-03	-0.20439 54743 877925 E-04
4	0.79760 35987 759812 E-09	0.19754 88789 534709 E-10
i	**c_i**	**d_i**
0	-0.12750 04684 511672 E+00	
1	-0.11637 35571 810356 E+00	0.23578 19796 287997 E+00
2	0.24746 50965 042455 E-02	-0.11381 93550 158312 E-01
3	-0.26593 05339 394706 E-04	0.18709 00085 917186 E-03
4		-0.13173 02954 498964 E-05

Table 11. Coefficients of the approximation for $J_{10}(x)$

i	a_i	b_i
0	0.29570 68823 498488. E+02	
1	-0.36109 90629 755978 E+01	-0.82157 05066 334004 E-01
2	0.71010 86382 801846 E-01	0.15132 33634 668198 E-02
3	-0.57102 80107 822520 E-03	-0.11448 51750 726675 E-04
4	-0.42496 13732 732543 E-08	-0.85205 28787 437206 E-10
i	**c_i**	**d_i**
0	-0.10749 88134 249699 E+00	
1	-0.97842 57666 988060 E-01	0.19766 39631 961117 E+00
2	0.18499 40252 047095 E-02	-0.81166 06043 381508 E-02
3	-0.15723 45626 964698 E-04	0.11254 59366 156353 E-03
4		-0.63051 45371 285454 E-06

is minimal, where q is a suitably chosen large integer number. Substituting for φ_ν the expression (4), this becomes:

$$\max_{5\leq s\leq q}\left|j_{\nu,s}+\left(\frac{1}{4}-\frac{\nu}{2}-s\right)\pi+Q(j_{\nu,s})\right| \tag{16}$$

is minimal. This condition guarantees that $J_\nu(x)$ and its approximation (3) have nearly the same zeros (for $s\geq 5$). This construction of $Q(x)$ requires accurate values of $j_{\nu,s}$ which can be found in [9]. After the computation of $Q(x)$ and thus of $\varphi_\nu(x)$, the function $R(x)$ is constructed so that

$$\max_{5\leq i\leq q}\left|1+R(x_i)-\left(\frac{\pi x_i}{2}\right)^{1/2}J_\nu(x_i)/\sin\left[\varphi_\nu(x_i)\right]\right| \tag{17}$$

is minimal, where

$$x_i = (j_{\nu,i} + j_{\nu,i-1})/2.$$

The values of $J_\nu(x)$ required for the construction of $R(x)$, are obtained using a multiple precision FORTRAN version of the ALGOL algorithm Japlusn of Gautschi[10].

The two discrete minimax approximation problems (16) and (17) are solved using Loeb's algorithm[11].

In (13) we choose $k = 4$, in (14)$m = 3$ and in (16) and (17) $q = 90$. The coefficients a_i, b_i, c_i and d_i of the rational functions $Q(x)$ and $R(x)$ are listed in the Tables 1–11, for $\nu = 0, 1, \ldots, 10$.

The resulting approximations for $J_\nu(x)$ have an absolute error smaller than 10^{-14} for $0 \leq \nu \leq 10$ and $j_{\nu,5} \leq x < \infty$.

4. NUMERICAL TESTS

We consider the Hankel transform of the functions

$$f_1(x) = e^{-x}/x$$

and

$$f_2(x) = \ln x/x.$$

We have

$$\mathcal{H}_\nu[f_1; p] = p^\nu(1+p^2)^{-1/2}[1+(1+p^2)^{1/2}]^{-\nu}$$

and

$$\mathcal{H}_\nu[f_2; p] = \frac{1}{2}\left[\ln\frac{2}{p} + \psi\left(\frac{\nu+1}{2}\right)\right]$$

where $\psi(x)$ is the psi-function[12].

The finite-interval integral in (12) is computed using the subroutine DQAGS of QUADPACK[13] (this integrator is essentially the same as D01AJF of the NAG-library, mark 8[14]). The Fourier integrals in (12) are computed using the subroutine DQAWF of the same package. An absolute accuracy of 10^{-7} is required for the three integrals. In Tables 12 and 13

Table 12. Hankel transforms of e^{-x}/x

ν	p	$\mathcal{H}_\nu(f_1;p)$	absolute error ε	number of integrand evaluations n
0	1	0.7071067812	0.17 (-15)	213
	5	0.1961161351	0.84 (-9)	171
	10	0.0995037190	0.99 (-12)	296
	50	0.0199960012	0.14 (-8)	421
5	1	0.0086219713	0.35 (-17)	213
	5	0.0726211686	0.16 (-9)	213
	10	0.0604021455	0.11 (-8)	263
	50	0.0180932507	0.10 (-8)	513
10	1	0.0001051304	0.24 (-19)	213
	5	0.0268913832	0.65 (-11)	213
	10	0.0366661590	0.23 (-10)	213
	50	0.0163715594	0.13 (-9)	513

Table 13. Hankel transforms of ln x/x

ν	p	$\mathcal{H}_\nu(f_2;p)$	absolute error ε	number of integrand evaluations n
0	1	-1.2703628455	0.88 (-10)	731
	5	-0.5759601516	0.17 (-9)	781
	10	-0.3572947938	0.41 (-9)	781
	50	-0.1036477170	0.10 (-10)	981
5	1	1.6159315157	0.89 (-11)	638
	5	0.0012987206	0.66 (-10)	563
	10	-0.0686653577	0.47 (-9)	588
	50	-0.0459218298	0.19 (-9)	788
10	1	2.3042403291	0.95 (-10)	663
	5	0.1389604833	0.45 (-10)	663
	10	0.0001655236	0.50 (-9)	613
	50	-0.0321556535	0.21 (-8)	713

we give some results: ϵ is the actual absolute accuracy of the result and n is the total number of function evaluations for the three integrals in (12). The computations are carried out in double precision on a IBM 3033 computer.

In evaluating the efficiency of the new method, we have to take into account that the integrators DQAGS end DQAWF are automatic integrators, for which reliability was considered more important than efficiency.

CONCLUSION

Using a new type of approximations for the Bessel functions, the Hankel transform can be computed automatically, using standard integration routines.

Acknowledgment—This research is supported by the Belgian National Science Foundation (NFWO).

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