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Optimal Average Message Passing Density in Moore Graphs

A. A. NANAVATI AND S. S. IYENGAR
Department of Computer Science, Louisiana State University
Baton Rouge, LA 70803, U.S.A.

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Abstract—Moore graphs are compact graphs. As interconnection networks, they would minimize communication costs. We show that K_1, K_2 and the Petersen graph are the only Moore graphs that belong to $\mathcal{E}^2$, that is, are optimal in average message passing density. We also show that $\mathcal{E}^2$ is closed under the Cartesian product $(\times)$ operation. On the basis of these findings, a new architecture is proposed.

Keywords—Computer architecture, Distributed computing, Graph theory, Performance evaluation, Networks.

The *average message passing density* is an important parameter in the study of interconnection networks. It is a measure of the traffic on the links of a network. It is expressed by the ratio σ_d/e, where σ_d is the sum of the distances to all the vertices in the graph from a particular vertex (origin). The graphs considered are symmetric in that the choice of the origin does not alter σ_d. Such graphs are called distance degree regular graphs [1].

DEFINITION 1. *Let $d(i,j)$ denote the distance between vertex i and j and let o denote the origin. A graph $G = (V, E)$ belongs to $\mathcal{E}^2$ iff it satisfies $\sigma_d = e$, where*

$$\sigma_d = \sum_{i \in V} d(o,i)$$

and e is $|E|$.

A value of σ_d/e larger than 1 indicates congestion on the edges (inefficient), and a value less than 1 means that there are too many edges (not economic). A value of 1 is, therefore, optimal. We shall use $\mathcal{E}^2$ to denote the set of all graphs that are both economic and efficient in terms of layout, i.e., have the property $\sigma_d = e$. It is meaningful to talk about this property only in the context of distance degree regular graphs. The class of Generalized Petersen graphs has been studied in this regard. Among them, only $P(4,1)$ and $P(5,2)$ are in $\mathcal{E}^2$ (see [2]). Here, we inspect the class of Moore graphs [3] for this property.

DEFINITION 2. *A k-regular graph with diameter d having the maximum possible vertices is a Moore graph. Therefore, the Moore graph $M = (V_M, E_M)$ has*

$$|V_M| = 1 + k + k(k-1) + k(k-1)^2 + \cdots + k(k-1)^{d-1}.$$

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It is important to search for graphs having this property, because such graphs would be good candidates for interconnection network topologies. We explore Moore graphs here because they have the additional trait of being very compact. In Moore graphs, as many vertices as possible are "squeezed" within a given degree and diameter. Such graphs are attractive as they minimize distances and reduce communication costs. Networks based on Moore graphs have been proposed. The hypercube of dimension 1 and dimension 2 and the Petersen graph are trivial examples of Moore graphs (Figure 1). Several interesting extensions of these graphs have been studied extensively in literature [4–7].

We study the $\times$ (Cartesian product) operator, because extensions of several networks are based upon this operator.

DEFINITION 3. *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. The Cartesian product $G \times G'$ is defined as follows: The vertex set of $G \times G'$ is the Cartesian product $V \times V'$. The vertex (v, v') is joined to (u, u') in $G \times G'$ iff $\{v' = u' \ \wedge \ (v, u) \in E \ \vee \ v = u \ \wedge \ (v', u') \in E'\}$.*

However, the $\times$ operator does not preserve compactness, i.e., it is not necessary for $G \times G'$ to be a Moore graph, even if G and G' are. But, as we show later, the $\times$ operator does preserve the $\mathcal{E}^2$ property. Therefore, even though K_1, K_2 and the Petersen graph are the only Moore graphs in $\mathcal{E}^2$, the n-cube, the Hyper Petersen and the recursive Petersen graphs of all dimensions are in $\mathcal{E}^2$ (see [4,7,8]).

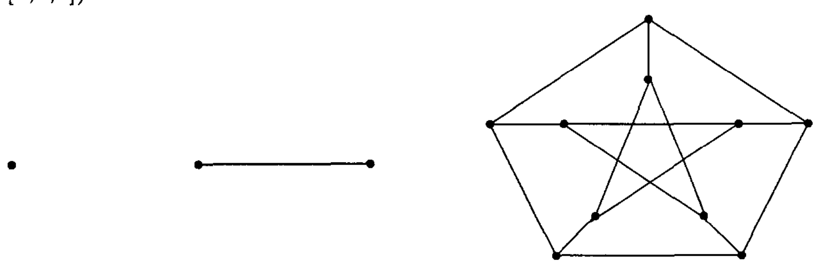

Figure 1.

THEOREM 1. *K_1, K_2 and the Petersen graph are the only Moore graphs in $\mathcal{E}^2$.*

PROOF. For a k-regular Moore graph of diameter d, to belong to $\mathcal{E}^2$, the following two equations must be satisfied:

$$1 + k + k(k-1) + k(k-1)^2 + k(k-1)^3 + \cdots + k(k-1)^{d-1} = n, \tag{1.1}$$

$$k + 2k(k-1) + 3k(k-1)^2 + \cdots + dk(k-1)^{d-1} = \frac{nk}{2}. \tag{1.2}$$

By inspection, $\{d = 0, k = 0, n = 1;\ d = 1, k = 1, n = 2\}$ are trivial solutions. Substituting n from (1.1) in (1.2), we get

$$k + 2k(k-1) + 3k(k-1)^2 + \cdots + dk(k-1)^{d-1} = \frac{k}{2}\left(1 + k + k(k-1) + \cdots + k(k-1)^{d-1}\right). \tag{1.3}$$

Dividing by $k (k \neq 0)$ and a rearrangement of terms yields

$$(k-2d)(k-1)^{d-1} + (k-2(d-1))(k-1)^{d-2} + \cdots + (k-1)(k-4) + k - 1 = 0. \tag{1.4}$$

Let $x \equiv (k-1)$. Then,

$$(k-2d)\, x^{d-1} + (k-2(d-1))\, x^{d-2} + \cdots + x(k-4) + k - 1 = 0, \tag{1.5}$$

$$k\sum_{i=1}^{d-1} x^i - 2\sum_{i=2}^{d} i x^{i-1} + k - 1 = 0, \tag{1.6}$$

which yields, for $x \neq 1 (k \neq 2)$,

$$k(x-1)\left(x^d-1\right) - 2(x-1)(d+1)\,x^d + 2\left(x^{d+1}-1\right) + (x-1)^2 = 0 \tag{1.7}$$

simplifying and solving which we get, for $k > 2$,

$$(k-1)^{d-1} = \frac{2}{(k-2d)(k-2)+2}. \tag{1.8}$$

Equation (1.8) has a positive integer RHS only for $\{d=2, k=3, n=10\}$. Therefore, Equation (1.3) has solutions only for $(k=0, d=0, n=1; k=1, d=1, n=2; k=3, d=2, n=10)$. ∎

THEOREM 2. *$\mathcal{E}^2$ is closed over $\times$.*

PROOF. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two arbitrary graphs. The vertices are labelled 1 through v_1 ($|V_1|$) and 1 through v_2 ($|V_2|$), respectively. e_1, e_2, e denote the number of edges, and σ_{d1}, σ_{d2}, σ_d denote the total distance sums of G_1, G_2, and $G_1 \times G_2$, respectively. Then, the vertices of $G_1 \times G_2$ are

$$\{(v,u) \mid v \in V_1, u \in V_2\}. \tag{2.1}$$

The number of edges in the final graph is [9]

$$e = v_1 e_2 + v_2 e_1. \tag{2.2}$$

Since, in the $\times$ operation, every vertex of G_1 is replaced with G_2 (or vice-versa, because $\times$ is commutative), there are v_2 copies of G_1. Let $v(i,j)$ denote the i^{th} vertex in the j^{th} copy of G_1. Therefore, $1 \leq i \leq v_1$ and $1 \leq j \leq v_2$. Let $v(o_1, j)$ denote the origin in the j^{th} copy of G_1. The total distances in all the copies from their respective origins is

$$\sum_{v \in V_2} \sigma_{d1} = v_2 \sigma_{d1}. \tag{2.3}$$

Also, let $v(o_1, o_2)$ be the origin of $G_1 \times G_2$. Now, the distance of $v(o_1, j)$ from $v(o_1, o_2)$ has to be added to the sum of distances. The set $v(o_1, j)$ $(1 \leq j \leq v_2)$ forms G_2, and therefore, the sum of distances of all $v(o_1, j)$ from $v(o_1, o_2)$ is σ_{d2}. Since there are v_1 vertices in each of the v_2 copies of G_1,

$$\sum_{v \in V_1} \sigma_{d2} = v_1 \sigma_{d2}. \tag{2.4}$$

Therefore, we have

$$\sigma_d = v_1 \sigma_{d2} + v_2 \sigma_{d1}. \tag{2.5}$$

Therefore, if $\sigma_{d1} = e_1$ and $\sigma_{d2} = e_2$, it follows from (1) and (4)

$$\sigma_d = v_1 \sigma_{d2} + v_2 \sigma_{d1} = v_1 e_2 + v_2 e_1 = e.$$ ∎

COROLLARY 1. *$(\mathcal{E}^2, \times, K_1)$ is an abelian semigroup.*

PROOF. By Theorem 2, $\mathcal{E}^2$ is closed. $\times$ is associative and commutative [9]. K_1 is the identity element. ∎

The hypercube, the hyper Petersen graphs and the recursive Petersen graphs are, therefore, in $\mathcal{E}^2$ (see [4,7,8]). It also obviously follows that $|\mathcal{E}^2| = \infty$.

COROLLARY 2(a). *If G_1 has the property $\sigma_{d1} < e_1$, then the Cartesian product $G_1 \times \mathcal{E}$ $(\mathcal{E} \in \mathcal{E}^2)$ also has $\sigma_d < e$.*

PROOF.

$$\sigma_d = v_1 e + v_2 \sigma_{d1} < v_1 e + v_2 e_1 = e.$$ ∎

COROLLARY 2(b). *If G has the property $\sigma_d > e$, then the Cartesian product $G \times \mathcal{E}(\mathcal{E} \in \mathcal{E}^2)$ also has $\sigma_d > e$.*

PROOF.

$$\sigma_d = v_1 e + v_2 \sigma_{d1} > v_1 e + v_2 e_1 = e. \qquad \blacksquare$$

From Theorems 1 and 2, it is clear that the Cartesian product of the Petersen graph with itself will have optimal average message passing density. We introduce and define the *recursive Petersen* architecture.

DEFINITION 4. *The recursive Petersen graph of dimension $n, (n > 1)$ is the Cartesian product of the Petersen graph with itself $(n-1)$ times. The Petersen graph is a recursive Petersen graph of dimension 1.*

A recursive Petersen network of dimension 2 has 100 vertices, each of degree 6, and a diameter as low as 4. A dimension 3 recursive Petersen network has 1000 vertices, each of degree 9, and is of diameter 6. In addition, such networks of all dimensions have optimal average message passing density [8].

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