Well posedness of a linearized fractional derivative fluid model

Arnaud Heibig, Liviu Iulian Palade *

Université de Lyon, CNRS, INSA-Lyon, Institut Camille Jordan UMR5208 & Pôle de Mathématiques, Bât. Léonard de Vinci no. 401, 21 avenue Jean Capelle, F-69621, Villeurbanne, France

A R T I C L E   I N F O

A B S T R A C T

The one-dimensional fractional derivative Maxwell model (e.g. Palade, et al., Rheol. Acta 35 (1996) 265), of importance in modeling the linear viscoelastic response in the glass transition region, has been generalized in Palade, et al., Internat. J. Engrg. Sci. 37 (1999) 315, to objective three-dimensional constitutive equations (CEs) for both fluids and solids. Regarding the rest state stability of the fluid CE, in Heibig and Palade, J. Math. Phys. 49 (2008) 043101, we gave a proof for the existence of weak solutions to the corresponding boundary value problem. The aim of this work is to achieve the study of the existence and uniqueness of the aforementioned solutions and to present smoothness results.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

Fractional derivative constitutive equations (CEs) have been found to accurately predict, among others, the stress relaxation of viscoelastic fluids in the glass transition and glassy (high frequency) states. The experimental behavior of storage G' and loss G'' moduli (obtained upon using the time–temperature superposition principle – see [30,36]) of a linear, narrow molecular weight series of polybutadienes is exceptionally well predicted by linearized fractional derivative models as can be reckoned from [31]. Polybutadienes are of utter importance for the tire industry, for manufacturing certain solid propergols, etc. Similar excellent agreements between frequency sweep experimental data obtained on other polymers (e.g. polystyrenes) and theoretical predictions of linear fractional derivative models are reported in [12,19,25].

The object of study is the below given objective, fractional derivative viscoelastic (incompressible) fluid constitutive equation (CE) (see [32])

$$\mathbf{S}(t) + \lambda^\alpha \mathbf{F}(t) \left\{ \int_{-\infty}^{t} \mu_1(t-\tau) \mathbf{F}^{-1}(\tau) \overset{\triangledown}{\mathbf{S}}(\tau) \left[\mathbf{F}^{-1}(\tau) \right]^T d\tau \right\} \mathbf{F}(t)^T$$

$$= G \lambda^\beta \mathbf{F}(t) \left\{ \int_{-\infty}^{t} \mu_2(t-\tau) \mathbf{F}^{-1}(\tau) \mathbf{A}_1(\tau) \left[\mathbf{F}^{-1}(\tau) \right]^T d\tau \right\} \mathbf{F}(t)^T \tag{1.1}$$

---

* Corresponding author. Fax: +33 472438529.
  E-mail address: liviu-iulian.palade@insa-lyon.fr (L.I. Palade).

Function $\mathbf{S}$ is the (objective) stress tensor and $\overset{\triangledown}{\mathbf{S}}$ its objective upper convected derivative defined by (with D/Dt denoting the material derivative and $\mathbf{L}$ the velocity gradient; see for example [16,28,46]):

$$\overset{\triangledown}{\mathbf{S}} = \frac{D\mathbf{S}}{Dt} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T \tag{1.2}$$

Function $\mathbf{F}$ is the strain gradient and $\mathbf{A}_1 = \nabla\mathbf{u} + (\nabla\mathbf{u})^T = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin–Ericksen tensor. The model parameters are such that $0 < \lambda$, $0 < \alpha < \beta < 1$. $\mu_{1,2}(t)$ are two memory kernels given by:

$$\mu_1(t - \tau) = \frac{(t - \tau)^{-\alpha}}{\Gamma(1 - \alpha)}, \qquad \mu_2(t - \tau) = \frac{(t - \tau)^{-\beta}}{\Gamma(1 - \beta)} \tag{1.3}$$

The stability of the rest state is now investigated using the linearized theory. As shown in [32] and [15], it is first assumed that the stress tensor $\mathbf{S} = O(\epsilon)$ and the deformation gradient $\mathbf{F}(t) = \mathbf{1} + \epsilon\mathbf{J}(t) + O(\epsilon^2)$. Since $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ (see for example [16, 17,28]), $\mathbf{L} = O(\epsilon)$. Hence the velocity $\mathbf{u} = O(\epsilon)$, and the first Rivlin–Ericksen tensor $\mathbf{A}_1 = O(\epsilon)$ as well. Therefore, keeping only terms of $O(\epsilon)$, within the linear response theory Eq. (1.1) reduces to:

$$\mathbf{S}(t) + \lambda^\alpha \int_{-\infty}^{t} \mu_1(t - \tau)\frac{\partial\mathbf{S}(\tau)}{\partial\tau}\,d\tau = G\lambda^\beta \int_{-\infty}^{t} \mu_2(t - \tau)\mathbf{A}_1(\tau)\,d\tau \tag{1.4}$$

Next, assume the fluid is contained in a bounded volume $\Omega \subset \mathbb{R}^3$ whose boundary $\partial\Omega$ is sufficiently smooth, and set in motion at $t = 0$. The CE in Eq. (1.4) then takes the form:

$$\mathbf{S}(t) + \lambda^\alpha \int_{0}^{t} \mu_1(t - \tau)\frac{\partial\mathbf{S}(\tau)}{\partial\tau}\,d\tau = G\lambda^\beta \int_{0}^{t} \mu_2(t - \tau)\mathbf{A}_1(\tau)\,d\tau \tag{1.5}$$

The above may be re-written in condensed form using the Caputo operators D_t^α and $I_t^{1-\beta}$ as:

$$\mathbf{S}(t) + \lambda^\alpha D_t^\alpha \mathbf{S} = G\lambda^\beta I_t^{1-\beta}\mathbf{A}_1 \tag{1.6}$$

where for an absolutely continuous function $f : \mathbb{R}_+ \to \mathbb{C}$:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_{0}^{t} \frac{f'(\tau)}{(t - \tau)^\alpha}\,d\tau \tag{1.7}$$

and for $f \in L_{\text{loc}}^1(\mathbb{R}_+)$,

$$I_t^{1-\beta} f(t) = \frac{1}{\Gamma(1 - \beta)} \int_{0}^{t} \frac{f(\tau)}{(t - \tau)^\beta}\,d\tau \tag{1.8}$$

As shown in [32,15], investigating the stability of the rest state is tantamount to studying the existence and the uniqueness of solutions to the following initial boundary value problem (IBVP):

$$\frac{\partial\mathbf{u}}{\partial t} = -\nabla p + \nabla \cdot \mathbf{S} \tag{1.9a}$$

$$\mathbf{S} + \lambda^\alpha D_t^\alpha \mathbf{S} = G\lambda^\beta I_t^{1-\beta}\mathbf{A}_1, \qquad \mathbf{A}_1 = \nabla\mathbf{u} + (\nabla\mathbf{u})^T \tag{1.9b}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } [0, +\infty[\times \Omega, \ \Omega \subset \mathbb{R}^3 \tag{1.9c}$$

$$\mathbf{u} = \mathbf{0}, \quad \text{in } [0, +\infty[\times \partial\Omega \tag{1.9d}$$

$$\mathbf{u}(t = 0) = \mathbf{u}_0, \qquad \mathbf{S}(t = 0) = \mathbf{S}_0 \tag{1.9e}$$

In the above system of equations we assume that $\mathbf{u} : [0, +\infty[\times \Omega \to \mathbb{R}^3$, $\nabla \cdot \mathbf{u}_0 = 0$, $p : [0, +\infty[\times \Omega \to \mathbb{R}$, $\mathbf{S} : [0, +\infty[\times \Omega \to \mathscr{M}_{3,3}(\mathbb{R})$, $0 < \alpha < \beta < 1$. Denote $\delta = \beta - \alpha > 0$.

A change of variables on $(\mathbf{x}, t)$ can be performed to eliminate the CE parameters λ and G (see [15]). This is carried out only for convenience; in no way the generality of this paper results is shrinked down. Therefore, from now on assume $\lambda = G = 1$.

At this stage recall that an existence result for the initial boundary value problem given in Eqs. (1.9) was presented in [15]. The present paper, which is a continuation of [15], is organized as follows:

- Section 2 presents the weak formulation of the boundary value problem.
- Section 3 is devoted to proving the existence and uniqueness of the solutions. We further on use the existence theorem obtained in [15] to state a general existence and uniqueness result.
- Section 4 deals with the functional framework within which the solution continuity at $t = 0$ is proved.
- Section 5 presents the proof of the solution continuity at $t = 0$.
- Section 6 contains results on the solution smoothness.

2. Weak formulation of the IBVP

All time-depending functions involved in the current stability analysis, save for when stated otherwise, are causal functions (i.e. set equal to zero on $\mathbb{R}_-$). Hence the convolution in time is simply $(f * g)(t) := \int_0^t f(s)g(t - s)\,ds$.

We first present the weak formulation of the boundary value problem Eqs. (1.9): find $\mathbf{u} \in \mathscr{C}^0([0, +\infty[, L^2(\Omega)^3) \cap L^1_{\text{loc}}(\mathbb{R}_+, H^1_0(\Omega)^3)$, $\nabla \cdot \mathbf{u} = 0$, $[\mathbf{S}]_{ij} \in \mathscr{C}^0(]0, +\infty[, L^2(\Omega) \cap L^1_{\text{loc}}(\mathbb{R}_+, L^2(\Omega)))$, $i, j = 1, 2, 3$, such that for any test-functions $\forall \boldsymbol{\theta} \in (H^1_0(\Omega))^3$, $\nabla \cdot \boldsymbol{\theta} = 0$, $\forall \mathbf{a} \in (\mathscr{D}(\Omega))^3$, $\forall \psi \in \mathscr{C}^\infty_{00}([0, +\infty[)$, where $\mathscr{C}^\infty_{00}([0, +\infty[)$ denotes the space of $\mathscr{C}^\infty$ class functions that vanish in a neighborhood of $+\infty$, the following equations hold true:

$$\psi(0) \int_\Omega \mathbf{u}_0(\mathbf{x}) \cdot \boldsymbol{\theta}(\mathbf{x})\,d\mathbf{x} + \int_0^{+\infty}\int_\Omega \mathbf{u}(t, \mathbf{x}) \cdot \boldsymbol{\theta}(\mathbf{x})\psi'(t)\,d\mathbf{x}\,dt = \int_0^{+\infty}\int_\Omega \big(\mathbf{S}(t, \mathbf{x}) : \nabla\boldsymbol{\theta}(\mathbf{x})\big)\psi(t)\,d\mathbf{x}\,dt \tag{2.1}$$

$$-\int_0^{+\infty}\int_\Omega \frac{\psi(\tau)}{\Gamma(1 - \alpha)\tau^\alpha}[\mathbf{S}_0]_{ij}(\mathbf{x})[\mathbf{a}]_j(\mathbf{x})\,d\mathbf{x}\,d\tau - \int_0^{+\infty}\int_\tau^{+\infty}\int_\Omega \frac{\psi'(t)}{\Gamma(1 - \alpha)\tau^\alpha}[\mathbf{S}]_{ij}(t - \tau, \mathbf{x})[\mathbf{a}]_j(\mathbf{x})\,d\mathbf{x}\,dt\,d\tau$$

$$+ \int_0^{+\infty}\int_\Omega [\mathbf{S}]_{ij}(t, \mathbf{x})[\mathbf{a}]_j(\mathbf{x})\psi(t)\,d\mathbf{x}\,dt$$

$$= -\frac{1}{\Gamma(1 - \beta)}\int_0^{+\infty}\int_0^t\int_\Omega \frac{\psi(t)}{(t - \tau)^\beta}\big\{(\nabla \cdot \mathbf{a})[\mathbf{u}]_i + [\mathbf{u} \cdot \nabla\mathbf{a}]_i\big\}(\tau, \mathbf{x})\,d\mathbf{x}\,d\tau\,dt \tag{2.2}$$

Summation over repeated indices is understood in Eqs. (2.1) and (2.2) above.

We now detail the functional framework. Let $V = \{\mathbf{h} \in (H^1_0(\Omega))^3 \text{ s.t. } \nabla \cdot \mathbf{h} = 0\}$ be the Hilbert space endowed with the inner product:

$$\langle \mathbf{f} | \mathbf{g} \rangle_V = \sum_{i,j=1}^3 \int_\Omega \frac{\partial f_i}{\partial x_j}\frac{\overline{\partial g_i}}{\partial x_j}(\mathbf{x})\,d\mathbf{x} \tag{2.3}$$

and denote $\| \ \|_V$ the corresponding norm.

The closure of V in $(L^2(\Omega))^3$ is denoted by H, the later space being endowed with the inner product:

$$\langle \mathbf{f} | \mathbf{g} \rangle_H = \sum_{i=1}^3 \int_\Omega f_i \bar{g}_i(\mathbf{x})\,d\mathbf{x} \tag{2.4}$$

with $\| \ \|_H$ being the corresponding norm. Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \lambda_n \leqslant \cdots \xrightarrow[n \to +\infty]{} +\infty$ and $\mathbf{w}_i \in V$, $i \in \mathbb{N}^*$, be the eigenvalues and the corresponding eigenfunctions of the Stokes operator in H, i.e.:

$$\forall \boldsymbol{\phi} \in V, \quad \langle \mathbf{w}_k | \boldsymbol{\phi} \rangle_V = \lambda_k \langle \mathbf{w}_k | \boldsymbol{\phi} \rangle_H, \quad \text{where } \|\mathbf{w}_k\|_H = 1 \tag{2.5}$$

3. The solution existence and uniqueness

To prove the solution uniqueness, we first eliminate $\mathbf{S}$ from Eqs. (2.1) and (2.2).

Denote $\mathscr{L} := \{f \in L^1_{\text{loc}}(\mathbb{R}_+) \text{ s.t. } \exists M > 0, \text{ so that } fe^{-Mt} \in L^1(\mathbb{R}_+)\} \subset L^1_{\text{loc}}(\mathbb{R}_+)$. Next, let $f \in L^1_{\text{loc}, \mathbb{R}_+}(\mathbb{R})$. For any $a \in \mathbb{R}$ and $\alpha \in \ ]0, 1[$, define $D^\alpha_{t,a} f$ by:

$$\langle D^\alpha_{t,a} f, \varphi \rangle = \frac{1}{\Gamma(1 - \alpha)}\left[-a \int_0^{+\infty} \frac{\varphi(\tau)}{\tau^\alpha}\,d\tau - \int_0^{+\infty} f(t)\left(\int_0^{+\infty} \frac{\varphi'(t + \tau)}{\tau^\alpha}\,d\tau \right)dt \right] \tag{3.1}$$

for any test function $\varphi \in \mathscr{D}(\mathbb{R})$.

Observe that $D_{t,a}^\alpha f \in \mathscr{D}'(\mathbb{R})$. Moreover, for any $f \in \mathscr{L}$, one easily sees that for $x \geqslant M$, $e^{-xt} D_{t,a}^\alpha f \in \mathscr{S}'(\mathbb{R})$.

The hat $\widehat{(\ )}$ notation to be used below stands for the usual Laplace transform.

Proposition 3.1. Let $f \in \mathscr{L}$. Then, for $\forall s \in \mathbb{C}$, with Re(s) large enough, one has $\widehat{D_{t,0}^\alpha f}(s) = s^\alpha \hat{f}(s)$.

Proof. Let $M > 0$ such that $e^{-Mt} f \in L^1(\mathbb{R}_+)$. For any $s \in \mathbb{C}$, Re$(s) \geqslant M$ and $\varphi(t) = e^{-st}$, Eq. (3.1) – still valid for this particular choice of $\varphi(t) = e^{-st} \notin \mathscr{D}(\mathbb{R})$ – gives:

$$\langle D_{t,0}^\alpha f, e^{-st} \rangle = -\frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} f(t) \left(\int_0^{+\infty} \frac{-s e^{s+\tau}}{\tau^\alpha} \, d\tau \right) dt \qquad (3.2)$$

As $\int_0^{+\infty} \frac{e^{-s\tau}}{\tau^\alpha} \, d\tau = \frac{\Gamma(1-\alpha)}{s^{1-\alpha}}$, for Re$(s) \geqslant M > 0$ one gets:

$$\widehat{D_{t,0}^\alpha f}(s) = \langle D_{t,0}^\alpha f, e^{-st} \rangle = -\frac{s}{\Gamma(1-\alpha)} \int_0^{+\infty} e^{-st} f(t) \Gamma(1-\alpha) s^{\alpha-1} \, dt = s^\alpha \hat{f}(s) \qquad \square \qquad (3.3)$$

The following classical result (see [7]) is stated here within our functional framework. Recall first that (see also Eqs. (14) and (15) in [15]):

$$W_0(t) = \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} \frac{e^{-rt} r^{\alpha-1}}{r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1} \, dr, \quad t \geqslant 0 \qquad (3.4)$$

$$E_\alpha(t) = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{r^\alpha e^{-rt}}{r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1} \, dr, \quad t > 0 \qquad (3.5)$$

Proposition 3.2. Let $F \in \mathscr{L}$. Then, for any $a \in \mathbb{R}$ and any $\alpha \in]0, 1[$, the equation

$$D_{t,a}^\alpha f + f = F \qquad (3.6)$$

has a unique solution $f \in \mathscr{L}$, given by $f = E_\alpha * F + aW_0$.

Proof. *Existence*: Assume $F \in \mathscr{C}^1(\mathbb{R}_+)$. Then, $f = E_\alpha * F + aW_0$ is a solution of Eq. (3.6) (cf. [7]). Now, if one assumes that $F \in \mathscr{L}$, then there exists $(F_n)_{n \in \mathbb{N}^*}$, $F_n \in \mathscr{C}^1(\mathbb{R}_+)$ such that $F_n \underset{n \to +\infty}{\longrightarrow} F$ in $L_{\mathrm{loc}}^1(\mathbb{R}_+)$. Since $E_\alpha \in L_{\mathrm{loc}}^1(\mathbb{R}_+)$, then $f_n = E_\alpha * F_n + aW_0 \underset{n \to +\infty}{\longrightarrow} E_\alpha \underset{(t)}{*} F + aW_0$ in $L_{\mathrm{loc}}^1(\mathbb{R}_+)$. Hence the equation $D_{t,a}^\alpha f_n + f_n = F_n$, for $n \to +\infty$, becomes $D_{t,a}^\alpha f + f = F$.

We must next show that $f \in \mathscr{L}$. Notice first that $\|aW_0\|_\infty \leqslant |a| W_0(0)$. Therefore $aW_0 \in \mathscr{L}$. Denote $g = e^{-Mt} F$; choose $M > 0$ so that $e^{-Mt} F \in L^1(\mathbb{R})$. Then $|E_\alpha \underset{(t)}{*} F| = |\int_0^t E_\alpha(t-s) e^{Ms} g(s) \, ds| \leqslant e^{Mt}[E_\alpha \underset{(t)}{*} g]$. Since $g \in L^1(\mathbb{R}_+)$, and $E_\alpha \in L^1(\mathbb{R}_+)$, then $E_\alpha * g \in L^1(\mathbb{R}_+)$. Finally $E_\alpha * F \in \mathscr{L}$ and $f = E_\alpha * F + aW_0 \in \mathscr{L}$.

Uniqueness: Let $f, g \in \mathscr{L}$ be two solutions of Eq. (3.6). Then $D_{t,0}^\alpha(f - g) + (f - g) = 0$, from which it follows that $(s^\alpha + 1) \widehat{f - g}(s) = 0$, for Re$(s)$ large enough. Therefore $\widehat{f - g}(s) = 0$, thus $f = g$. $\quad \square$

We shall use the following result to prove the uniqueness property:

Lemma 3.1. For any $g \in \mathscr{L}$, $I_t^{1-\beta} g \in \mathscr{L}$.

Proof. Since $g \in \mathscr{L}$, there exists $G \in L^1(\mathbb{R}_+)$ and $M > 0$ such that $g = Ge^{-Mt}$. Therefore, for $t \geqslant 0$ a.e., $|I_t^{1-\beta} g(t)| \leqslant K|\int_0^t \frac{g(t-u)}{u^\beta} \, du| \leqslant K \int_0^t |G(t-u)| \frac{e^{-M(t-u)}}{u^\beta} \, du \leqslant Ke^{-Mt}(|G| * u_\beta e^{-Mt})$. Now, $G \in L^1(\mathbb{R}_+)$, $u_\beta e^{-Mt} \in L^1(\mathbb{R}_+)$ leads to $(|G| * u_\beta e^{-Mt}) \in L^1(\mathbb{R}_+)$. Therefore, $|I_t^{1-\beta} g(t)| \leqslant e^{Mt} H(t)$, with $H \in L^1(\mathbb{R}_+)$, which gives $I_t^{1-\beta} g \in \mathscr{L}$. $\quad \square$

Making use of Proposition 3.2 and Lemma 3.1, we get:

Corollary 3.1 (Solution uniqueness). Let $\mathbf{u}_0 \in H$ and $\mathbf{S}_0 \in L^2(\Omega)^9$. The system of Eqs. (2.1)–(2.2) has at most one solution that belongs to the functional space $\mathscr{F} := \{(\mathbf{u}, \mathbf{S}) \in [\mathscr{C}^0(\mathbb{R}_+, H) \cap L_{\mathrm{loc}}^1(\mathbb{R}_+, V)] \times \mathscr{C}^0(]0, +\infty[, L^2(\Omega)^9)$, such that $\|\nabla\mathbf{u}\|_{L^2(\Omega)^9} \in \mathscr{L}$, $\|\mathbf{S}\|_{L^2(\Omega)^9} \in \mathscr{L}\}$.

Proof. Let $(\mathbf{u}, \mathbf{S}) \in \mathscr{F}$ be a solution to Eqs. (2.1)–(2.2). For any test function $\boldsymbol{\varphi} \in \mathscr{D}(\Omega)^9$, as a consequence of Eq. (2.2) and of the fact that $\mathbf{u} \in L^1_{\mathrm{loc}}(\mathbb{R}_+, V)$, one has

$$D^{\alpha}_{t, \langle \mathbf{S}_0 | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9}} \langle \mathbf{S} | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} + \langle \mathbf{S} | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} = I^{1-\beta}_t \big(\langle \mathbf{A}_1 | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} \big) \tag{3.7}$$

However, $|\langle \mathbf{S} | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9}| \leqslant \|\mathbf{S}\|_{L^2(\Omega)^9} \|\boldsymbol{\varphi}\|_{L^2(\Omega)^9}$. Since $\|\mathbf{S}\|_{L^2(\Omega)^9} \in \mathscr{L}$ and $\|\nabla \mathbf{u}\|_{L^2(\Omega)^9} \in \mathscr{L}$, we infer that $\langle \mathbf{S} | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} \in \mathscr{L}$, and $\langle \mathbf{A}_1 | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} \in \mathscr{L}$. Now Lemma 3.1 implies $I^{1-\beta}_t \langle \mathbf{A}_1 | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} \in \mathscr{L}$, and Proposition 3.2 leads to

$$\begin{aligned}
\langle \mathbf{S} | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} &= E_\alpha * I^{1-\beta}_t \big(\langle \mathbf{A}_1 | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} \big) + \langle \mathbf{S}_0 | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} W_0 \\
&= \rho * \langle \mathbf{A}_1 | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} + \langle \mathbf{S}_0 | \boldsymbol{\varphi} \rangle_{L^2(\Omega)^9} W_0
\end{aligned} \tag{3.8}$$

Notice that Eq. (3.8) still holds true for $\boldsymbol{\varphi} \in L^2(\Omega)^9$.

Let $\boldsymbol{\theta} \in V$, $\psi \in \mathcal{C}^{+\infty}_{00}([0, +\infty[)$. We deduce from (3.8) and (2.1) that:

$$\begin{aligned}
\psi(0) \int\limits_\Omega \mathbf{u}_0(\mathbf{x}) \cdot \boldsymbol{\theta}(\mathbf{x}) \, d\mathbf{x} &+ \int\limits_0^{+\infty} \int\limits_\Omega \mathbf{u}(t, \mathbf{x}) \cdot \boldsymbol{\theta}(\mathbf{x}) \psi'(t) \, d\mathbf{x} \, dt \\
&= - \int\limits_0^{+\infty} \big(\rho * \langle \mathbf{A}_1 | \nabla \boldsymbol{\theta} \rangle_{L^2(\Omega)^9} + \langle \mathbf{S}_0 | \nabla \boldsymbol{\theta} \rangle_{L^2(\Omega)^9} W_0 \big)(t) \psi(t) \, dt
\end{aligned} \tag{3.9}$$

We search for $\mathbf{u} \in L^1_{\mathrm{loc}}(\mathbb{R}_+, V)$. In this case, for almost every $t > 0$, $\mathbf{u}$ can be expressed as

$$\mathbf{u}(t) = \sum_{q=1}^{+\infty} \alpha_q(t) \mathbf{w}_q \tag{3.10}$$

the series being convergent in V. It follows, by taking $\boldsymbol{\theta} = \mathbf{w}_k$ and $\psi \in \mathscr{D}(]0, +\infty[)$ in Eq. (3.9), that $\alpha'_k = -\lambda_k (\rho * \alpha_k) - b_k \sqrt{\lambda_k} W_0$, with $b_k := \int_\Omega (\mathbf{S}_0 : \nabla \mathbf{w}_k)(\mathbf{x}) \, d\mathbf{x}$, the equality holding true in $\mathscr{D}'(]0, +\infty[)$. Recall that $\alpha_k = \langle \mathbf{u} | \mathbf{w}_k \rangle_H \in \mathscr{C}^0(\mathbb{R}_+)$. As $W_0 \in \mathscr{C}^0(\mathbb{R}_+)$, then necessarily $\alpha_k \in \mathscr{C}^1(\mathbb{R}_+)$. However (cf. [15]), the Cauchy's initial value problem

$$\alpha'_k(t) = -\lambda_k (\rho * \alpha_k)(t) - b_k \sqrt{\lambda_k} W_0(t) \tag{3.11a}$$
$$\alpha_k(0) = \alpha^0_k \tag{3.11b}$$

has a unique solution in $\mathscr{C}^1(\mathbb{R}_+)$. The uniqueness of the solution $\mathbf{u}$ is thus proved, and that of $\mathbf{S}$ follows. $\square$

We now state an existence and uniqueness result. Denote $\mathscr{C}^0_b(\mathbb{R}^*_+, V) := \{\mathbf{u} \in \mathscr{C}^0(\mathbb{R}^*_+, V) \text{ s.t. } \sup_{t \geqslant 1} \|\mathbf{u}(t)\|_V < +\infty\}$. The functional space $\mathscr{C}^0_b(\mathbb{R}^*_+, L^2(\Omega)^9)$ is defined in a similar way.

Theorem 3.1 *(First existence and uniqueness theorem). Let $\mathbf{u}_0 \in H$, $\mathbf{S}_0 \in L^2(\Omega)^9$. Then the boundary value problem given by the system of Eqs.* (2.1)–(2.2) *has a unique solution*

$$(\mathbf{u}, \mathbf{S}) \in \big[\mathscr{C}^0(\mathbb{R}_+, H) \cap \mathscr{C}^0_b(\mathbb{R}^*_+, V) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+, V) \big] \times \big[\mathscr{C}^0_b(\mathbb{R}^*_+, L^2(\Omega)^9) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+, L^2(\Omega)^9) \big] \tag{3.12}$$

Moreover, $\mathbf{u}(0) = \mathbf{u}_0$.

Proof. The existence of at least one solution

$$(\mathbf{u}, \mathbf{S}) \in \big[\mathscr{C}^0(\mathbb{R}_+, H) \cap \mathscr{C}^0(\mathbb{R}^*_+, V) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+, V) \big] \times \big[\mathscr{C}^0(\mathbb{R}^*_+, L^2(\Omega)^9) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+, L^2(\Omega)^9) \big]$$

follows from Theorem 8.4 in [15].

It remains to be proved that the solution $(\mathbf{u}, \mathbf{S})$ obtained in [15] satisfies

$$\sup_{t \geqslant 1} \|\mathbf{u}(t)\|_V + \sup_{t \geqslant 1} \|\mathbf{S}(t)\|_{L^2(\Omega)^9} < +\infty \tag{3.13}$$

This is essentially contained in the arguments presented in [15]. Indeed, since $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, V)$, we write $\mathbf{u} = \sum_{k=1}^{+\infty} \alpha_k(t) \mathbf{w}_k$, $t > 0$. From Eq. (125) in [15] one gets

$$\lambda_k |\alpha_k(t)|^2 \leqslant M \left\{ \left[\left(\frac{1}{t^{\delta/2}} + \frac{1}{t^{1-\delta/2}} \right)^2 + \frac{(t\lambda_k^{1/(2-\delta)})^{2(1-\delta/2)}}{t^{2(1-\delta/2)}} e^{-2\gamma t \lambda_k^{1/(2-\delta)}} \right] |\alpha_k^0|^2 \right.$$
$$\left. + \left(\frac{1}{t^{2(1-\delta)}} + \frac{(t\lambda_k^{1/(2-\delta)})^{2(1-\delta)}}{t^{2(1-\delta)}} e^{-2at\lambda_k^{1/(2-\delta)}} \right) |b_k|^2 \right\} \tag{3.14}$$

In the above, $\delta = (\alpha - \beta) \in {]0,1[}$, $a > 0$, $\gamma > 0$, and $\alpha_k^0 = \langle \mathbf{u}_0 | \mathbf{w}_k \rangle_H$. Clearly $\mathbf{u}_0 \in L^2(\Omega)^3$, $\mathbf{S}_0 \in L^2(\Omega)^9$ implies that $\sum_{k=1}^{+\infty} |\alpha_k^0|^2 < +\infty$ and $\sum_{k=1}^{+\infty} |b_k|^2 < +\infty$. Hence

$$\|\mathbf{u}\|_V^2 = \sum_{k=1}^{+\infty} \lambda_k |\alpha_k(t)|^2 \leqslant \frac{M}{t^{\inf(\delta, 2-2\delta)}} \underset{t \to +\infty}{\longrightarrow} 0 \tag{3.15}$$

and $\mathbf{u}_0 \in \mathscr{C}_b^0(\mathbb{R}_+^*, V)$.

We use the equation that defines $\mathbf{S}$ given right below (see Eq. (137) in [15]). Then:

$$\|\mathbf{S}\|_{L^2(\Omega)^9} \leqslant M \left[\sum_{k=1}^{+\infty} \lambda_k |\rho * \alpha_k(t)|^2 + |W_0(t)| \|\mathbf{S}_0\|_{L^2(\Omega)^9} \right] \tag{3.16}$$

From (3.4) we see that

$$|W_0(t)| \underset{t \to +\infty}{\longrightarrow} 0 \tag{3.17}$$

Moreover (see Eq. (130) in [15])

$$\lambda_k |\rho * \alpha_k(t)|^2 \leqslant M \left[\left(\frac{1}{t^\delta} + \frac{(t\lambda_k^{1/(2-\delta)})^\delta}{t^\delta} \exp\left[-at\lambda_k^{1/(2-\delta)} \right] \right) |\alpha_k^0|^2 + \left(\frac{1}{t^{2\epsilon}} + \exp(-2at) \right) |b_k|^2 \right], \quad \epsilon > 0, \ a > 0$$

Hence

$$\sum_{k=1}^{+\infty} \lambda_k |\alpha_k(t)|^2 \leqslant \frac{M}{t^{\inf(\delta, 2\epsilon)}}, \quad \text{for } t \geqslant 1 \tag{3.18}$$

Finally (see (3.16)–(3.18)): $\|\mathbf{S}\|_{L^2(\Omega)^9} \underset{t \to +\infty}{\longrightarrow} 0$.

It follows that $\mathbf{S} \in \mathscr{C}_b^0(\mathbb{R}_+^*, L^2(\Omega)^9)$. The existence of at least one solution belonging to the functional space of (3.12) is thus proved.

The uniqueness of such a solution results from Corollary 3.1 and from the fact that

$$\left[\mathscr{C}^0(\mathbb{R}_+, H) \cap \mathscr{C}_b^0(\mathbb{R}_+^*, V) \cap L^1_{\text{loc}}(\mathbb{R}_+, V) \right] \times \left[\mathscr{C}_b^0(\mathbb{R}_+^*, L^2(\Omega)^9) \cap L^1_{\text{loc}}(\mathbb{R}_+, L^2(\Omega)^9) \right] \subset \mathscr{F} \qquad \square$$

4. Functional spaces

In order to prove the continuity of the $(\mathbf{u}, \mathbf{S})$ at $t = 0$ we recall several classical functional spaces (see also [23] and [44]).

Denote $\boldsymbol{\epsilon}_k = \frac{\nabla \mathbf{w}_k}{\sqrt{\lambda_k}}$, $k \in \mathbb{N}^*$. Let $\Pi : L^2(\Omega)^9 \to L^2(\Omega)^9$ be the orthogonal projection operator of $L^2(\Omega)^9$ onto $[\text{Vect}(\boldsymbol{\epsilon}_k)_{k \in \mathbb{N}^*}]^\perp$, $\theta \geqslant 0$.

For any $\mathbf{f} \in L^2(\Omega)^9$, denote $\|\mathbf{f}\|_{D_\theta}^2 := \sum_{q=1}^{+\infty} \lambda_q^\theta |\langle \mathbf{f} | \boldsymbol{\epsilon}_q \rangle_{L^2(\Omega)^9}|^2 + \|\Pi(\mathbf{f})\|_{L^2(\Omega)^9}^2$. Let $D_\theta := \{ \mathbf{f} \in L^2(\Omega)^9 \text{ s.t. } \|f\|_{D_\theta} < +\infty \}$.

For any $\mathbf{f}, \mathbf{g} \in D_\theta$, denote $\langle \mathbf{f} | \mathbf{g} \rangle_{D_\theta} := \sum_{q=1}^{+\infty} \lambda_q^\theta \langle \mathbf{f} | \boldsymbol{\epsilon}_q \rangle_{L^2(\Omega)^9} \overline{\langle \mathbf{g} | \boldsymbol{\epsilon}_q \rangle_{L^2(\Omega)^9}} + \langle \Pi(\mathbf{f}) | \Pi(\mathbf{g}) \rangle_{L^2(\Omega)^9}$. The functional space $(D_\theta, \langle \ | \ \rangle_{D_\theta})$ is a Hilbert space.

For any $\mathbf{f} \in H$, let $\|\mathbf{f}\|_{H_\theta}^2 := \sum_{q=1}^{+\infty} \lambda_q^\theta |\langle \mathbf{f} | \mathbf{w}_q \rangle_H|^2$. Next, let $H_\theta := \{ \mathbf{f} \in H, \ \|\mathbf{f}\|_{H_\theta} < +\infty \}$. For any $f, g \in H_\theta$, $\langle f | g \rangle_{H_\theta} := \sum_{k=1}^{+\infty} \lambda_q^\theta \langle f | \mathbf{w}_k \rangle_H \overline{\langle g | \mathbf{w}_k \rangle_H}$. As the sequence $(\mathbf{w}_k)_{k \in \mathbb{N}^*}$ is complete in $L^2(\Omega)^9$, the functional space $(H_\theta, \langle \ | \ \rangle_{H_\theta})$ is a Hilbert space.

Remark that, for any $0 \leqslant \theta \leqslant \theta' \leqslant 1 \leqslant \theta''$, one has:

$$H = H_0 \hookleftarrow H_\theta \hookleftarrow H_{\theta'} \hookleftarrow H_1 = V \hookleftarrow H_{\theta''} \tag{4.1}$$
$$L^2(\Omega)^9 = D_0 \hookleftarrow D_\theta \hookleftarrow D_{\theta'} \hookleftarrow D_1 \hookleftarrow D_{\theta''} \tag{4.2}$$

The above injections are dense; use of them will be often made from now on.

The following Δ_θ spaces are closely related to the D_θ ones. Let $P : L^2(\Omega)^9 \to L^2(\Omega)^9$ be the orthogonal projection operator from $L^2(\Omega)^9$ onto $[\bigcup_{k=1}^{+\infty} \{ \boldsymbol{\epsilon}_k, \boldsymbol{\epsilon}_k^T \}]^\perp$.

Let $\theta \in \mathbb{R}$. For any element $\mathbf{f} \in L^2(\Omega)^9$, denote

$$\|\mathbf{f}\|^2_{\Delta_\theta} = \sum_{q=1}^{+\infty} \lambda_q^\theta \big|\langle \mathbf{f}|\boldsymbol{\epsilon}_q\rangle_{L^2(\Omega)^9}\big|^2 + \sum_{q=1}^{+\infty} \lambda_q^\theta \big|\langle \mathbf{f}|\boldsymbol{\epsilon}_q^T\rangle_{L^2(\Omega)^9}\big|^2 + \|P(\mathbf{f})\|^2_{L^2(\Omega)^9} \tag{4.3}$$

For any $\theta \geqslant 0$, $\Delta_\theta := \{\mathbf{f} \in L^2(\Omega)^9 \text{ s.t. } \|\mathbf{f}\|_{\Delta_\theta} < +\infty\}$. The functional space Δ_θ endowed with the inner product defined as: $\forall \mathbf{f} \in \Delta_\theta$, $\forall \mathbf{g} \in \Delta_\theta$,

$$\langle \mathbf{f}|\mathbf{g}\rangle_{\Delta_\theta} = \sum_{q=1}^{+\infty} \lambda_q^\theta \langle \mathbf{f}|\boldsymbol{\epsilon}_q\rangle_{L^2(\Omega)^9} \overline{\langle \mathbf{g}|\boldsymbol{\epsilon}_q\rangle}_{L^2(\Omega)^9} + \sum_{q=1}^{+\infty} \lambda_q^\theta \langle \mathbf{f}|\boldsymbol{\epsilon}_q^T\rangle_{L^2(\Omega)^9} \overline{\langle \mathbf{g}|\boldsymbol{\epsilon}_q^T\rangle}_{L^2(\Omega)^9} + \langle P(\mathbf{f})|P(\mathbf{g})\rangle_{L^2(\Omega)^9} \tag{4.4}$$

is a Hilbert space.

For any $\theta < 0$, let Δ_θ denote the topological dual space of $\Delta_{-\theta}$, i.e. $\Delta_\theta = (\Delta_{-\theta})'$. The space Δ_θ is the completion of $(L^2(\Omega)^9, \|\ \|_{\Delta_\theta})$. Next, note that whenever $0 \leqslant \gamma \leqslant \tilde{\gamma}$, the following injections $L^2(\Omega)^9 = \Delta_0 \hookleftarrow \Delta_\gamma \hookleftarrow \Delta_{\tilde{\gamma}}$ are dense. It results that $\Delta_{-\tilde{\gamma}} \hookleftarrow \Delta_{-\gamma} = (\Delta_\gamma)' \hookleftarrow (L^2(\Omega)^9)' \simeq L^2(\Omega)^9 = \Delta_0 \hookleftarrow \Delta_\gamma \hookleftarrow \Delta_{\tilde{\gamma}}$, invoking the fact that $L^2(\Omega)^9$ and $(L^2(\Omega)^9)'$ are isomorphic to each other.

Next, for any $\mathbf{f} \in H_0^1(\Omega)^9$, one has:

$$\begin{aligned}
\|\mathbf{f}\|^2_{\Delta_1} &= \sum_{q=1}^{+\infty} \lambda_q \big|\langle \mathbf{f}|\boldsymbol{\epsilon}_q\rangle_{L^2(\Omega)^9}\big|^2 + \sum_{q=1}^{+\infty} \lambda_q \big|\langle \mathbf{f}|\boldsymbol{\epsilon}_q^T\rangle_{L^2(\Omega)^9}\big|^2 + \|P(\mathbf{f})\|^2_{L^2(\Omega)^9} \\
&= \sum_{q=1}^{+\infty} \big|\langle \nabla \cdot \mathbf{f}|\mathbf{w}_q\rangle_{L^2(\Omega)^3}\big|^2 + \big|\langle \nabla \cdot \mathbf{f}^T|\mathbf{w}_q\rangle_{L^2(\Omega)^3}\big|^2 + \|P(\mathbf{f})\|^2_{L^2(\Omega)^3} \\
&\leqslant K\|\mathbf{f}\|^2_{H_0^1(\Omega)^9}
\end{aligned} \tag{4.5}$$

due to the Poincaré's inequality. Consequently $H_0^1(\Omega)^9 \hookrightarrow \Delta_1$ and the restriction $r : \Delta_1' \to H^{-1}(\Omega)^9$, such that $T \overset{r}{\mapsto} T|_{H_0^1(\Omega)^9}$ is continuous.

Lemma 4.1. *Let $\theta \in \mathbb{R}_+$.*

(a) *The sequence $(\boldsymbol{\epsilon}_k)_{k \in \mathbb{N}^*}$ is orthogonal in D_θ. Moreover, $\|\boldsymbol{\epsilon}_k\|_{D_\theta} = \lambda_k^{\theta/2}$.*
(b) *The sequence $(\boldsymbol{\epsilon}_k^T)_{k \in \mathbb{N}^*}$ is orthogonal in D_θ.*
(c) *The sequence $(\boldsymbol{\epsilon}_k + \boldsymbol{\epsilon}_k^T)_{k \in \mathbb{N}^*}$ is orthogonal in D_θ, and $\|\boldsymbol{\epsilon}_k + \boldsymbol{\epsilon}_k^T\|_{D_\theta} = (1 + \lambda_k^\theta)^{1/2}$.*
(d) *The sequence $(\mathbf{w}_k)_{k \in \mathbb{N}^*}$ is orthogonal in H_θ, and $\|\mathbf{w}_k\|_{H_\theta} = \lambda_k^{\theta/2}$.*
(e) *Let $\mathbf{f} \in L^2(\Omega)^9$. Denote $\mathbf{f} = \sum_{k=1}^{+\infty} a_k \boldsymbol{\epsilon}_k + \sum_{k=1}^{+\infty} b_k \boldsymbol{\epsilon}_k^T + P(\mathbf{f})$. Then:*

$$\|\mathbf{f}\|^2_{\Delta_{-\theta}} = \sum_{k=1}^{+\infty} \lambda_k^{-\theta} \big(|a_k|^2 + |b_k|^2\big) + \|P(\mathbf{f})\|^2_{L^2(\Omega)^9}.$$

Proof. Observe that (cf. Eq. (2.5)) for any $(k, q) \in \mathbb{N}^{*2}$:

$$\langle \boldsymbol{\epsilon}_k|\boldsymbol{\epsilon}_q\rangle_{L^2} = \langle \mathbf{w}_k|\mathbf{w}_q\rangle_{L^2} = \delta_{kq} \tag{4.6}$$

On the other hand, since $\nabla \cdot \mathbf{w}_k = 0$,

$$\langle \boldsymbol{\epsilon}_k|\boldsymbol{\epsilon}_q^T\rangle_{L^2} = \sum_{i,j} \int_\Omega \frac{\partial (\mathbf{w}_k)_i}{\partial x_j}(\mathbf{x}) \overline{\frac{\partial (\mathbf{w}_q)_j}{\partial x_i}}(\mathbf{x})\, d\mathbf{x} = \int_\Omega \left(\sum_i \frac{\partial (\mathbf{w}_k)_i}{\partial x_j}(\mathbf{x})\right) \left(\overline{\sum_j \frac{\partial (\mathbf{w}_q)_j}{\partial x_i}}(\mathbf{x})\right) d\mathbf{x} = 0 \tag{4.7}$$

Hence

$$\langle \boldsymbol{\epsilon}_k|\boldsymbol{\epsilon}_q^T\rangle_{L^2} = 0 \tag{4.8}$$

The statements (a) to (e) result from Eqs. (4.6)–(4.8). $\quad\square$

Except for the injection $H_{2k} \hookrightarrow H^{2k}(\Omega)^3$ (see below), the following description of the spaces will not be used in this paper.

Let first $\theta \in [0, 1]$.

Let $\Lambda : V \to H$, such that $\Lambda(\sum_{k \in \mathbb{N}^*} a_k \mathbf{w}_k) = \sum_{k \in \mathbb{N}^*} a_k \sqrt{\lambda_k} \mathbf{w}_k$. Then, for any $(\mathbf{u}, \mathbf{v}) \in V^2$, $\langle \Lambda \mathbf{u} | \Lambda \mathbf{v} \rangle_H = \langle \mathbf{u} | \mathbf{v} \rangle_V$, and $H_\theta = D(\Lambda^\theta) = [V, H]_{1-\theta}$, where $[V, H]_{1-\theta}$ stands for the holomorphic interpolation of spaces V and H, and $D(\Lambda^\theta)$ for the domain of Λ^θ. Denote $H_0^0(\Omega) \equiv L^2(\Omega)$. Let the canonical injection $H \overset{i}{\hookrightarrow} H_0^0(\Omega)^3$ and $V \overset{i|_V}{\hookrightarrow} H_0^1(\Omega)^3$ be its restriction.

Then $H_\theta = [V, H]_{1-\theta} \overset{i|_{[V,H]_{1-\theta}}}{\hookrightarrow} [H_0^1(\Omega)^3, H_0^0(\Omega)^3]_{1-\theta} \hookrightarrow H_0^\theta(\Omega)^3$ (the last continuous injection $\hookrightarrow$ boils down to an equality $=$ whenever $\theta \neq 1/2$). Let now γ_n denote the normal-trace application. It is well known that $H = \{\mathbf{u} \in H^0(\Omega)^3,$ s.t. $\nabla \cdot \mathbf{u} = 0,$ $\gamma_n(\mathbf{u}) = 0\}$. From the preceding arguments it results that we have the continuous injection $H_\theta \hookrightarrow H \cap H_0^\theta(\Omega)^3 = \{\mathbf{u} \in H_0^\theta(\Omega)^3,$ s.t. $\nabla \cdot \mathbf{u} = 0,$ $\gamma_n(\mathbf{u}) = 0\}$, with the space $H \cap H_0^\theta(\Omega)^3$ being endowed with the $H_0^\theta(\Omega)^3$ topology.

Let now $\theta \in \mathbb{N}^*$.

As quoted on p. 106 in [44], $H_2 = D(\Lambda^2) = H^2(\Omega)^3 \cap V$. Also, invoking Agmon–Douglis–Nirenberg's Theorem as stated on p. 832 in [8], leads to $H_{2k} = D(\Lambda^{2k}) \hookrightarrow H^{2k}(\Omega)^3 \cap V$, $k \in \mathbb{N}^*$. Here $H^{2k}(\Omega)$ are classical Sobolev spaces.

5. The continuity of the solution (u, S) at $t = 0$

From now on $(\mathbf{u}, \mathbf{S})$ denotes the solution to Eqs. (2.1)–(2.2), with initial data $(\mathbf{u}_0, \mathbf{S}_0) \in H \times L^2(\Omega)^9$ (see Theorem 3.1). In order to prove continuity results we recall several representation formulas for $\mathbf{u}$ and $\mathbf{S}$. First, functions α_k, $k \in \mathbb{N}^*$, are defined by Eqs. (3.11a)–(3.11b). Equivalently, for $x \in \mathbb{R}_+$ (see [15]),

$$\alpha_k(t) = \frac{1}{2\pi} \left[\lim_{A \to +\infty} \int_{-A}^{+A} T_{\lambda_k}(x+iy) e^{(x+iy)t} \, dy \, \alpha_k^0 - \lim_{A \to +\infty} \int_{-A}^{+A} (T_{\lambda_k} w)(x+iy) e^{(x+iy)t} \, dy \, \sqrt{\lambda_k} b_k \right] \tag{5.1}$$

with $T_\mu(s) = \frac{s^{1-\beta}(s^\alpha+1)}{s^{2-\beta}(s^\alpha+1)+\mu}$, $w(s) = \frac{1}{s^{1-\alpha}(s^\alpha+1)}$, $\mu \in \mathbb{R}_+$, $s \in \mathbb{C} - \mathbb{R}_-$, and $\alpha_k^0 = \langle \mathbf{u}_0 | \mathbf{w}_k \rangle_H$, $b_k = \int_\Omega (\mathbf{S}_0 : \nabla \mathbf{w}_k)(\mathbf{x}) \, d\mathbf{x}$.

Notice that Eq. (5.1) is given in [15] only for $x \geqslant M$. The general result ($x \in \mathbb{R}_+$) follows from a simple use of the Cauchy formula; details are omitted. Regarding function $\mathbf{S}$, recall the following formula from [15]:

$$\mathbf{S} = \sum_{k=1}^{+\infty} (\rho * \alpha_k) \otimes (\nabla \mathbf{w}_k + \nabla^T \mathbf{w}_k) + W_0 \otimes \mathbf{S}_0 \tag{5.2}$$

Notation $h = f \otimes g$ means $h(x, y) = f(x)g(y)$. As quoted in [15], the series in (5.2) converges in $\mathscr{C}^0(\mathbb{R}_+^*, L^2(\Omega)^9)$ and in $L^2_{\text{loc}}(\mathbb{R}_+, L^2(\Omega)^9)$. Here $\rho = E_\alpha * \frac{t^{-\beta}}{\Gamma(1-\beta)}$, and $0 < \rho(t) \leqslant k t^{-\delta}$ (see [15]), $\delta = \beta - \alpha$.

The following estimate will give the continuity at $t = 0$ of $(\mathbf{u}, \mathbf{S})$.

Lemma 5.1. *For any $\mu_0 > 0$, $\exists M > 0$, such that $\forall (\mathbf{x}, t) \in (\mathbb{R}_+)^3 \times \mathbb{R}_+$, and $\forall \mu \geqslant \mu_0$, we have:*

$$\lim_{A \to \infty} \left| \int_{-A}^{+A} T_\mu w(x+iy) e^{(x+iy)t} \, dy \right| \leqslant \int_{-\infty}^{+\infty} |(T_\mu w)(iy)| \, dy \leqslant \frac{M}{\mu^{1/(2-\delta)}} \tag{5.3}$$

Proof. Whenever $y > 0$, $|(T_\mu w)(iy)| = \frac{1}{|y|^\delta} \frac{1}{|\mu+(ye^{i\pi/2})^{2-\delta}+(ye^{i\pi/2})^{2-\beta}|}$. Therefore,

$$\begin{aligned} \left| \mu + (ye^{i\pi/2})^{2-\delta} + (ye^{i\pi/2})^{2-\beta} \right| &\geqslant \left| \text{Im}[\mu e^{i\pi(\beta-2)/2} + y^{2-\delta} e^{i\pi\alpha/2} + y^{2-\beta}] \right| \\ &\geqslant \left| -\mu \sin(\pi\beta/2) + y^{2-\delta} \sin(\pi\alpha/2) \right| \\ &\geqslant \mu \sin(\pi\beta/2) - y^{2-\delta} \sin(\pi\alpha/2) \\ &\geqslant K\mu, \quad \text{for } y \leqslant \mu^{1/(2-\delta)} \end{aligned} \tag{5.4}$$

The constant $K = \sin(\pi\beta/2) - \sin(\pi\alpha/2) > 0$ is independent of μ. Moreover,

$$\begin{aligned} \left| \mu + (ye^{i\pi/2})^{2-\delta} + (ye^{i\pi/2})^{2-\beta} \right| &\geqslant \left| \text{Im}[\mu + (ye^{i\pi/2})^{2-\delta} + (ye^{i\pi/2})^{2-\beta}] \right| \\ &= \left| y^{2-\delta} \sin(\pi - \pi\delta/2) + y^{2-\beta} \sin(\pi - \pi\beta/2) \right| \\ &= \left| y^{2-\delta} \sin(\pi\delta/2) + y^{2-\beta} \sin(\pi\beta/2) \right| \\ &\geqslant K y^{2-\delta}, \quad \text{for } y \geqslant \mu_0^{1/(2-\delta)} \end{aligned} \tag{5.5}$$

From the above estimates we infer that:

$$
\int\limits_0^{+\infty} \left|(T_\mu w)(iy)\right| \mathrm{d}y = \int\limits_0^{\mu^{1/(2-\delta)}} \frac{\mathrm{d}y}{y^\delta |\mu + (ye^{i\pi/2})^{2-\delta} + (ye^{i\pi/2})^{2-\beta}|} + \int\limits_{\mu^{1/(2-\delta)}}^{+\infty} \frac{\mathrm{d}y}{y^\delta |\mu + (ye^{i\pi/2})^{2-\delta} + (ye^{i\pi/2})^{2-\beta}|}
$$

$$
\leqslant \int\limits_0^{\mu^{1/(2-\delta)}} \frac{K}{\mu y^\delta}\,\mathrm{d}y + \int\limits_{\mu^{1/(2-\delta)}}^{+\infty} \frac{K}{y^\delta y^{2-\delta}}\,\mathrm{d}y
$$

$$
\leqslant \frac{M}{\mu^{1/(2-\delta)}} \tag{5.6}
$$

A similar estimate can be obtained for $\int_{-\infty}^0 |(T_\mu w)(iy)|\,\mathrm{d}y$. Combining these results achieves the proof. □

Denote, $\delta = \beta - \alpha$, $\omega = \delta/(2-\delta)$ and notice that $0 < \omega < \delta < 1$. From now on we shall sometimes write $\alpha_k(\mathbf{u}_0, \mathbf{S}_0)$ instead of α_k; of course α_k is linear w.r.t. initial data $(\mathbf{u}_0, \mathbf{S}_0)$. Most of the following estimates are already proved in [15], save for those derived from Lemma 5.1.

Proposition 5.1. *Let* $\mathbf{u}_0 \in H$, $\mathbf{S}_0 \in L^2(\Omega)^9$. *Then exists* $\exists M > 0$, *such that,* $\forall t \in \mathbb{R}_+$ *and* $\forall k \in \mathbb{N}^*$,

(i) $|\alpha_k(t)|^2 \leqslant M(|\alpha_k^0|^2 + \lambda_k^{-\omega}|b_k|^2)$;

(ii) $\lambda_k|\alpha_k(t)|^2 \leqslant M(\frac{|\alpha_k^0|^2}{t^{2-\delta}} + \frac{|b_k|^2}{t^{2-2\delta}})$;

(iii) *for any* $\mu \in [0, 1]$ *and any* $\tau \in [0, 1]$,

$$
\left|\alpha_k(t)\right|^2 \leqslant M\left(\frac{|\alpha_k^0|^2}{\lambda_k^\mu t^{\mu(2-\delta)}} + \frac{|b_k|^2}{\lambda_k^{\tau+(1-\tau)\omega} t^{2\tau(1-\delta)}}\right)
$$

(iv) *for any* $\mu \in [0, 1]$ *and any* $\tau \in [0, 1]$,

$$
\lambda_k|\rho * \alpha_k|^2(t) \leqslant M\left(\frac{|\alpha_k^0|^2}{\lambda_k^{\mu-1} t^{\mu(2-\delta)+2(\delta-1)}} + \frac{|b_k|^2}{\lambda_k^{-(1-\tau)(1-\omega)} t^{-2(1-\tau)(1-\delta)}}\right)
$$

Proof. (i) Since α_k is linear with respect to $(\mathbf{u}_0, \mathbf{S}_0)$, we have, by Eq. (121) in Theorem 8.1 in [15] and Eq. (5.1):

$$
\left|\alpha_k(\mathbf{u}_0, \mathbf{S}_0)\right|^2 \leqslant 2\left|\alpha_k(\mathbf{u}_0, 0)\right|^2 + 2\left|\alpha_k(0, \mathbf{S}_0)\right|^2 \leqslant M\left(\left|\alpha_k^0\right|^2 + \left|\int\limits_{-\infty}^{+\infty} (T_{\lambda_k} w)(iy)e^{iyt}\,\mathrm{d}y\sqrt{\lambda_k}b_k\right|^2\right) \tag{5.7}
$$

Invoking Lemma 5.1 we get

$$
\left|\alpha_k(\mathbf{u}_0, \mathbf{S}_0)\right|^2 \leqslant M\left(\left|\alpha_k^0\right|^2 + \frac{|b_k|^2 \lambda_k}{\lambda_k^{2(2-\delta)}}\right) \tag{5.8}
$$

which gives (i).

(ii) Estimate (ii) is obtainable right away from Eq. (122) in Theorem 8.1 in [15], with M_T instead of M. The proof that M can be chosen independently of T is deferred until Corollary 6.1 in Section 6. Hence we take here M independent of T and proceed further on.

(iii) Notice first that (ii) above gives

$$
\left|\alpha_k(\mathbf{u}_0, 0)\right|^2(t) \leqslant \frac{M|\alpha_k^0|^2}{\lambda_k t^{2-\delta}} \tag{5.9}
$$

and

$$
\left|\alpha_k(0, \mathbf{S}_0)\right|^2(t) \leqslant \frac{M|b_k|^2}{\lambda_k t^{2-2\delta}} \tag{5.10}
$$

Next, combining Eqs. (5.9) and (5.8) with $\mathbf{S}_0 = \mathbf{0}$ on one hand, and Eqs. (5.10) and (5.8) with $\mathbf{u}_0 = \mathbf{0}$ on the other, making further use of Eq. (5.7) leads to estimate (iii).

(iv) Since $0 \leqslant \rho(t) \leqslant K/t^\delta$, estimate (iii) gives

$$\sqrt{\lambda_k}|\rho * \alpha_k|(t) \leqslant \sqrt{\lambda_k}\big(\rho * |\alpha_k|\big)(t) \leqslant M\left(\frac{\sqrt{\lambda_k}|\alpha_k^0|}{\lambda_k^{\mu/2} t^{\mu(2-\delta)/2+\delta-1}} + \frac{\sqrt{\lambda_k}|b_k|}{\lambda_k^{[\tau+\omega(1-\tau)]/2} t^{\tau(1-\delta)+\delta-1}}\right)$$

from which (iv) is obtained. $\quad\square$

In order to work on spaces H_θ and D_γ, we need to reformulate Proposition 5.1. Let $[\ ]_+$ denote the positive part of a real number.

Lemma 5.2. *Let* $\mathbf{u}_0 \in H$, $\mathbf{S}_0 \in L^2(\Omega)^9$, *and* $0 \leqslant \gamma \leqslant \theta \leqslant \gamma + 1$. *Then* $\exists M > 0$, *so that* $\forall t \geqslant 0$,

(i) $\qquad \lambda_k^\theta |\alpha_k(t)|^2 \leqslant M\left(\lambda_k^\theta |\alpha_k^0|^2 + \dfrac{|\lambda_k|^\gamma |b_k|^2}{t^{2(1-\delta)[(\theta-\gamma-\omega)/(1-\omega)]_+}}\right)$

(ii) $\qquad \lambda_k^\gamma \big\langle \lambda_k |\rho * \alpha_k(t)|^2 \big\rangle(t) \leqslant M\big(\lambda_k^\theta |\alpha_k^0|^2 t^{\delta(\theta-\gamma-\omega)/\omega} + \lambda_k^\gamma |b_k|^2\big)$

Proof. (i) Assume $0 \leqslant \gamma \leqslant \theta \leqslant \gamma + \omega$. Multiplying (i) of Theorem 5.1 by λ_k^θ leads to $\lambda_k^\theta |\alpha_k(t)|^2 \leqslant A[\lambda_k^\theta |\alpha_k^0|^2 + \lambda_k^{\theta-\omega} |b_k|^2]$. Since $\min_{k \geqslant 1} \lambda_k > 0$ and $\theta - \omega \leqslant \gamma$, one gets $\lambda_k^\theta |\alpha_k(t)|^2 \leqslant M[\lambda_k^\theta |\alpha_k^0|^2 + \lambda_k^\gamma |b_k|^2]$.

Assume now that $0 \leqslant \gamma + \omega \leqslant \theta \leqslant \gamma + 1$. Use part (iii) of Theorem 5.1 with $\mu = 0$ and $\tau = \frac{\theta-\gamma-\omega}{1-\omega} \in [0, 1]$. A simple calculation leads to $\tau + (1-\tau)\omega = \tau(1-\omega) + \omega = (\theta - \gamma - \omega) + \omega = \theta - \gamma$. Henceforth,

$$\big|\alpha_k(t)\big|^2 \leqslant M\left(\big|\alpha_k^0\big|^2 + \frac{|b_k|^2}{\lambda_k^{\theta-\gamma} t^{2(1-\delta)(\theta-\gamma-\omega)/(1-\omega)}}\right)$$

which leads to (i).

(ii) Assume $0 \leqslant \gamma \leqslant \theta \leqslant \gamma + 1$. Letting $\mu = \gamma - \theta + 1 \in [0, 1]$ and $\tau = 1$ in part (iv) of Lemma 5.1, gives $\mu(2 - \delta) + 2(\delta - 1) = (\gamma - \theta)(2 - \delta) + \delta = -\delta(\theta - \gamma - \omega)/\omega$. Hence

$$\lambda_k |\rho * \alpha_k|^2(t) \leqslant M\left(\frac{|\alpha_k^0|^2}{\lambda_k^{\gamma-\theta} t^{-\delta(\theta-\gamma-\omega)/\omega}} + |b_k|^2\right)$$

which gives (ii). $\quad\square$

As a consequence, we have:

Corollary 5.1. *Let* $0 \leqslant \gamma \leqslant \theta \leqslant \gamma + 1$, $\mathbf{u}_0 \in H_\theta$, $\mathbf{S}_0 \in D_\gamma$. *Then* $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+^*, H_{\gamma+1})$. *In addition*:

(a) *Whenever* $0 \leqslant \gamma \leqslant \theta \leqslant \gamma + \omega$, $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, H_\theta)$, *and* $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+^*, D_\gamma)$; *moreover,* $\mathbf{u}(0) = \mathbf{u}_0$.
(b) *Whenever* $0 \leqslant \gamma + \omega \leqslant \theta$, $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, H_{\gamma+\omega})$, *and* $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, D_\gamma)$; *moreover,* $\mathbf{u}(0) = \mathbf{u}_0$ *and* $\mathbf{S}(0) = \mathbf{S}_0$.

In both cases, for any $t \geqslant 0$,

$$\big\|\mathbf{u}(t)\big\|_{H_\theta} \leqslant M\left(\|\mathbf{u}_0\|_{H_\theta} + \frac{\|\mathbf{S}_0\|_{D_\gamma}}{t^{(1-\delta)[(\theta-\gamma-\omega)/(1-\omega)]_+}}\right) \tag{5.11}$$

$$\big\|\mathbf{S}(t)\big\|_{D_\gamma} \leqslant M\big(t^{\delta(\theta-\gamma-\omega)/(2\omega)} \|\mathbf{u}_0\|_{H_\theta} + \|\mathbf{S}_0\|_{D_\gamma}\big) \tag{5.12}$$

Proof. We first prove that $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+^*, H_{\gamma+1})$.

From (d) of Lemma 4.1 and (ii) in Proposition 5.1, we reckon that, for any $N \leqslant M$ and $t \in [T_1, T_2]$, where $0 < T_1 < T_2$,

$$\left\|\sum_{k=N}^M \alpha_k(t)\mathbf{w}_q\right\|_{H_{\gamma+1}}^2 = \sum_{k=N}^M |\alpha_k(t)|^2 \lambda_k^{\gamma+1} \leqslant M_T \sum_{k=N}^{+\infty}\left[\frac{\lambda_k^\gamma |\alpha_k^0|^2}{T_1^{2-\delta}} + \frac{\lambda_k^\gamma |b_k|^2}{T_1^{2(1-\delta)}}\right] \tag{5.13}$$

Since $\gamma \leqslant \theta$, we have that $\mathbf{u}_0 \in H_\theta \subset H_\gamma$. Also $\mathbf{S}_0 \in H_\gamma$. Hence Eq. (5.13) implies that

$$\sup_{t \in [T_1, T_2]} \left\|\sum_{k=N}^M \alpha_k(t)\mathbf{w}_q\right\|_{H_{\gamma+1}} \underset{N \to +\infty}{\longrightarrow} 0 \tag{5.14}$$

Finally, as $\alpha_k \in \mathscr{C}^0(\mathbb{R}_+)$, from (5.14) above we deduce that $\mathbf{u} = (\sum_{k=N}^{+\infty} \alpha_k \otimes \mathbf{w}_q) \in \mathscr{C}^0(\mathbb{R}_+^*, H_{\gamma+1})$.

Next we proceed with the rest of the proof.

(a) Let us prove that $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+^*, D_\gamma)$. For any $N \leqslant M$, by (c) of Lemma 4.1 and by (ii) of Lemma 5.2,

$$\left\| \sum_{k=N}^{M} (\rho * \alpha_k)(t)\left(\nabla \mathbf{w}_k + \nabla^T \mathbf{w}_k\right) \right\|_{D_\gamma}^2 = \sum_{k=N}^{M} \left| \rho \underset{(t)}{*} \alpha_k \right|^2 \left(1 + \lambda_k^\gamma\right)\lambda_k \leqslant K \sum_{k=N}^{+\infty} \left[\lambda_k^\theta |\alpha_k^0|^2 t^{\delta(\theta-\gamma-\omega)/\omega} + \lambda_k^\gamma |b_k|^2\right] \quad (5.15)$$

Since $\mathbf{u}_0 \in H_\theta$, and $\mathbf{S}_0 \in D_\gamma$, then $\sum_{k=1}^{+\infty} (\rho * \alpha_k)(t)(\nabla \mathbf{w}_k + \nabla^T \mathbf{w}_k)$ is uniformly convergent w.r.t. t on any compact subset $[T_0, T_1] \subset \mathbb{R}_+^*$, in D_γ. Given that $\alpha_k \in \mathscr{C}^0(\mathbb{R}_+)$, and that $\rho \in L^1_{\text{loc}}(\mathbb{R}_+)$ and hence $(\rho * \alpha_k) \in \mathscr{C}^0(\mathbb{R}_+)$, we conclude that $\sum_{k=1}^{+\infty} (\rho * \alpha_k) \otimes (\nabla \mathbf{w}_k + \nabla^T \mathbf{w}_k) \in \mathscr{C}^0([T_0, T_1], D_\gamma)$. As $W_0 \in \mathscr{C}^0(\mathbb{R}_+)$, one gets that (see (5.2))

$$\mathbf{S} = \sum_{k=1}^{+\infty} (\rho * \alpha_k) \otimes \left(\nabla \mathbf{w}_k + \nabla^T \mathbf{w}_k\right) + W_0 \otimes \mathbf{S}_0 \in \mathscr{C}^0(\mathbb{R}_+, D_\gamma) \quad (5.16)$$

The inequality (5.12) results from Eq. (5.15) by letting $N = 1$, $M = +\infty$, and from the fact that $\|W_0(t)\mathbf{S}_0\|_{D_\gamma} \leqslant \|W_0\|_\infty \|\mathbf{S}_0\|_{D_\gamma}$ for any $t \in \mathbb{R}_+$, since $W_0 \in L^\infty(\mathbb{R}_+) \cap \mathscr{C}^0(\mathbb{R}_+)$. In a similar way we prove that $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, H_\theta)$.

The inequality (5.11) is a consequence of (i) in Lemma 5.2 and of the fact that $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, H_\theta)$. Finally, as $H_\theta \hookrightarrow H$ and $\lim_{t \to 0} \mathbf{u}(t) = \mathbf{u}_0$ in H (see Theorem 8.4 in [15]), $\lim_{t \to 0} \mathbf{u}(t) = \mathbf{u}_0$ in H_θ.

(b) Whenever $\theta \geqslant \omega + \gamma$, $\mathbf{u}_0 \in H_{\gamma+\omega}$ and $\mathbf{S}_0 \in D_\gamma$. The inequalities (i) and (ii) of Lemma 5.2, for $\theta' = \omega + \gamma$ and $\gamma' = \gamma$, read

$$\lambda_k^{\theta'} |\alpha_k(t)|^2 \leqslant M\left(\lambda_k^{\theta'} |\alpha_k^0|^2 + \lambda_k^{\gamma'} |b_k|^2\right)$$

$$\lambda_k^{\gamma'} \left(\lambda_k |\rho * \alpha_k|^2(t)\right) \leqslant M\left(\lambda_k^{\theta'} |\alpha_k^0|^2 + \lambda_k^{\gamma'} |b_k|^2\right)$$

The proof of (i) of Lemma 5.2 entails the uniform convergence (with respect to t on $[0, +\infty[$) of $\sum_{k=1}^{+\infty} (\rho * \alpha_k)(t)(\nabla \mathbf{w}_k + \nabla^T \mathbf{w}_k)$ in D_γ.

Therefore $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, D_\gamma)$. Moreover, as $W_0(0) = 1$, we get (see (5.2)) $\mathbf{S}(0) = \sum_{k=1}^{+\infty} (\rho * \alpha_k)(0)(\nabla \mathbf{w}_k + \nabla^T \mathbf{w}_k) + W_0(0)\mathbf{S}_0 = \mathbf{S}_0$. Arguing as in (a) above one proves that: $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, H_{\gamma+\omega})$ and $\mathbf{u}(0) = \mathbf{u}_0$. □

In the case $\gamma \geqslant 0$ and $\theta = \gamma + \omega$ we get the continuity of $(\mathbf{u}, \mathbf{S})$ at $t = 0$:

Theorem 5.1 (Existence theorem). *Let $\gamma \geqslant 0$. Assume that $\mathbf{u}_0 \in H_{\gamma+\omega}$ and $\mathbf{S}_0 \in D_\gamma$. Then the system of Eqs. (2.1)–(2.2) has at least one solution $(\mathbf{u}, \mathbf{S}) \in [\mathscr{C}^0(\mathbb{R}_+, H_{\gamma+\omega}) \cap \mathscr{C}^0(\mathbb{R}_+^*, V) \cap L^p_{\text{loc}}(\mathbb{R}_+, V)] \times \mathscr{C}^0(\mathbb{R}_+, D_\gamma)$, $p \in [1, 2/(2-\delta)[$. Moreover, $\exists A > 0$, such that $\forall t \in \mathbb{R}_+$,*

$$\left\| \mathbf{u}(t) \right\|_{H_{\gamma+\omega}} + \left\| \mathbf{S}(t) \right\|_{D_\gamma} \leqslant A\left(\|\mathbf{u}_0\|_{H_{\gamma+\omega}} + \|\mathbf{S}_0\|_{D_\gamma} \right)$$

The statement (a) of Corollary 5.1 says that, for any $0 \leqslant \gamma \leqslant \theta \leqslant \gamma + \omega$, $\mathbf{u}_0 \in H_\theta$, $\mathbf{S}_0 \in D_\gamma$, and $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+^*, H_\gamma)$. This does not ensure continuity at $t = 0$. Nevertheless, for $0 \leqslant \gamma \leqslant \theta \leqslant \gamma + \omega$, $\theta \geqslant \omega$ and still holding on the assumptions $\mathbf{u}_0 \in H_\theta$ and $\mathbf{S}_0 \in D_\gamma$, we have $\mathbf{S}_0 \in D_{\theta-\omega}$. Therefore (see Theorem 5.1) $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, D_{\theta-\omega})$, and $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, D_\theta)$.

From now on we shall focus on the case $0 \leqslant \gamma \leqslant \theta \leqslant \omega < 1$. Proceeding as previously we get:

Lemma 5.3. *Assume that $0 \leqslant \gamma \leqslant \theta \leqslant \omega \leqslant 1$, $\mathbf{u}_0 \in H_\theta$ and $\mathbf{S}_0 \in D_\gamma$. Then $\exists M > 0$, such that for any $t \geqslant 0$,*

$$\lambda_k^\theta |\alpha_k(t)|^2 \leqslant M\left(\lambda_k^\theta |\alpha_k^0|^2 + \lambda_k^{\theta-\omega} |b_k|^2\right) \quad (5.17)$$

and

$$\lambda_k^{\theta-\omega} \left(\lambda_k |\rho * \alpha_k|^2(t)\right) \leqslant M\left[\lambda_k^\theta |\alpha_k^0|^2 + \lambda_k^{\theta-\omega} |b_k|^2\right] \quad (5.18)$$

Proof. Eq. (5.17) follows from part (i) in Proposition 5.1. Next, we use part (iv) in Lemma 5.1 with $\mu = 1 - \omega$ and $\tau = 1$. It gives $\mu - 1 = -\omega$ and $\mu(2-\delta) + 2(\delta-1) = [1 - \delta/(2-\delta)](2-\delta) + 2\delta - 2 = 0$. Henceforth,

$$\lambda_k \left(|\rho * \alpha_k|^2(t)\right) \leqslant M\left(\lambda_k^\omega |\alpha_k^0|^2 + |b_k|^2\right)$$

which ends the proof. □

Hence:

Corollary 5.2. *Assume* $0 \leqslant \gamma \leqslant \theta$, $\mathbf{u}_0 \in H_\theta$ *and* $\mathbf{S}_0 \in D_\gamma$.

(a) *Whenever* $\theta \geqslant \omega$, $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, H_\theta)$, *and* $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, D_{\theta-\omega})$. *Moreover,* $\exists A > 0$ *s.t.* $\forall t \geqslant 0$:

$$\big\| \mathbf{u}(t) \big\|_{H_\theta} + \big\| \mathbf{S}(t) \big\|_{D_{\theta-\omega}} \leqslant A \big(\| \mathbf{u}_0 \|_{H_\theta} + \| \mathbf{S}_0 \|_{D_{\theta-\omega}} \big) \tag{5.19}$$

(b) *whenever* $\theta \leqslant \omega$, $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, H_\theta)$, *and* $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, \Delta_{\theta-\omega})$. *Moreover,* $\exists A > 0$ *s.t.* $\forall t \geqslant 0$:

$$\big\| \mathbf{u}(t) \big\|_{H_\theta} + \big\| \mathbf{S}(t) \big\|_{\Delta_{\theta-\omega}} \leqslant A \big(\| \mathbf{u}_0 \|_{H_\theta} + \| \mathbf{S}_0 \|_{\Delta_{\theta-\omega}} \big) \tag{5.20}$$

Proof. (a) The proof is a direct consequence of the discussion preceding Lemma 5.3.

(b) That $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, H_\theta)$ is a consequence of part (a) of Corollary 5.1. Next, Lemma 5.3 and part (e) of Lemma 4.1 imply that, for any $M \leqslant N$,

$$\left\| \sum_{k=M}^{N} (\rho * \alpha_k)(t) \big(\nabla \mathbf{w}_k + \nabla^T \mathbf{w}_k \big) \right\|_{\Delta_{\theta-\omega}}^2 = 2 \sum_{k=M}^{N} \lambda_k^{\theta-\omega} \big(\lambda_k \big| \rho \underset{(t)}{*} \alpha_k \big|^2 \big) \leqslant M \sum_{k=M}^{N} \big(\lambda_k^{\theta} |\alpha_k^0|^2 + \lambda_k^{\theta-\omega} |b_k|^2 \big) \tag{5.21}$$

Since $\mathbf{u}_0 \in H_\theta$, $\mathbf{S}_0 \in D_\gamma \hookrightarrow L^2(\Omega)^9 \hookrightarrow \Delta_{\theta-\omega}$, we get by Eq. (5.21) and Eq. (5.2) that

$$(\mathbf{S} - W_0 \otimes \mathbf{S}_0) \in \mathscr{C}^0(\mathbb{R}_+, \Delta_{\theta-\omega}) \tag{5.22}$$

Moreover, $W_0 \in \mathscr{C}^0(\mathbb{R}_+)$ and $\mathbf{S}_0 \in D_\gamma \hookrightarrow \Delta_{\theta-\omega}$. Therefore, by Eq. (5.22), $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, \Delta_{\theta-\omega})$. Inequality (5.20) follows right away after invoking Eqs. (5.17), (5.2), and (5.21) with $M = 1$ and $N = +\infty$, and that $W_0 \in L^\infty(\mathbb{R}_+)$ and $\mathbf{S}_0 \in \Delta_{\theta-\omega}$. $\quad\square$

Remark 5.1. Using part (b) in Corollary 5.2 and by a density argument one may prove that, for $\mathbf{u}_0 \in H$ and $\mathbf{S}_0 \in \Delta_{-\omega}$, the system of Eqs. (1.9) has a weak solution $(\mathbf{u}, \mathbf{S}) \in \mathscr{C}^0(\mathbb{R}_+, H) \times \mathscr{C}^0(\mathbb{R}_+, \Delta_{-\omega})$. Of course the integrals have to be replaced by inner product functionals.

Corollary 5.3. *Assume* $0 \leqslant \gamma \leqslant \theta \leqslant \omega \leqslant 1$, $\mathbf{u}_0 \in H_\theta$ *and* $\mathbf{S}_0 \in D_\gamma$. *Then* $\mathbf{u} \in \mathscr{C}^0(\mathbb{R}_+, H_\theta)$, *and* $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, H^{-1}(\Omega)^9)$. *Moreover* $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{S}(0) = \mathbf{S}_0$.

Proof. Corollary 5.2 states that $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, \Delta_{\theta-\omega})$. Therefore (see Section 4), the mapping of $\mathbb{R}_+$ into $H^{-1}(\Omega)^9$ defined by $\mathbb{R}_+ \overset{\mathbf{S}}{\to} \Delta_{\theta-\omega} \overset{i}{\hookrightarrow} \Delta_{-1} \overset{r}{\to} H^{-1}(\Omega)^9$, is continuous; the corollary statement follows right away. $\quad\square$

Hence, to the first existence and uniqueness theorem, we can add the following conclusion: $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, \Delta_{-\omega}) \hookrightarrow \mathscr{C}^0(\mathbb{R}_+, H^{-1}(\Omega)^9)$ and $\mathbf{S}(0) = \mathbf{S}_0$.

We now give a second existence and uniqueness theorem in $H_{\gamma+\omega} \times D_\gamma$ spaces.

Theorem 5.2 (*Second existence and uniqueness theorem*). *Let* $\gamma \geqslant 0$. *Assume that* $\mathbf{u}_0 \in H_{\gamma+\omega}$, $\mathbf{S}_0 \in D_\gamma$. *Then the boundary value problem given by the system of Eqs. (2.1)–(2.2) has a unique solution*

$$(\mathbf{u}, \mathbf{S}) \in \big[\mathscr{C}^0(\mathbb{R}_+, H_{\gamma+\omega}) \cap \mathscr{C}_b^0(\mathbb{R}_+^*, V) \cap L_{\mathrm{loc}}^1(\mathbb{R}_+, V) \big] \times \big[\mathscr{C}^0(\mathbb{R}_+, D_\gamma) \cap L^\infty(\mathbb{R}_+, D_\gamma) \big] \tag{5.23}$$

Moreover, there exists $A > 0$, *independent of* $\mathbf{u}$, *such that, for any* $t \geqslant 0$,

$$\big\| \mathbf{u}(t) \big\|_{H_{\gamma+\omega}} + \big\| \mathbf{S}(t) \big\|_{D_\gamma} \leqslant A \big(\| \mathbf{u}_0 \|_{H_{\gamma+\omega}} + \| \mathbf{S}_0 \|_{D_\gamma} \big) \tag{5.24}$$

Lastly, $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{S}(0) = \mathbf{S}_0$.

Proof. The solution uniqueness is a consequence of the following inclusions: $H_{\gamma+\omega} \hookrightarrow H$, $D_\gamma \hookrightarrow L^2(\Omega)^9$, $[\mathscr{C}^0(\mathbb{R}_+, H_{\gamma+\omega}) \cap \mathscr{C}_b^0(\mathbb{R}_+^*, V) \cap L_{\mathrm{loc}}^1(\mathbb{R}_+, V)] \times [\mathscr{C}^0(\mathbb{R}_+, D_\gamma) \cap L^\infty(\mathbb{R}_+, D_\gamma)] \subset \mathscr{F}$, and of Corollary 3.1.

The existence of a solution $(\mathbf{u}, \mathbf{S}) \in [\mathscr{C}^0(\mathbb{R}_+, H_{\gamma+\omega}) \cap \mathscr{C}_b^0(\mathbb{R}_+^*, V) \cap L_{\mathrm{loc}}^1(\mathbb{R}_+, V)] \times [\mathscr{C}^0(\mathbb{R}_+, D_\gamma)]$ follows from Theorem 5.1. In addition, the last estimate in Theorem 5.1 grants that $\mathbf{S} \in L^\infty(\mathbb{R}_+, D_\gamma)$. Next, that $\|\mathbf{u}(t)\|_V \underset{t \to +\infty}{\longrightarrow} 0$ was proved in the first existence and uniqueness theorem presented above. Based on this fact, we infer that $\mathbf{u} \in \mathscr{C}_b^0(\mathbb{R}_+^*, V)$, which ends the proof of solution existence.

The estimate (5.24) is a consequence of Theorem 5.1. $\quad\square$

6. The smoothness of solutions.

The following estimates will be used in proving the smoothness of solutions. They generalize those previously obtained in [15].

Proposition 6.1. $\exists M > 0$, s.t. $\forall (\tau, \chi) \in [0, 1]^2$, $\forall (x, t) \in \mathbb{R}_+$, $\forall \mu \geqslant \lambda_1$, one has:

(a)
$$\left| \lim_{A \to +\infty} \int_{-A}^{+A} T_\mu(x + iy) e^{(x+iy)t} \, dy \right| \leqslant \frac{M}{\mu^{\chi/(2-\delta)} t^\chi}$$

(b)
$$\sqrt{\mu} \left| \lim_{A \to +\infty} \int_{-A}^{+A} T_\mu w(x + iy) e^{(x+iy)t} \, dy \right| \leqslant \frac{M}{\mu^{[\omega(1-\tau)+\tau]/2} t^{\tau(1-\delta)}}$$

Proof. We only have to prove these estimates for $\chi = 0$ and $\chi = 1$, $\tau = 0$ and $\tau = 1$.

(a) The case $\chi = 0$ has already been addressed in Lemmas 7.4 and 7.2 in [15].

We now prove the case $\chi = 1$. Lemma 7.4 and inequality (70) in [15] give, for suitable $\kappa > 0$ and $B > 0$,

$$\left| \lim_{A \to +\infty} \int_{-A}^{+A} T_\mu(x + iy) e^{(x+iy)t} \, dy \right| \leqslant K \left[\int_0^{+\infty} \left| T_\mu(z e^{i\pi}) - T_\mu(z e^{-i\pi}) \right| e^{-zt} \, dz + e^{-\kappa t \mu^{1/(2-\delta)}} \right]$$

$$\leqslant K \left[\int_0^{+\infty} \frac{u^{1-\delta} + \lambda_1^{-(1-\delta)/(2-\delta)}}{(u^{2-\delta} + B)^2} e^{-ut\mu^{1/(2-\delta)}} \, du + e^{-\kappa t \mu^{1/(2-\delta)}} \right] \tag{6.1}$$

It implies that:

$$\left| \lim_{A \to +\infty} \int_{-A}^{+A} T_\mu(x + iy) e^{(x+iy)t} \, dy \right| \leqslant K \left[\int_0^{+\infty} e^{-ut\mu^{1/(2-\delta)}} \, du + \frac{1}{t\mu^{1/(2-\delta)}} \right] \leqslant \frac{A}{t\mu^{1/(2-\delta)}} \tag{6.2}$$

which gives the statement in (a) for $\chi = 1$.

(b) The case $\tau = 0$ is addressed in Lemma 5.1. The case $\tau = 1$: from (iii) in Lemma 7.5 in [15] we get

$$\sqrt{\mu} \left| \lim_{A \to +\infty} \int_{-A}^{+A} T_\mu w(x + iy) e^{(x+iy)t} \, dy \right| \leqslant \frac{K}{\mu t^{1-\delta}} + \frac{K e^{-at\mu^{1/(2-\delta)}}}{\mu^{1/2-\delta}} \leqslant \frac{M}{\mu t^{1-\delta}} \tag{6.3}$$

as $\sup_{t\mu^{1/(2-\delta)} \geqslant 0} |(t\mu^{1/(2-\delta)})^{1-\delta} e^{-at\mu^{1/(2-\delta)}}| \leqslant M < +\infty$. □

As a consequence we have the following extensions of estimates (iii) and (iv) of Theorem 5.1.

Corollary 6.1. Let $\mathbf{u}_0 \in H$, $\mathbf{S}_0 \in L^2(\Omega)^9$. Then:

(a) $\exists M > 0$, $\forall T \geqslant 0$, s.t. $\forall \gamma \in [0, \frac{1}{2-\delta}]$, $\forall \tau \in [0, 1]$, $\forall t \in [0, T]$ and $\forall k \in \mathbb{N}^*$,

$$\left| \alpha_k(t) \right|^2 \leqslant M \left(\frac{|\alpha_k^0|^2}{\lambda_k^{2\gamma} t^{2\gamma(2-\delta)}} + \frac{|b_k|^2}{\lambda_k^{\tau + \omega(1-\tau)} t^{2\tau(1-\delta)}} \right)$$

(b) $\exists M > 0$, $\forall T \geqslant 0$, s.t. $\forall \gamma \in [0, 1]$, $\forall \tau \in [0, \frac{1}{2-\delta}]$, $\forall t \in [0, T]$ and $\forall k \in \mathbb{N}^*$,

$$\lambda_k |\rho * \alpha_k|^2(t) \leqslant M \left(\frac{|\alpha_k^0|^2}{\lambda_k^{2\gamma-1} t^{2[\gamma(2-\delta)+\delta-1]}} + \frac{|b_k|^2}{\lambda_k^{-(1-\tau)(1-\omega)} t^{-2(1-\tau)(1-\delta)}} \right)$$

Proof. (a) The statement in (a) follows from (5.1) and Proposition 6.1 (with $\gamma = \frac{\chi}{2-\delta}$).

(b) The statement follows from (a) above by convolution. □

We deduce from Corollary 6.1:

Proposition 6.2. *Let $\gamma \geqslant 0$, $\eta > 0$. Then*:

(a) *Assume $\mathbf{u}_0 \in H_{1+\gamma+\omega}$ and $\mathbf{S}_0 \in D_{1+\gamma}$. Then $\mathbf{u} \in \mathscr{C}^1(\mathbb{R}_+, H_\gamma)$.*
(b) *Assume $\mathbf{u}_0 \in H_{3+\gamma-\omega+\eta}$ and $\mathbf{S}_0 \in D_{3+\gamma}$. Then $\mathbf{u} \in \mathscr{C}^2(\mathbb{R}_+^*, H_\gamma) \cap W_{\mathrm{loc}}^{2,p}(\mathbb{R}_+, H_\gamma)$, $p \in [1, 1/(1-\alpha)[$.*

Proof. (a) Based on Eq. (3.11a) and $W_0 \in L^\infty(\mathbb{R}_+)$, we infer that:

$$\lambda_k^\gamma \left| \alpha_k'(t) \right|^2 \leqslant \lambda_k^{1+\gamma} \left(\lambda_k |\rho * \alpha_k|^2(t) \right) + \lambda_k^{1+\gamma} |b_k|^2 \|W_0\|_\infty^2 \tag{6.4}$$

Now, Lemma 5.2 with $\theta = \gamma + \omega$, leads to

$$\lambda_k^{1+\gamma} \left(\lambda_k |\rho * \alpha_k|^2(t) \right) \leqslant A \lambda_k \left(\lambda_k^{\gamma+\omega} |\alpha_k^0|^2 + \lambda_k^\gamma |b_k|^2 \right) \tag{6.5}$$

From Eqs. (6.4)–(6.5),

$$\lambda_k^\gamma \left| \alpha_k'(t) \right|^2 \leqslant A \left(\lambda_k^{1+\gamma+\omega} |\alpha_k^0|^2 + \lambda_k^{1+\gamma} |b_k|^2 \right) \tag{6.6}$$

Since $\mathbf{u}_0 \in H_{1+\gamma+\omega}$ and $\mathbf{S}_0 \in D_{1+\gamma}$, $\lambda_k^\gamma |\alpha_k'(t)|^2 < +\infty$. Hence $\sum_{k=1}^{+\infty} \alpha_k' \otimes \mathbf{w}_k$ converges in $\mathscr{C}^0(\mathbb{R}_+, H_\gamma)$. Since by Lemma 5.2 $\sum_{k=1}^{+\infty} \alpha_k \otimes \mathbf{w}_k$ converges in $\mathscr{C}^0(\mathbb{R}_+, H_{1+\gamma})$, it also converges in $\mathscr{C}^0(\mathbb{R}_+, H_\gamma)$. Finally $\mathbf{u} \in \mathscr{C}^1(\mathbb{R}_+, H_\gamma)$.

(b) Observe that $1 + \gamma + \omega \leqslant 3 + \gamma - \omega + \eta$ and $1 + \gamma \leqslant 3 + \gamma$. Consequently $\mathbf{u}_0 \in H_{1+\gamma+\omega}$, $\mathbf{S}_0 \in H_{1+\gamma}$. Next, (a) above ensures that $\mathbf{u} \in \mathscr{C}^1(\mathbb{R}_+, H_\gamma)$. We now deduce several estimates for the second order derivatives. From Eq. (3.11a) it follows that, for $t > 0$, $\alpha_k''(t) = -\lambda_k(\rho * \alpha_k')(t) - \sqrt{\lambda_k} b_k W_0'(t) = \lambda_k^2(\rho * \rho * \alpha_k)(t) + \lambda_k^{3/2} b_k(\rho * W_0)(t) - \sqrt{\lambda_k} b_k W_0'(t) \in \mathscr{C}^0(\mathbb{R}_+^*)$ since $\alpha_k \in \mathscr{C}^0(\mathbb{R}_+)$, $W_0 \in \mathscr{C}^0(\mathbb{R}_+) \cap \mathscr{C}^1(\mathbb{R}_+^*)$ and $0 \leqslant \rho(t) \leqslant kt^{-\delta}$.

Let now $\epsilon > 0$ be small enough. Part (a) in Corollary 6.1 with $\gamma = (1-\epsilon)/(2-\delta) \in [0, \frac{1}{2-\delta}]$, $\tau = 1$ and $|W_0'(t)| \leqslant \frac{k}{t^{1-\alpha}}$ (see [15]), gives

$$\lambda_k^{\gamma/2} \left| \alpha_k''(t) \right| \leqslant M_T \lambda_k^{2+\gamma/2} \left[t^{1-2\delta} * \left(\frac{|\alpha_k^0|}{\lambda_k^{(1-\epsilon)/(2-\delta)} t^{1-\epsilon}} + \frac{|b_k|}{\lambda_k^{1/2} t^{1-\delta}} \right) \right] + M_T \lambda_k^{(1+\gamma)/2} \left(\frac{\lambda_k}{t^{\delta-1}} + \frac{1}{t^{1-\alpha}} \right) |b_k|$$

$$\leqslant M_T \left(\frac{\lambda_k^{(3+\gamma)/2}}{t^{\delta-1}} + \frac{\lambda_k^{(1+\gamma)/2}}{t^{1-\alpha}} \right) |b_k| + M_T \frac{\lambda_k^{2-(1-\epsilon)/(2-\delta)+\gamma/2}}{t^{2\delta-\epsilon-1}} |\alpha_k^0| \tag{6.7}$$

Observe that, for $\epsilon > 0$ small enough, $2(3+\gamma)/2 \leqslant 3 + \gamma$ and $2(\gamma+1)/2 \leqslant 3 + \gamma$. Also, for $\epsilon > 0$ small enough, $2[2 + \gamma/2 - (1-\epsilon)/(2-\delta)] \leqslant 4 + \gamma - 2/(2-\delta) + \eta$.

Hence, by (6.7), and since $\mathbf{u}_0 \in H_{3+\gamma-\omega+\eta} = H_{4+\gamma-2/(2-\delta)+\eta}$ and $\mathbf{S}_0 \in D_{3+\gamma}$, we get $\mathbf{u} \in \mathscr{C}^2(\mathbb{R}_+^*, H_\gamma)$. $\quad\square$

Proceeding as before (see Proposition 6.2) one obtains the following smoothness properties:

Proposition 6.3. *Let $\gamma \geqslant 0$, $\eta > 0$.*

(a) *Assume $\mathbf{u}_0 \in H_{2+\gamma-\omega+\eta}$, $\mathbf{S}_0 \in D_{2+\gamma}$. Then $\mathbf{S} \in \mathscr{C}^1(\mathbb{R}_+^*, D_\gamma) \cap W_{\mathrm{loc}}^{1,p}(\mathbb{R}_+, D_\gamma)$, for any $p \in [1, 1/(1-\alpha)[$.*
(b) *Assume $\mathbf{u}_0 \in H_{4+\gamma-\omega+\eta}$, $\mathbf{S}_0 \in D_{4+\gamma}$. Then $\mathbf{S} \in \mathscr{C}^2(\mathbb{R}_+^*, D_\gamma)$.*

Proof. (a) We limit the proof to the case $0 < \eta < \omega$. Denote $\tilde{\omega} = \omega - \eta$; hence $0 < \tilde{\omega} < \omega < 1$.

Since $\mathbf{u}_0 \in H_{2+\gamma-\tilde{\omega}}$, $\mathbf{S}_0 \in D_{2+\gamma-\tilde{\omega}-\omega}$, based on Theorem 5.1, we have that $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, D_{2+\gamma-\tilde{\omega}-\omega})$. Next, as $0 < \tilde{\omega} < \omega < 1$ entails $\gamma < 2 + \gamma - \tilde{\omega} - \omega$, one gets $\mathbf{S} \in \mathscr{C}^0(\mathbb{R}_+, D_\gamma)$.

Next we obtain an estimate for $\mathbf{S}'(t)$. Observe first that $W_0 \in L^\infty(\mathbb{R}_+)$ and that $\alpha_k'(t) = -\lambda_k(\rho * \alpha_k)(t) - \sqrt{\lambda_k} b_k W_0(t)$. Therefore:

$$\lambda_k^{\gamma/2} \sqrt{\lambda_k} |\rho * \alpha_k'| \leqslant A \lambda_k^{(\gamma+1)/2} \left[\lambda_k \left(\rho * \rho * |\alpha_k| \right) + \sqrt{\lambda_k} \left(\rho * |b_k| \right) \right](t)$$

Now, part (a) of Corollary 6.1, with $\tilde{\delta} = 2\tilde{\omega}/(1+\tilde{\omega}) < 2\omega/(1+\omega) = \delta$, $\gamma = 1/(2-\tilde{\delta})$, $\tilde{\delta} \in]0, \delta[$, and $\tau = 1$, gives

$$\left| \alpha_k(t) \right| \leqslant M_T \left(\frac{|\alpha_k^0|}{\lambda_k^{1/(2-\tilde{\delta})} t^{(2-\delta)/(2-\tilde{\delta})}} + \frac{|b_k|}{\sqrt{\lambda_k} t^{1-\delta}} \right)$$

However, $|\rho * \rho|(t) \leqslant K t^{1-2\delta}$. Also, since $0 < \tilde{\delta} < \delta$, then $(2-\delta)/(2-\tilde{\delta}) \in [0, 1[$. One infers that

$$\lambda_k^{\gamma/2} \sqrt{\lambda_k} |\rho * \alpha_k'|(t) \leqslant M_T \lambda_k^{1+\gamma/2} \left[\sqrt{\lambda_k} \left(\frac{1}{t^{2\delta-1}} * \frac{|\alpha_k^0|}{\lambda_k^{1/(2-\tilde{\delta})} t^{(2-\delta)/(2-\tilde{\delta})}} \right) + \sqrt{\lambda_k} \left(\frac{1}{t^{2\delta-1}} * \frac{|b_k|}{\sqrt{\lambda_k} t^{1-\delta}} \right) + \frac{|b_k|}{t^{\delta-1}} \right]$$

$$\leqslant M_T \lambda_k^{1+\gamma/2} \left(\lambda_k^{-\tilde{\omega}/2} t^{2-2\delta-(2-\delta)/(2-\tilde{\delta})} |\alpha_k^0| + t^{1-\delta} |b_k| \right) \tag{6.8}$$

Let $a = 2 - 2\delta - (2-\delta)/(2-\tilde{\delta}) = 1 - 2\delta + (\delta - \tilde{\delta})/(2-\tilde{\delta})$, and $b = 1 - \delta > 0$. Then:

$$\left|\lambda_k^{\gamma/2}\sqrt{\lambda_k}(\rho * \alpha_k')\right|^2(t) \leqslant M_T\left[\lambda_k^{2+\gamma-\tilde{\omega}}|\alpha_k^0|^2 t^{2a} + \lambda_k^{2+\gamma}t^{2b}|b_k|^2\right] \tag{6.9}$$

Recall that – as stated in (c) of Lemma 4.1 – that $(\nabla\mathbf{w}_k + \nabla^T\mathbf{w}_k)_{k\in\mathbb{N}^*}$ is an orthogonal sequence of functions that belongs to D_γ, and $\|\nabla\mathbf{w}_k + \nabla^T\mathbf{w}_k\|_{D_\gamma}^2 = (1 + \lambda_k^\gamma)\lambda_k$. Consequently, using the estimate given above and that $\mathbf{u}_0 \in H_{2+\gamma-\tilde{\omega}}$, $\mathbf{S}_0 \in D_{2+\gamma}$, leads to the fact that $\sum_{k=1}^{+\infty}(\rho * \alpha_k') \otimes (\nabla\mathbf{w}_k + \nabla^T\mathbf{w}_k)$ converges in $\mathscr{C}^0(\mathbb{R}_+^*, D_\gamma)$. Next, from equation in [15], $W_0' \otimes \mathbf{S}_0 \in \mathscr{C}^0(\mathbb{R}_+^*, D_\gamma)$. Therefore $\sum_{k=1}^{+\infty}(\rho * \alpha_k') \otimes (\nabla\mathbf{w}_k + \nabla^T\mathbf{w}_k) + W_0' \otimes \mathbf{S}_0 \in \mathscr{C}^0(\mathbb{R}_+^*, D_\gamma)$. Hence $\mathbf{S} \in \mathscr{C}^1(\mathbb{R}_+^*, D_\gamma)$.

Whenever $a \geqslant 0$, by Eq. (6.9), $\sum_{k=1}^{+\infty}(\rho * \alpha_k') \otimes (\nabla\mathbf{w}_k + \nabla^T\mathbf{w}_k) + W_0' \otimes \mathbf{S}_0$ belongs to $\mathscr{C}^0(\mathbb{R}_+, D_\gamma)$, thus belongs to $L_{\text{loc}}^p(\mathbb{R}_+, D_\gamma)$ for any $1 \leqslant p < +\infty$.

Now, whenever $a < 0$, $-a - (1-\alpha) = 2\delta - 2 - (\delta - \tilde{\delta})/(2-\delta) - 1 + \alpha = [-3 - (\delta - \tilde{\delta})/(2-\delta) + (2\delta + \alpha)] < 0$. We conclude that $\sum_{k=1}^{+\infty}(\rho * \alpha_k') \otimes (\nabla\mathbf{w}_k + \nabla^T\mathbf{w}_k)$ converges in $L_{\text{loc}}^q(\mathbb{R}_+, D_\gamma)$, for any $q \in [1, 1/(1-\alpha)]$. Moreover, from [15] we observe that $\|W_0'(t)\mathbf{S}_0\|_{D_\gamma} \leqslant K/t^{1-\alpha}$. It implies that $W_0' \otimes \mathbf{S}_0 \in L_{\text{loc}}^q(\mathbb{R}_+, D_\gamma)$ for any $q \in [1, 1/(1-\alpha)[$. Therefore

$$\sum_{k=1}^{+\infty}(\rho * \alpha_k') \otimes (\nabla\mathbf{w}_k + \nabla^T\mathbf{w}_k) + W_0' \otimes \mathbf{S}_0 \in L_{\text{loc}}^q(\mathbb{R}_+, D_\gamma)$$

for any $q \in [1, 1/(1-\alpha)[$ and irrespective of whether a is positive or negative.

Eventually $\mathbf{S} \in W_{\text{loc}}^{1,q}(\mathbb{R}_+, D_\gamma)$ for any $q \in [1, 1/(1-\alpha)[q \in [1, 1/(1-\alpha)[$.

(b) The proof is omitted. $\square$

From Proposition 6.2 we can infer the existence of smooth solutions to Eqs. (1.9). Assume that $\mathbf{u}_0 \in H_{5+\omega}$ and $\mathbf{S}_0 \in D_5 \cap \mathscr{C}^1(\overline{\Omega})^9$. Then, the solution $(\mathbf{u}, \mathbf{S})$ the existence of which is granted by Theorem 3.1 of Section 3, complies with the statement (a) of Proposition 6.2, that is $\mathbf{u} \in \mathscr{C}^1(\mathbb{R}_+, H_4)$. Since $H_4 \hookrightarrow H^4(\Omega) \hookrightarrow \mathscr{C}^2(\overline{\Omega})$ (see Section 4 and by Sobolev's injection), one has $\mathbf{u} \in \mathscr{C}^1(\mathbb{R}_+, \mathscr{C}^2(\overline{\Omega})^3)$ and $(\nabla\mathbf{u} + \nabla^T\mathbf{u}) \in \mathscr{C}^1(\mathbb{R}_+, \mathscr{C}^1(\overline{\Omega})^9)$. One also has $\mathbf{S}_0 \in \mathscr{C}^1(\overline{\Omega})^9$, $W_0 \in \mathscr{C}^0(\mathbb{R}_+) \cap \mathscr{C}^1(\mathbb{R}_+^*)$ and $\rho \in L_{\text{loc}}^1(\mathbb{R}_+)$. Consequently $\mathbf{S} = \rho * (\nabla\mathbf{u} + \nabla^T\mathbf{u}) + W_0 \otimes \mathbf{S}_0 \in \mathscr{C}^0(\mathbb{R}_+, \mathscr{C}^1(\overline{\Omega})^9) \cap \mathscr{C}^1(\mathbb{R}_+^*, \mathscr{C}^1(\overline{\Omega})^9)$. All the precedent arguments eventually lead to the conclusion that $(\mathbf{u}, \mathbf{S}) \in \mathscr{C}^1(\mathbb{R}_+, \mathscr{C}^2(\overline{\Omega})^3) \times [\mathscr{C}^0(\mathbb{R}_+, \mathscr{C}^1(\overline{\Omega})^9) \cap \mathscr{C}^1(\mathbb{R}_+^*, \mathscr{C}^1(\overline{\Omega})^9)]$, whenever $\mathbf{u}_0 \in H_{5+\omega}$ and $\mathbf{S}_0 \in D_5 \cap \mathscr{C}^1(\overline{\Omega})^9$.

7. Final comments

Fractional calculus has a long history that parallels the classical analysis [27,35,38]. It has long been used in modeling natural phenomena: for a quick glimpse see for example [1–7,9–11,13,14,20–22,24,26,29,35,37,39,43,47–49], and references cited therein. In particular, fractional derivative CEs have been found to accurately predict stress relaxation of viscoelastic fluids in the glass transition and glassy (high frequency) states.

The results presented here enrich and complement the linear stability analysis within the framework of variational/weak solutions initiated in [15]. We have proved results regarding existence, uniqueness, smoothness and continuity at $t = 0$ of the solution to the initial boundary value problem stated in Section 1. Moreover, this work is related to that of Shaw, Whiteman and co-workers on the well-posedness, existence and uniqueness of weak solutions for similar in nature hereditary-type integral models (see for example [18,40–42]), as well as to that reported in [33,34,45].

The matter of the stability of the original nonlinear CE is an open question on which future work shall focus.

Acknowledgments

References

[1] K. Adolfsson, M. Enelund, Fractional derivative viscoelasticity at large deformations, Nonlinear Dynam. 33 (2003) 301–321.
[2] K. Adolfsson, Nonlinear fractional order viscoelasticity at large strains, Nonlinear Dynam. 38 (2004) 233–246.
[3] K. Adolfsson, M. Enelund, P. Olsson, On the fractional order model of viscoelasticity, Mech. Time-Depend. Mater. 9 (2005) 15–34.
[4] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Appl. Math. 109 (2010) 973–1033.
[5] O.P. Agrawall, Solution for a fractional diffusion-wave equation defined in a bounded domain, Nonlinear Dynam. 29 (2002) 145–155.
[6] D. Baleanu, S.I. Muslih, K. Tas, Fractional Hamiltonian analysis of higher order derivatives systems, J. Math. Phys. 47 (2006) 103503.
[7] A. Carpinteri, F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, New York, 1997.
[8] R. Dautray, J.-L. Lions, Analyse mathématique et calcul numérique pour les sciences et les techniques, vol. 8, Evolution: semi-groupe, variationnel, INSTN CEA Collect. Enseign., Masson, Paris, 1988.
[9] J.F. Douglas, Some applications of fractional calculus to polymer science, Adv. Chem. Phys. 102 (1997) 121.
[10] J.F. Douglas, Polymer science applications of path-integration, integral equations, and fractional calculus, in: R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[11] A.D. Drozdov, Fractional differential models in finite viscoelasticity, Acta Mech. 124 (1997) 155–180.

[12] C. Friedrich, Relaxation functions of rheological constitutive equations with fractional derivatives: Thermodynamical constraints, in: J. Casas-Vasquez, D. Jou (Eds.), Lecture Notes in Phys., vol. 381, Springer-Verlag, New York, 1991.

[13] D. Guyomar, B. Ducharne, G. Sébald, Time fractional derivatives for voltage creep in ferroelectric materials: theory and experiment, J. Phys. D: Appl. Phys. 41 (2008) 125410.

[14] A. Hanyga, Fractional-order relaxation laws in non-linear viscoelasticity, Contin. Mech. Thermodyn. 19 (2007) 25–36.

[15] A. Heibig, L.I. Palade, On the rest state stability of an objective fractional derivative viscoelastic fluid model, J. Math. Phys. 49 (2008) 043101.

[16] R.R. Huilgol, N. Phan-Thien, Fluid Mechanics of Viscoelasticity, Elsevier, Amsterdam, 1997.

[17] D.D. Joseph, Fluid Dynamics of Viscoelastic Fluids, Springer-Verlag, Berlin, 1990.

[18] M. Karamanou, M.K. Warby, J.R. Whiteman, Computational modelling of thermoforming processes in the case of finite viscoelastic materials, Comput. Methods Appl. Mech. Engrg. 195 (2006) 5220.

[19] H.H. Kausch, N. Heymans, C.J. Plummer, P. Decroly, Matériaux Polymères: Propriétés Mécaniques et Physiques, Presses Polytechniques et Universitaires Romandes, Lausanne, 2001, p. 195.

[20] G.R. Kneller, K. Hinsen, Fractional Brownian dynamics in proteins, J. Chem. Phys. 121 (2004) 10278–10283.

[21] R.C. Koeller, A theory relating creep and relaxation for linear materials with memory, Trans. ASME J. Appl. Mech. 77 (2010) 031008.

[22] A. Lion, On the thermodynamics of fractional damping elements, Contin. Mech. Thermodyn. 9 (1997) 83.

[23] J.-L. Lions, E. Magenes, Problèmes aux Limites Non-Homogènes et Applications, vol. 1, Dunod, Paris, 1968.

[24] K. Logvinova, M.C. Neel, A fractional equation for anomalous diffusion in a randomly heterogeneous porous medium, Chaos 14 (2004) 982–987.

[25] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity. An Introduction to Mathematical Models, Imperial College Press, London, 2010.

[26] N. Makris, Three-dimensional constitutive viscoelastic laws with fractional order time derivatives, J. Rheol. 41 (1997) 1007.

[27] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, J. Wiley & Sons, New York, 1993.

[28] F.A. Morrison, Understanding Rheology, Oxford University Press, 2001.

[29] N. Özdemir, D. Karadeniz, Fractional diffusion-wave problem in cylindrical coordinates, Phys. Lett. A 372 (2008) 5968–5972.

[30] L.I. Palade, V. Verney, P. Attané, Time–temperature superposition and linear viscoelasticity of polybutadienes, Macromol. 28 (1995) 7051–7057.

[31] L.I. Palade, V. Verney, P. Attané, A modified fractional model to describe the entire viscoelastic behavior of polybutadienes from flow to glassy regime, Rheol. Acta 35 (1996) 265.

[32] L.I. Palade, P. Attané, R.R. Huilgol, B. Mena, Anomalous stability behavior of a properly invariant constitutive equation which generalises fractional derivative models, Internat. J. Engrg. Sci. 37 (1999) 315.

[33] L.I. Palade, J.R. Walton, A. Farina, A new constitutive equation that models extensional flow strain hardening based on evolving natural configurations: stability analysis, Internat. J. Non-Linear Mech. 39 (2004) 379–387.

[34] L.I. Palade, An integral constitutive law for viscoelastic fluids based on the concept of evolving natural configurations: stability analysis, Internat. J. Non-Linear Mech. 39 (2004) 1275–1287.

[35] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Academic Press, New York, 1999.

[36] C.G. Robertson, L.I. Palade, Unified application of the coupling model to segmental, Rouse, and terminal dynamics of entangled polymers, J. Non-Cryst. Solids 352 (2006) 342–348.

[37] O. Robles-Vasquez, A. Gonzalez-Alvarez, J.E. Puig, O. Manero, A composition rule to predict the linear viscoelastic properties of polybutadienes with varying microstructure, Rubber Chem. Technol. 79 (2006) 859–869.

[38] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, New York, 1993.

[39] M. Seredyńska, A. Hanyga, Nonlinear differential equations with fractional damping with applications to the 1dof and 2dof pendulum, Acta Mech. 176 (2005) 169–183.

[40] S. Shaw, J.R. Whiteman, A posteriori error estimates for space-time finite element approximation of quasistatic hereditary linear viscoelasticity problems, Comput. Methods Appl. Mech. Engrg. 193 (2004) 5551.

[41] S. Shaw, J.R. Whiteman, Adaptive space-time finite element solution for Volterra equations arising in viscoelasticity problems, J. Comput. Appl. Math. 125 (2000) 337.

[42] S. Shaw, J.R. Whiteman, Applications and numerical analysis of partial differential Volterra equations: A brief survey, Comput. Methods Appl. Mech. Engrg. 150 (1997) 397.

[43] W.C. Tan, W.X. Pan, M.Y. Xu, A note on unsteady flows of a viscoelastic fluid with the fractional Maxwell model between two parallel plates, Internat. J. Non-Linear Mech. 38 (2003) 645–650.

[44] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Appl. Math. Sci., vol. 68, Springer-Verlag, New York, 1997.

[45] E. Voyiatzis, C.J. Tsenoglu, A.G. Boudouvis, On Hadamard stability and dissipative stability of the molecular stress function model of non-linear viscoelasticity, Internat. J. Non-Linear Mech. 44 (2009) 727–734.

[46] S. Cleja-Ţigoiu, V. Ţigoiu, Rheology and Thermodynamics, Part I – Rheology, Editura Universităţii din Bucureşti, 1998.

[47] H. Xu, S.-J. Liao, X.-C. You, Analysis of nonlinear fractional partial differential equations with the homotopy analysis method, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 1152–1156.

[48] D. Yang, K.Q. Zhu, Start-up flow of a viscoelastic fluid in a pipe with a fractional Maxwell's model, Comput. Math. Appl. 60 (2010) 2231.

[49] Y.B. Yin, K.Q. Zhu, Oscillating flow of a viscoelastic fluid in a pipe with the fractional Maxwell model, Appl. Math. Comput. 173 (2006) 231–242.