J. Math. Anal. Appl. 356 (2009) 322–327

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Generalized Kato decomposition, single-valued extension property and approximate point spectrum

Qiaofen Jiang [*,1], Huaijie Zhong [1]

School of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, China

A R T I C L E I N F O

Article history:
Received 19 September 2008
Available online 10 March 2009
Submitted by M. Putinar

Keywords:
Banach space
Generalized Kato decomposition
Single-valued extension property

A B S T R A C T

In this paper, we define the generalized Kato spectrum of an operator, and obtain that the generalized Kato spectrum differs from the semi-regular spectrum on at most countably many points. We study the localized version of the single-valued extension property at the points which are not limit points of the approximate point spectrum, as well as of the surjectivity spectrum. In particular, we shall characterize the single-valued extension property at a point $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ admits a generalized Kato decomposition. From this characterization we shall deduce several results on cluster points of some distinguished parts of the spectrum.

1. Introduction

Let $B(X)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex Banach space X. For $T \in B(X)$, let $\sigma(T)$ denote the spectrum of T and $\rho(T) := \mathbb{C} \setminus \sigma(T)$ be the resolvent set of T. $R(T) = T(X)$ denotes the range of T, and $\ker T = N(T)$ denotes the kernel of T.

Two important subspaces are the *hyperrange* of T, defined by $T^\infty(X) = \bigcap_{n=1}^{\infty} R(T^n)$, and the *hyperkernel* of T defined by $N^\infty(T) = \bigcup_{n=1}^{\infty} \ker(T^n)$. The algebraic core $C(T)$ of T is defined to be the greatest subspace M of X for which $T(M) = M$.

Associated with $T \in B(X)$, there are another two linear subspaces of X, the analytical core $K(T)$ of T defined by

$$K(T) := \{x \in X\colon \text{ there exist a sequence } \{x_n\}_{n\geqslant 1} \text{ in } X \text{ and a constant } \delta > 0 \text{ such that } Tx_1 = x,$$
$$Tx_{n+1} = x_n \text{ and } \|x_n\| \leqslant \delta^n \|x\| \text{ for all } n \in \mathbb{N}\},$$

and the quasi-nilpotent part $H_0(T)$ of T defined by

$$H_0(T) := \left\{x \in X\colon \lim_{n\to\infty} \left\|T^n x\right\|^{1/n} = 0\right\}.$$

The reduced minimum modulus of a non-zero operator T is defined to be $\gamma(T) := \inf_{x \notin \ker T} \frac{\|Tx\|}{dist(x, \ker T)}$ (for $T = 0$ we define formally $\gamma(T) = \infty$). Note that $\gamma(T) = \gamma(T^*)$ for every $T \in B(X)$ where T^* denotes the dual of T; $\gamma(T) > 0$ if and only if $R(T)$ is closed [1, Theorem 1.23].

The lattice of invariant subspaces of an operator T is denoted as $Lat(T)$. A pair of closed subspaces (M, N) is said to reduce T (denoted as $(M, N) \in Red(T)$), if $X = M \oplus N$ and $M, N \in Lat(T)$. For $M \in Lat(T)$, $T|_M$ denotes the restriction of T to M. $T \in B(X)$ is said to be semi-regular if $T(X)$ is closed and $\ker T \subset R(T^n)$ for all $n \in \mathbb{N}$. T is said to admit a generalized

* Corresponding author.
E-mail address: bj001_ren@163.com (Q. Jiang).
1 Supported by National Natural Science Foundation of China 10771034 and Foundation of the Education Department of Fujian Province JB07047.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.03.017

Kato decomposition, abbreviated as GKD, if there exists $(M, N) \in Red(T)$ such that $T|_M$ is semi-regular and $T|_N$ is quasi-nilpotent. If we assume in the definition above that $T|_N$ is nilpotent, then there exists $d \in \mathbb{N}$ for which $(T|_N)^d = 0$. In this case T is said to be of Kato type of order d. An operator is said to be essentially semi-regular if it admits a GKD(M, N) such that N is finite-dimensional.

For every bounded operator $T \in B(X)$, let us define the semi-regular spectrum, the essentially semi-regular spectrum, the Kato type spectrum and the generalized Kato spectrum as follows respectively:

$$\sigma_{se}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \text{ is not semi-regular}\};$$

$$\sigma_{es}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \text{ is not essentially semi-regular}\};$$

$$\sigma_{k}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \text{ is not of Kato type}\};$$

$$\sigma_{gk}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \text{ does not admit a generalized Kato decomposition}\}.$$

And define the semi-regular resolvent set, the essentially semi-regular resolvent set, the Kato type resolvent set and the generalized Kato resolvent set as follows respectively: $\rho_{se}(T) = \mathbb{C} \setminus \sigma_{se}(T)$; $\rho_{es}(T) = \mathbb{C} \setminus \sigma_{es}(T)$; $\rho_k(T) = \mathbb{C} \setminus \sigma_k(T)$; $\rho_{gk}(T) = \mathbb{C} \setminus \sigma_{gk}(T)$.

Evidently $\sigma_{gk}(T) \subset \sigma_k(T) \subset \sigma_{es}(T) \subset \sigma_{se}(T)$.

A very detailed and far-reaching account of these notations can be seen in [1,7]. Discussions of operators which admit a generalized decomposition may be found in [8,9,11].

Recall that $T \in B(X)$ is bounded below if T is injective and has closed range $T(X)$. T is said to be Fredholm if $\dim \mathit{ker}\, T < \infty$ and $\operatorname{codim} R(T) < \infty$.

The approximate point spectrum is defined by

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \text{ is not bounded below}\},$$

the essential spectrum is defined by

$$\sigma_{e}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \text{ is not Fredholm operator}\},$$

the surjectivity spectrum is defined by

$$\sigma_{su}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \text{ is not surjective}\}.$$

By the closed range theorem we easily know that the approximate point spectrum and the surjectivity spectrum are dual to each other, in the sense that $\sigma_{su}(T) = \sigma_{ap}(T^*)$ and $\sigma_{ap}(T) = \sigma_{su}(T^*)$.

An operator $T \in B(X)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (*SVEP at* λ_0 *for brevity*), if for every neighborhood $\mathcal{U}$ of λ_0 the only analytic function $f : \mathcal{U} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ is the function $f(\lambda) \equiv 0$.

An operator $T \in B(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

Trivially, an operator $T \in B(X)$ has the SVEP at every point of the resolvent set $\rho(T)$. Moreover, from the identity theorem for analytical functions and $\sigma(T) = \sigma(T^*)$ it easily follows that T and T^* have the SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T and T^* have the SVEP at every isolated point of the spectrum. Hence, we have the implication

$$\sigma(T) \text{ does not cluster at } \lambda_0 \Rightarrow T \text{ and } T^* \text{ have the SVEP at } \lambda_0. \tag{1}$$

Also by means of the identity theorem for analytical functions we have

$$\sigma_{ap}(T) \text{ does not cluster at } \lambda_0 \Rightarrow T \text{ has the SVEP at } \lambda_0 \tag{2}$$

and dually

$$\sigma_{su}(T) \text{ does not cluster at } \lambda_0 \Rightarrow T^* \text{ has the SVEP at } \lambda_0. \tag{3}$$

Note that none of the implications (1)–(3) may be reversed. Indeed, $\partial\sigma(T)$ is contained in $\sigma_{ap}(T)$, as well as in $\sigma_{su}(T)$, see [7, Proposition 3.1.6]. Consequently, if λ_0 is a non-isolated boundary point of $\sigma(T)$ then $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ cluster at λ_0, but as observed before, T and T^* have the SVEP at λ_0. An example of operator T having the SVEP and such that every spectral point is limit of points of $\sigma_{ap}(T)$ may be found among the unilateral weighted right shift operators. Indeed, there exist unilateral weighted right shift operators T on $l^p(\mathbb{N})$ for which $\sigma(T) = \sigma_{ap}(T) = \sigma_{su}(T)$ and $\sigma(T)$ is a closed ball centered at 0 with radius $r > 0$, see the remarks after Theorem 6.1 of [2].

P. Aiena and E. Rosas give characterizations of the SVEP at λ_0 in the case that $\lambda_0 I - T$ is of Kato type. Precisely, they show that if $\lambda_0 I - T$ is of Kato type then the implications (1)–(3) can be reversed [5].

The basic role of SVEP arises in local spectral theory, since a decomposable operator T enjoys this property, as well as its dual T^* [1, Theorem 6.22]. The notion of the localized SVEP at a point dates back to Finch [6] and it has been pursued further in the more recent papers [3,4,10,13]. In particular, it has been shown that if $\lambda_0 I - T$ admits a generalized Kato decomposition then the SVEP at a point $\lambda_0 \in \mathbb{C}$ is equivalent to a variety of conditions that involve some kernel-type and range-type subspaces of $\lambda_0 I - T$, as the analytical core and quasi-nilpotent part [1,3].

In this paper we shall give further characterizations of the SVEP at λ_0, always in the case that $\lambda_0 I - T$ admits a generalized Kato decomposition. Precisely, we shall see that if $\lambda_0 I - T$ admits a generalized Kato decomposition then T has the SVEP at λ_0 if and only if $\sigma_{ap}(T)$ does not cluster at λ_0. A dual result shows that, always if $\lambda_0 I - T$ admits a generalized Kato decomposition, T^* has the SVEP at λ_0 precisely when $\sigma_{su}(T)$ does not cluster at λ_0. That is, the implications (1)–(3) can be reversed in the case that $\lambda_0 I - T$ admits a generalized Kato decomposition. As consequence we shall deduce several results on cluster points of distinguished parts of the spectrum. These results are applied to some concrete cases, as unilateral weighted right shift operators on $l^p(\mathbb{N})$.

If T is of Kato type, then T admits a generalized Kato decomposition. So our results extend works of P. Aiena and E. Rosas in [5].

We also show that the generalized Kato spectrum of an operator is closed, and obtain that the generalized Kato spectrum differs from the semi-regular spectrum on at most countably many isolated points (Corollary 2.3).

2. Generalized Kato spectrum

Using the property $K(T) = T^\infty(X)$ of an operator which is of Kato type [1, Theorem 1.44, Corollary 1.45] shows that the Kato type spectrum of an operator is closed and differs from the semi-regular spectrum on at most countably many isolated points.

But for an operator which admits a generalized Kato decomposition, the property $K(T) = T^\infty(X)$ does not necessarily hold.

For example, let T denote the Volterra operator on the Banach space $X = C[0,1]$ defined by

$$(Tf)(t) := \int_0^t f(s)\,ds \quad \text{for all } f \in C[0,1] \text{ and } t \in [0,1].$$

T is injective and quasi-nilpotent. Consequently $K(T) = \{0\}$. It is easy to check that $T^\infty(X) = \{f \in C^\infty[0,1]\colon\ f^{(n)}(0) = 0,\ n \in \mathbb{N}\}$, thus $T^\infty(X)$ is not closed and hence is strictly larger than $K(T)$. Hence $K(T) \neq T^\infty(X)$.

Using rather direct technique different from [1], we extend the results to operators which admit a generalized Kato decomposition.

Proposition 2.1. *(See [12].) For $(M,N) \in Red(T)$, T is semi-regular if and only if both $T|_M$, $T|_N$ are semi-regular.*

Denote the open disc centered at λ_0 with radius ε in $\mathbb{C}$ by $D(\lambda_0, \varepsilon)$. For the set E, let $\partial E, \overline{E}$ denote the boundary and the closure of E, respectively.

Theorem 2.2. *Suppose that $T \in B(X)$ admits a GKD(M,N). Then there exists a constant $\varepsilon > 0$ (if $M = \{0\}$, take ε to be any constant number $C > 0$; if $M \neq \{0\}$, take $\varepsilon = \gamma(T|_M)$), such that for all $\lambda \in D(0,\varepsilon) \setminus \{0\}$, $\lambda I - T$ is semi-regular.*

Proof. (1) If $M = \{0\}$, precisely, T is quasi-nilpotent, then for all $0 \neq \lambda \in \mathbb{C}, \lambda I - T$ is invertible, obviously $\lambda I - T$ is semi-regular.

(2) If $M \neq \{0\}$, for the GKD(M,N) of T, $T = \begin{pmatrix} T|_M & 0 \\ 0 & T|_N \end{pmatrix}$, so $\lambda I - T = \begin{pmatrix} (\lambda I - T)|_M & 0 \\ 0 & (\lambda I - T)|_N \end{pmatrix}$. Since $T|_M$ is semi-regular, then $R(T|_M)$ is closed and we have $\gamma(T|_M) > 0$. We also know that for $|\lambda| < \gamma(T|_M)$, $(\lambda - T)|_M$ is semi-regular [1, Theorem 1.31]. As $T|_N$ is quasi-nilpotent, for all $0 \neq \lambda \in \mathbb{C}$, we know that $(\lambda I - T)|_N$ is invertible, then $(\lambda I - T)|_N$ is semi-regular. By Proposition 2.1, for all $\lambda \in D(0, \gamma(T|_M)) \setminus \{0\}$, $\lambda I - T$ is semi-regular. □

According to Theorem 2.2, it follows that the generalized Kato spectrum $\sigma_{gk}(T)$ is closed.

Since $\sigma_{gk}(T) \subset \sigma_k(T) \subset \sigma_{es}(T) \subset \sigma_{se}(T)$, as a straightforward consequence of Theorem 2.2, we easily obtain that these spectra differ from each other on at most countably many isolated points.

Corollary 2.3. *$\sigma_{gk}(T)$ is compact subset of $\mathbb{C}$. Moreover $\sigma_{se}(T) \setminus \sigma_{gk}(T)$, $\sigma_{es}(T) \setminus \sigma_{gk}(T)$, $\sigma_k(T) \setminus \sigma_{gk}(T)$ consist of at most countably many isolated points.*

Proof. Obviously, $\rho(T) \subset \rho_{gk}(T)$. By Theorem 2.2, $\rho_{gk}(T) = \mathbb{C} \setminus \sigma_{gk}(T)$ is open. Then $\sigma_{gk}(T)$ are compact.

Furthermore, if $\lambda_0 \in \sigma_{se}(T) \setminus \sigma_{gk}(T)$ then $\lambda_0 I - T$ admits a GKD(M,N). By Theorem 2.2 there exists $\varepsilon > 0$ such that for all $\lambda \in D(\lambda_0, \varepsilon) \setminus \{0\}$, $\lambda I - T$ is semi-regular. Hence λ_0 is an isolated point of $\sigma_{se}(T)$. From this it follows that $\sigma_{se}(T) \setminus \sigma_{gk}(T)$ consists of at most countably many isolated points.

Since $\sigma_{gk}(T) \subset \sigma_k(T) \subset \sigma_{es}(T) \subset \sigma_{se}(T)$, this corollary can be completed by the method analogous to that used above. □

$\sigma_{gk}(T)$ is not necessarily non-empty. For example, the quasi-nilpotent operator T has empty generalized Kato spectrum.

3. SVEP and approximate point spectrum

If $\lambda_0 I - T$ admits a generalized Kato decomposition then the SVEP at a point $\lambda_0 \in \mathbb{C}$ is equivalent to a variety of conditions that involve some kernel-type and range-type subspaces of $\lambda_0 I - T$, as the analytical core and quasi-nilpotent part [1,3].

Proposition 3.1. *(See [1].) Suppose that $\lambda_0 I - T \in B(X)$ admits a* $\mathrm{GKD}(M, N)$. *Then the following assertions are equivalent*:

(1) *T has the SVEP at λ_0*;
(2) *$T|_M$ has the SVEP at λ_0*;
(3) *$(\lambda_0 I - T)|_M$ is injective*;
(4) $H_0(\lambda_0 I - T) = N$;
(5) *$H_0(\lambda_0 I - T)$ is closed*;
(6) $H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$.

Proposition 3.2. *(See [1].) Suppose that $\lambda_0 I - T$ admits a* $\mathrm{GKD}(M, N)$. *Then the following assertions are equivalent*:

(1) *T^* has the SVEP at λ_0*;
(2) *$(\lambda_0 I - T)|_M$ is surjective*;
(3) $K(\lambda_0 I - T) = M$;
(4) $X = H_0(\lambda_0 I - T) + K(\lambda_0 I - T)$.

In this section, we give further characterizations of the SVEP at λ_0 by means of the approximate point spectrum in the case that $\lambda_0 I - T$ admits a generalized Kato decomposition.

P. Aiena and E. Rosas also give characterizations of the SVEP at λ_0 by means of approximate point spectrum in the case that $\lambda_0 I - T$ is of Kato decomposition. Precisely, if $\lambda_0 I - T$ is of Kato type, T has the SVEP at λ_0 if and only if $\sigma_{ap}(T)$ does not cluster at λ_0 [5]. Note that they use the closeness of $T^\infty(X)$ in the proof. As we know, for an operator which admits a generalized Kato decomposition, it does not necessarily hold. For example, let T be the Volterra operator defined in the second section.

In fact, we extend the results of P. Aiena and E. Rosas in [5] to the operators which admit a generalized Kato decomposition. But our method of proof is rather direct and different.

Suppose that $(M, N) \in Red(T)$, consider the relation of the property of bounded below among T and $T|_M, T|_N$:

Lemma 3.3. *Suppose that $(M, N) \in Red(T)$. Then T is bounded below if and only if both $T|_M$ and $T|_N$ are bounded below.*

Proof. First we show that $R(T)$ is closed if and only if both $R(T|_M), R(T|_N)$ are closed. Let P be the projection of X onto M along N.

Suppose that $(T|_M)(x_n) \in R(T|_M) \subset M$, $T(x_n) = (T|_M)(x_n) \to y$. If $R(T)$ is closed, then there exists $x = m + r$, $x \in X$, $m \in M$, $r \in N$ such that $y = Tx = T(m + r) = Tm + Tr$. It follows that $y = Py = P(Tm + Tr) = Tm = (T|_M)m$, that is, $y \in R(T|_M)$, so $R(T|_M)$ is closed. The closeness of $R(T|_N)$ can be proved in a similar way.

On the other hand, if both $R(T|_M), R(T|_N)$ are closed, suppose $x_n = y_n + z_n$, $y_n \in M$, $z_n \in N$, $T(x_n) \to x$, then $PT(x_n) = TP(x_n) = (T|_M)(P(x_n)) = (T|_M)(y_n) \to Px$, $(I - P)T(x_n) = (T|_N)(z_n) \to (I - P)x$. Since $R(T|_M), R(T|_N)$ are closed, there exist $y \in M$, $z \in N$ such that $(T|_M)y = Px$, $(T|_N)z = (I - P)x$, so $x = (T|_M)y + (T|_N)z = T(y + z) \in R(T)$, that is, $R(T)$ is closed.

It is easy to verify that T is injective if and only both $T|_M, T|_N$ are injective by simple calculation.

Thus we have derived that T is bounded below if and only if both $T|_M$ and $T|_N$ are bounded below. □

Proposition 3.4. *(See [1].) Suppose that $T \in B(X)$, $(M, N) \in Red(T)$. Then T has the SVEP at* 0 *if and only if both $T|_M, T|_N$ have the SVEP at* 0.

Theorem 3.5. *Suppose that $\lambda_0 I - T \in B(X)$ admits a* $\mathrm{GKD}(M, N)$. *Then the following statements are equivalent*:

(1) *T has the SVEP at λ_0*;
(2) *$\sigma_{ap}(T)$ does not cluster at λ_0*;
(3) *λ_0 is not an interior point of $\sigma_{ap}(T)$.*

Proof. Assume that $\lambda_0 = 0$.

The implication (2) ⇒ (3) is obvious. According to the identity theorem for analytical functions, the implication (3) ⇒ (1) is clear.

We need only to prove the implication (1) ⇒ (2).

Since T admits a GKD(M,N), then $T|_M$ is semi-regular and $T|_N$ is quasi-nilpotent. According to Proposition 3.4, as T has the SVEP at 0, it follows that $T|_M$ has the SVEP at 0, hence $T|_M$ is bounded below[1, Theorem 2.49]. Thus there exists $\varepsilon > 0$ such that for all $|\lambda| < \varepsilon$, $(\lambda I - T)|_M$ is bounded below [1, Lemma 1.30]. As $T|_N$ is quasi-nilpotent, then for all $\lambda \neq 0$, $(\lambda I - T)|_N$ is bounded below. According to Lemma 3.3, it follows that for $\lambda \in D(0,\varepsilon) \setminus \{0\}$, $\lambda I - T = \begin{pmatrix} (\lambda I-T)|_M & 0 \\ 0 & (\lambda I-T)|_N \end{pmatrix}$ is bounded below, that is, $\sigma_{ap}(T)$ does not cluster at 0. □

Theorem 3.5 generalizes [5, Theorem 2.2] to operators which admit a generalized Kato decomposition. Moreover, we give rather direct proof.

This theorem also generalizes [3, Theorem 2.14]: Suppose that $\lambda_0 I - T$ admits a generalized Kato decomposition, then T does not have the SVEP at λ_0 if and only if λ_0 is a limit point of $\sigma_p(T)$.

Observe that $\sigma_p(T) \subset \sigma_{ap}(T)$. If $\sigma_{ap}(T)$ does not cluster at λ_0, then $\sigma_p(T)$ does not cluster at λ_0. On the other hand, if $\sigma_p(T)$ does not cluster at λ_0, it is not necessary that $\sigma_{ap}(T)$ does not cluster at λ_0. Let us consider the unilateral shift T. For all $\lambda \in \partial D(0,1)$, $\sigma_p(T)$ does not cluster at λ, but λ is a cluster point of $\sigma_{ap}(T)$. According to Theorem 3.5, we can see that $\lambda I - T$ does not admit a generalized Kato decomposition for all $\lambda \in \partial D(0,1)$.

For a bounded operator $T \in B(X)$, let us consider the following parts of the spectrum: $\sigma_{se}(T)$, $\sigma_k(T)$ and $\sigma_{gk}(T)$. It is known that the three sets are closed, for the first set see [7, Proposition 3.1.9], for the second set see [1, Corollary 1.45], and the third set see Corollary 2.3 in this paper.

Clearly $\sigma_{gk}(T) \subset \sigma_{ap}(T)$ and $\sigma_{gk}(T) \subset \sigma_{su}(T)$. The result of Theorem 3.5 is quite useful for establishing the membership of cluster points of some distinguished parts of the spectrum to the spectrum $\sigma_{gk}(T)$. A first application is given from the following result, which improves a classical Putnam theorem about the non-isolated boundary points of the spectrum as a subset of the Fredholm spectrum.

Corollary 3.6. *For every $T \in B(X)$, every non-isolated boundary point of $\sigma(T)$ belongs to $\sigma_{gk}(T)$. In particular, every non-isolated boundary point of $\sigma(T)$ belongs to the Fredholm spectrum $\sigma_e(T)$.*

Proof. Assume $\lambda_0 \in \partial\sigma(T)$, λ_0 is non-isolated in $\partial\sigma(T)$, then T has the SVEP at λ_0. If $\lambda_0 I - T$ admits a generalized Kato decomposition, $\sigma_{ap}(T)$ does not cluster at λ_0 according to Theorem 3.5. Since $\partial\sigma(T) \subset \sigma_{ap}(T)$, then $\sigma_{ap}(T)$ cluster at λ_0. This leads to a contradiction. So $\lambda_0 \in \sigma_{gk}(T) \subset \sigma_e(T)$. □

Corollary 3.7. *Suppose that $T \in B(X)$ has the SVEP. Then all cluster points of $\sigma_{ap}(T)$ belong to $\sigma_{gk}(T)$.*

Proof. Suppose that $\lambda_0 \notin \sigma_{gk}(T)$. Since T has the SVEP, and in particular has the SVEP at λ_0. Then $\sigma_{ap}(T)$ does not cluster at λ_0 by Theorem 3.5. □

The next result gives a clear description of the points $\lambda_0 \notin \sigma_{gk}(T)$ which belong to the boundary of $\sigma(T)$.

Theorem 3.8. *Let $T \in B(X)$, and suppose that $\lambda_0 \in \partial\sigma(T)$. Then $\lambda_0 I - T$ admits a generalized Kato decomposition if and only if λ_0 is an isolated point of $\sigma(T)$.*

Proof. For $\lambda_0 \in \partial\sigma(T)$, if λ_0 is non-isolated in $\sigma(T)$, then λ_0 is non-isolated in $\partial\sigma(T)$. According to Corollary 3.6, $\lambda_0 \in \sigma_{gk}(T)$, this leads to a contradiction. So λ_0 is an isolated point of $\sigma(T)$.

On the other hand, suppose that λ_0 is an isolated point of $\sigma(T)$, then $X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$, $(\lambda_0 I - T)(H_0(\lambda_0 I - T)) \subset H_0(\lambda_0 I - T)$ and $(\lambda_0 I - T)(K(\lambda_0 I - T)) = K(\lambda_0 I - T)$. Hence $(\lambda_0 I - T)|_{H_0(\lambda_0 I - T)}$ is quasi-nilpotent, $(\lambda_0 I - T)|_{K(\lambda_0 I - T)}$ is surjective. Moreover, $(\lambda_0 I - T)|_{K(\lambda_0 I - T)}$ is semi-regular. Thus $\lambda_0 I - T$ admits a generalized Kato decomposition. □

The next result is dual, in a sense, to Theorem 3.5.

Theorem 3.9. *Suppose that $\lambda_0 I - T$ admits a* GKD(M,N), *then the following statements are equivalent*:

(1) *T^* has the SVEP at λ_0*;
(2) *$\sigma_{su}(T)$ does not cluster at λ_0*;
(3) *λ_0 is not an interior point of $\sigma_{su}(T)$.*

Proof. Suppose that $\lambda_0 I - T$ admits a generalized Kato decomposition, then $(\lambda_0 I - T)^*$ also admits a generalized Kato decomposition [1, Theorem 1.43]. Observe that $\sigma_{ap}(T^*) = \sigma_{su}(T)$, and the equivalences immediately follow from Theorem 3.5. □

Corollary 3.10. *Let $T \in B(X)$, suppose that T^* has the SVEP. Then all cluster points of $\sigma_{su}(T)$ belong to $\sigma_{gk}(T)$.*

Proof. Suppose that $\lambda_0 \notin \sigma_{gk}(T)$. Since T^* has the SVEP at λ_0, then $\sigma_{su}(T)$ does not cluster at λ_0 by Theorem 3.9. □

According to Theorems 3.5 and 3.9, we can easily obtain that:

Theorem 3.11. *Suppose that $T \in B(X)$, $\lambda_0 \in \sigma(T)$, T and T^* have the SVEP at λ_0. Then*

$$\lambda_0 I - T \text{ admits a generalized Kato decomposition} \quad \Leftrightarrow \quad \lambda_0 \text{ is isolated in } \sigma(T).$$

Proof. If $\lambda_0 I - T$ admits a generalized Kato decomposition, by Theorems 3.5 and 3.9, there exist $r_1, r_2 > 0$ such that $D(\lambda_0, r_1) \setminus \{\lambda_0\} \subset \mathbb{C} \setminus \sigma_{ap}(T)$, $D(\lambda_0, r_2) \setminus \{\lambda_0\} \subset \mathbb{C} \setminus \sigma_{su}(T)$. Let $r = \min\{r_1, r_2\}$, then $D(\lambda_0, r) \setminus \{\lambda_0\} \subset \rho(T)$, so λ_0 is isolated in $\sigma(T)$.

Using the same argument as in the proof of Theorem 3.8, we can easily carry out the proof of this theorem. □

In particular, suppose that T is a Riesz operator which has infinite points in $\sigma(T)$, then $\sigma_{gk}(T) = \{0\}$.

Since T and T^* have the SVEP at $\lambda_0 \in \partial\sigma(T)$, we can easily obtain Theorem 3.8 according to Theorem 3.11.

All results established above have a number of interesting applications.In the next theorem we consider a situation which occurs in some concrete cases.

Theorem 3.12. *Let $T \in B(X)$ be an operator for which $\sigma_{ap}(T) = \partial\sigma(T)$, and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$, then $\sigma_{ap}(T) = \sigma_{se}(T) = \sigma_{es}(T) = \sigma_k(T) = \sigma_{gk}(T)$.*

Proof. Since $\lambda \in \partial\sigma(T)$ is non-isolated, according to Corollary 3.6, $\sigma_{ap}(T) = \partial\sigma(T) \subset \sigma_{gk}(T) \subset \sigma_k(T) \subset \sigma_{es}(T) \subset \sigma_{se}(T) \subset \sigma_{ap}(T)$, that is, $\sigma_{ap}(T) = \sigma_{se}(T) = \sigma_{es}(T) = \sigma_k(T) = \sigma_{gk}(T)$. □

Dually,

Corollary 3.13. *Suppose that T in $B(X)$ is an operator for which $\sigma_{su}(T) = \partial\sigma(T)$, and every $\lambda \in \partial\sigma(T)$is non-isolated in $\sigma(T)$. Then $\sigma_{su}(T) = \sigma_{se}(T) = \sigma_{es}(T) = \sigma_k(T) = \sigma_{gk}(T)$.*

Proof. Suppose that $\lambda_0 \in \sigma_{su}(T) = \partial\sigma(T)$, since $\lambda_0 \in \partial\sigma(T)$ is non-isolated, then $\sigma_{su}(T)$ cluster in λ_0. Observe that T^* has the SVEP at $\lambda_0 \in \partial\sigma(T)$, then $\lambda_0 I - T$ does not admit a generalized Kato decomposition by Theorem 3.9, that is, $\lambda_0 \in \sigma_{gk}(T)$. So $\sigma_{su}(T) = \partial\sigma(T) \subset \sigma_{gk}(T) \subset \sigma_k(T) \subset \sigma_{es}(T) \subset \sigma_{se}(T) \subset \sigma_{su}(T)$. Thus we have $\sigma_{su}(T) = \sigma_{se}(T) = \sigma_{es}(T) = \sigma_k(T) = \sigma_{gk}(T)$. □

Theorem 3.5 applies to the unilateral weighted right shift operators defined on $l^p(\mathbb{N})$.

Example 3.14. Let T be the unilateral weighted right shift operator defined on $l^p(\mathbb{N})(1 \leqslant p < \infty)$ with weight (ω_n).

Suppose that $c(T) = \lim_{n\to\infty} inf\,(\omega_1\omega_2\cdots\omega_n)^{\frac{1}{n}} = 0$, and T has closed range. Thus $r(T) > 0$ and T has the SVEP. Moreover, $\sigma(T) = \sigma_{ap}(T) = \sigma_{su}(T) = \overline{D(0, r(T))}$ [1, Corollary 3.118].

Then $\sigma_{ap}(T)$ clusters at every point in $\sigma(T)$. By Theorem 3.5, $\lambda I - T$ does not admit a generalized Kato decomposition. That means $\sigma(T) = \sigma_{gk}(T)$.

Acknowledgments

The authors are grateful to the referees for helpful comments concerning this paper.

References

[1] P. Aiena, Fredholm and Local Spectral Theory, with Applications to Multipliers, Kluwer Academic Pub., 2004.
[2] P. Aiena, T. Miller, M. Neumann, On a localised single-valued extension property, Proc. R. Ir. Acad. 104 (1) (2004) 17–34.
[3] P. Aiena, O. Monsalve, Operators which do not have the single-valued extension property, J. Math. Anal. Appl. 250 (2) (2000) 435–448.
[4] P. Aiena, O. Monsalve, The single valued extension property and the generalized Kato decomposition property, Acta Sci. Math. (Szeged) 67 (46) (2001) 1–77.
[5] P. Aiena, E. Rosas, Single-valued extension property at the points of the approximate point spectrum, J. Math. Anal. Appl. 279 (1) (2003) 180–188.
[6] J. Finch, The single-valued extension property on a Banach space, Pacific J. Math. 58 (1) (1975) 61–69.
[7] K. Laursen, M. Neumann, An Introduction to Local Spectral Theory, Oxford Univ. Press, 2000.
[8] M.Ó. Searcóid, T.T. West, Continuity of the generalized kernel and range of semi-Fredholm operators, Math. Proc. Cambridge Philos. Soc. 105 (1989) 513–522.
[9] M. Mbekhta, Généralisation de la décomposition de kato aux opérateurs paranormaux et spectraux, Glasgow Math. J. 29 (1987) 159–175.
[10] M. Mbekhta, Sur la théorie spectrale locale et limite des nilpotents, Proc. Amer. Math. Soc. 110 (1990) 621–631.
[11] M. Mbekhta, Sur l'unicité de la dcomposition de kato généralisée, Acta Sci. Math. (Szeged) 54 (1990) 367–377.
[12] M. Mbekhta, V. Muller, On the axiomatic theory of spectrum ii, Studia Math. 119 (1996) 129–147.
[13] C. Schmoeger, Semi-Fredholm operators and local spectral theory, Demonstratio Math. 28 (1995) 997–1004.