Common fixed point theorems of Gregus type for weakly compatible mappings satisfying generalized contractive conditions

A. Aliouche

Department of Mathematics, University of Larbi Ben M’Hidi, Oum-El-Bouaghi 04000, Algeria

Received 24 May 2007
Available online 7 November 2007
Submitted by T.D. Benavides

Abstract


© 2007 Elsevier Inc. All rights reserved.

Keywords: Weakly compatible mappings; Common fixed point; Metric space; Property (E.A); Common property (E.A); Generalized contractive condition

1. Introduction

Let $S$ and $T$ be self-mappings of a metric space $(X, d)$. $S$ and $T$ are commuting if $STx = TSx$ for all $x \in X$.

Sessa [21] defined $S$ and $T$ to be weakly commuting if for all $x \in X$

$$d(STx, TSx) \leq d(Tx, Sx).$$

Jungck [8] defined $S$ and $T$ to be compatible as a generalization of weakly commuting if

$$\lim_{n \to \infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

E-mail address: alioumath@yahoo.fr.
It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper, see [8] and [21].

Jungck et al. [9] defined $S$ and $T$ to be compatible mappings of type (A) if
\[
\lim_{n \to \infty} d(STx_n, T^2x_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(TSx_n, S^2x_n) = 0,
\]
whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

Examples are given to show that the two concepts of compatibility are independent, see [9].

Recently, Pathak and Khan [14] defined $S$ and $T$ to be compatible mappings of type (B) as a generalization of compatible mappings of type (A) if
\[
\lim_{n \to \infty} d(TSx_n, S^2x_n) = \frac{1}{2}\left[\lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, T^2x_n)\right]
\quad \text{and}
\lim_{n \to \infty} d(STx_n, T^2x_n) = \frac{1}{2}\left[\lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, S^2x_n)\right],
\]
whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true, see [14]. However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if $S$ and $T$ are continuous, see [14].

Pathak et al. [15] defined $S$ and $T$ to be compatible mappings of type (P) if
\[
\lim_{n \to \infty} d(S^2x_n, T^2x_n) = 0,
\]
whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if $S$ and $T$ are continuous, see [15].

Pathak et al. [16] defined $S$ and $T$ to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if
\[
\lim_{n \to \infty} d(TSx_n, S^2x_n) = \frac{1}{3}\left[\lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, S^2x_n) + \lim_{n \to \infty} d(Tt, T^2x_n)\right]
\quad \text{and}
\lim_{n \to \infty} d(STx_n, T^2x_n) = \frac{1}{3}\left[\lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, T^2x_n) + \lim_{n \to \infty} d(St, S^2x_n)\right],
\]
whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

Compatibility, compatibility of type (A) and compatibility of type (C) are equivalent if $S$ and $T$ are continuous, see [16].

2. Preliminaries

**Definition 2.1.** (See [10].) $S$ and $T$ are said to be weakly compatible if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

**Lemma 2.2.** (See [8,9,14–16].) If $S$ and $T$ are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then they are weakly compatible.

It was shown in [6] that the converse is not true in general.

**Definition 2.3.** (See [13].) $S$ and $T$ are said to be $R$-weakly commuting if there exists an $R > 0$ such that
\[
d(STx, TSx) \leq Rd(Tx, Sx) \quad \text{for all } x \in X.
\quad (2.1)
\]

**Definition 2.4.** (See [13].) $S$ and $T$ are pointwise $R$-weakly commuting if for all $x \in X$, there exists an $R > 0$ such that (2.1) holds.
It was proved in [13] that $R$-weak commutativity is equivalent to commutativity at coincidence points; i.e., $S$ and $T$ are pointwise $R$-weakly commuting if and only if they are weakly compatible.

**Definition 2.5.** (See [1].) Let $S, T : X \to X$. The pair $(S, T)$ satisfies property (E.A) if there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \in X. \quad (2.2)$$

It is clear from the definition of compatibility that the pair $(S, T)$ of a metric space $(X, d)$ is noncompatible if there exists at least one sequence $\{x_n\}$ in $X$ such that (2.2) holds but, $\lim_{n \to \infty} d(STx_n, TSx_n)$ is either nonzero or does not exist. Therefore, two noncompatible mappings of a metric space $(X, d)$ satisfy property (E.A).

**Definition 2.6.** (See [12].) Let $A, S, B, T : X \to X$. The pairs $(A, S)$ and $(B, T)$ satisfy a common property (E.A) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = t \in X. \quad (2.3)$$

If $B = A$ and $T = S$ in (2.3), we obtain the definition of property (E.A).

Several authors have proved fixed point theorems for mappings satisfying contractive conditions of integral type. See [2–4,6,19,20,23]. Recently, Zhang [24] proved a common fixed point theorem using a new generalized contractive condition for a pair of self-mappings in a metric space. This theorem extend results in [4,19] and [20].

Let $A \in (0, \infty]$, $R_A^+ = [0, A)$ and $F : R_A^+ \to \mathbb{R}$ satisfying

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A)$,

(ii) $F$ is nondecreasing on $R_A^+$,

(iii) $F$ is continuous.

Define $F [0, A) = \{F : F$ satisfies (i)–(iii)$\}.$

The following examples were given in [24].

(1) Let $F(t) = t$, then $F \in F [0, A)$ for each $A \in (0, +\infty]$.

(2) Suppose that $\varphi$ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^\epsilon \varphi(t) \, dt > 0 \quad \text{for each } \epsilon \in (0, A).$$

Let $F(t) = \int_0^t \varphi(s) \, ds$, then $F \in [0, A)$.

(3) Suppose that $\psi$ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^\epsilon \psi(t) \, dt > 0 \quad \text{for each } \epsilon \in (0, A)$$

and $\varphi$ is nonnegative, Lebesgue integrable on $[0, \int_0^A \psi(s) \, ds)$ and satisfies

$$\int_0^\epsilon \varphi(t) \, dt > 0 \quad \text{for each } \epsilon \in \left(0, \int_0^A \psi(s) \, ds\right).$$

Let $F(t) = \int_0^t \varphi(u) \, du$, then $F \in F [0, A)$.
Lemma 2.7. (See [24].) Let $A \in (0, +\infty]$, $F \in F[0, A]$. If $\lim_{n \to \infty} F(\epsilon_n) = 0$ for $\epsilon_n \in R_A^+$, then $\lim_{n \to \infty} \epsilon_n = 0$.

Let $A \in (0, +\infty]$, $\psi : R_A^+ \to R_+$ satisfying

(i) $\psi(t) < t$ for each $t \in (0, A)$,

(ii) $\psi$ is nondecreasing and upper semi-continuous.

Define $\Psi[0, A] = \{\psi : \psi$ satisfies (i) and (ii) above$\}$.

Lemma 2.8. (See [24].) If $\psi \in \Psi[0, A]$, then $\psi(0) = 0$.

Lemma 2.9. (See [22].) For any $t \in (0, A)$, $\psi(t) < t$ iff $\lim_{n \to +\infty} \psi^n(t) = 0$, where $\psi^n$ denotes the $n$-times repeated composition of $\psi$ with itself.

Lemma 2.10. (See [17].) If $\psi_i \in \Psi(0, A)$ for all $i \in I$, where $I$ is a finite indexing set, then there exists some $\psi \in \Psi$ such that:

$$\max\{\psi_i(t), \ i \in I\} \leq \psi(t) \quad \text{for all} \ t > 0.$$

Theorem 2.11. (See [7].) Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $T$ be a mapping from $C$ into itself satisfying the inequality

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$$

for all $x, y$ in, where $a > 0, b, c \geq 0, a + b + c = 1$. Then, $T$ has a unique fixed point.

Several authors have generalized Theorem 2.11, see [5,6,11,14,16].

In [5], there is a problem in the proof of $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$ if $n$ is odd. In fact, assume that $y_n \neq y_{n+1}$ for all $n$. Applying the following inequality

$$d^p(Ax, By) \leq \psi\left[ad^p(Sx, Ty) + (1 - a)\max\{d(Ax, Sx), d(By, Ty), (d(Ax, Sx))^\frac{1}{2}, (d(Ax, Ty))^\frac{1}{2}\}\right]$$

for $x = x_{2n+2}$ and $y = x_{2n+1}$, we have

$$d^p(Ax_{2n+2}, By_{2n+1}) = d^p(y_{2n+1}, y_{2n+2})$$

$$\leq \psi\left(ad^p(y_{2n}, y_{2n+1})
+ (1 - a)\max\{d^p(y_{2n}, y_{2n+1}), d^p(y_{2n+1}, y_{2n+2})
\right.$$}

$$\left.d^\frac{p}{2} (y_{2n+1}, y_{2n+2} \cdot d^\frac{p}{2} (y_{2n}, y_{2n+2})\right).$$

Therefore

$$d^p(y_{2n+1}, y_{2n+2}) \leq \psi\left(ad^p(y_{2n+1}, y_{2n+2})
+ (1 - a)\max\{d^p(y_{2n+1}, y_{2n+2}), d^p(y_{2n}, y_{2n+1})
\right.$$}

$$\left.[d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}) + d(y_{2n+1}, y_{2n+2})]^\frac{p}{2}\right).$$
If \( d(y_{2n}, y_{2n+1}) \leq d(y_{2n+1}, y_{2n+2}) \) in the above inequality, then

\[
d^p(y_{2n+1}, y_{2n+2}) \leq \psi(ad^p(y_{2n+1}, y_{2n+2}) + (1 - a) \max\{d^p(y_{2n+1}, y_{2n+2}), 2\frac{e}{a}d^p(y_{2n+1}, y_{2n+2})\}).
\]

Since \( 2\frac{e}{a}d^p(y_{2n+1}, y_{2n+2}) > d^p(y_{2n+1}, y_{2n+2}) \), it follows that

\[
d^p(y_{2n+1}, y_{2n+2}) \leq \psi((a + 2\frac{e}{a} (1 - a))d^p(y_{2n+1}, y_{2n+2}))
= \psi((a + 2\frac{e}{a} (1 - a))d^p(y_{2n+1}, y_{2n+2}))
< (a + 2\frac{e}{a} (1 - a))d^p(y_{2n+1}, y_{2n+2}).
\]

As \( a + 2\frac{e}{a} (1 - a) \geq 1 \), we cannot get a contradiction. Therefore, the term \( d^p(Ax, Sx) \cdot d^p(Ax, Ty) \) should be replaced by

\[
\min\{d^p(Ax, Sx) \cdot d^p(Ax, Ty), d^p(By, Ty) \cdot d^p(By, By)\}.
\]

Then, in [6], the term \( \left(\int_0^{d(Ax, Sx)} \psi(t) dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Ty)} \psi(t) dt\right)^{\frac{1}{2}} \) should be replaced by

\[
\min\left\{\left(\int_0^{d(Ax, Sx)} \psi(t) dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Ty)} \psi(t) dt\right)^{\frac{1}{2}}, \left(\int_0^{d(By, Ty)} \psi(t) dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(By, By)} \psi(t) dt\right)^{\frac{1}{2}}\right\}
\]

and so in the revised paper, the term \( \left(\int_0^{d(Ax, Sx)} \psi(t) dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Ty)} \psi(t) dt\right)^{\frac{1}{2}} \) should be replaced by

\[
\min\{\left(\int_0^{d(Ax, Sx)} \psi(t) dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Ty)} \psi(t) dt\right)^{\frac{1}{2}}, \left(\int_0^{d(By, Ty)} \psi(t) dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(By, By)} \psi(t) dt\right)^{\frac{1}{2}}\}.
\]

The same problem appears in [14] with the term \( \frac{d^p(Ax, Ty) + d^p(By, By)}{2} \) if \( n \) is even and \( n \) is odd and so this term should be deleted or replaced by \( \min\{d^p(Ax, Sx) \cdot d^p(Ax, Ty), d^p(By, Ty) \cdot d^p(By, By)\} \).

### 3. Main results

Let \( D = \sup\{d(x, y): x, y \in X\} \). Set \( A = D \) if \( D = \infty \) and \( A > D \) if \( D < \infty \). Let \( A, B, S \) and \( T \) be mappings from a metric space \( (X, d) \) into itself satisfying

\[
A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X),
\]
(3.1)

\[
(F(d(Ax, By)))^p \leq \psi(F(d(Sx, Ty)))^p + (1 - a) \max\{F(d(Ax, Sx)), F(d(By, Ty))\}
\]
\[
\min\{\left(F(d(Ax, Sx))\right)^{\frac{1}{2}} \cdot \left(F(d(Ax, Ty))\right)^{\frac{1}{2}}, \left(F(d(By, Ty))\right)^{\frac{1}{2}} \cdot \left(F(d(By, By))\right)^{\frac{1}{2}}\},
\]
\[
(F(d(Sx, By)))^p \leq \psi(F(d(Ax, Ty)))^p.
\]
(3.2)

for all \( x, y \) in \( X \), where \( 0 \leq a \leq 1, \ p \geq 1, \ F \in F[0, A] \) and \( \psi \in \Psi[0, F(A - 0)] \).

By (3.1), we can define inductively a sequence \( \{y_n\} \) in \( X \) such that

\[
y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}
\]
(3.3)

for all \( n = 0, 1, 2, \ldots \).

**Lemma 3.1.** Let \( A, B, S \) and \( T \) be mappings from a metric space \( (X, d) \) into itself satisfying (3.1) and (3.2). Then, the sequence \( \{y_n\} \) defined by (3.3) is a Cauchy sequence in \( X \).
Proof. First of all, assume that \(y_n \neq y_{n+1}\) for all \(n\). Applying (3.2) and (3.3) we have
\[
(F(d(y_{2n}, y_{2n+1})))^p = (F(d(Ax_{2n}, Bx_{2n+1})))^p \\
\leq \psi \bigg[ a \bigg( F(d(y_{2n-1}, y_{2n})) \bigg)^p + (1 - a) \max \bigg\{ F(d(y_{2n-1}, y_{2n})), F(d(y_{2n}, y_{2n+1})) \bigg\}^p \bigg].
\] (3.4)

If \(F(d(y_{2n-1}, y_{2n})) \leq F(d(y_{2n}, y_{2n+1}))\) in (3.4), then
\[
(F(d(y_{2n}, y_{2n+1})))^p \leq \psi \bigg( (F(d(y_{2n}, y_{2n+1})))^p \bigg) < (F(d(y_{2n}, y_{2n+1})))^p
\]
which is a contradiction. Therefore
\[
(F(d(y_{2n}, y_{2n+1})))^p \leq \psi \bigg( (F(d(y_{2n-1}, y_{2n}))^p \bigg).
\]

Similarly, we get
\[
(F(d(y_{2n+1}, y_{2n+2})))^p \leq \psi \bigg( (F(d(y_{2n}, y_{2n+1}))^p \bigg).
\]

By induction, we obtain
\[
(F(d(y_n, y_{n+1})))^p \leq \psi \bigg( (F(d(y_{n-1}, y_n)))^p \bigg) \leq \cdots \leq \psi^n \bigg( (F(d(y_0, y_1)))^p \bigg).
\]

Using Lemma 2.9, it follows that
\[
\lim_{n \to \infty} F(d(y_n, y_{n+1})) = 0,
\] (3.5)
and Lemma 2.7 implies that
\[
\lim_{n \to \infty} d(y_n, y_{n+1}) = 0.
\] (3.6)

Now, we show that \(\{y_n\}\) is a Cauchy sequence in \(X\). By (3.6), it suffices to show that the subsequence \(\{y_{2n}\}\) of \(\{y_n\}\) is a Cauchy sequence in \(X\). Suppose not. As in [5] we have
\[
d(y_{2n(k)}, y_{2m(k)}) \to \varepsilon \quad \text{as} \quad k \to \infty,
\] (3.7)
\[
d(y_{2n(k)}, y_{2m(k)-1}) \to \varepsilon \quad \text{and} \quad d(y_{2n(k)+1}, y_{2m(k)-1}) \to \varepsilon \quad \text{as} \quad k \to \infty.
\] (3.8)

Using (3.3) we get
\[
d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2n(k)+1}).
\]

By (3.2) and (3.6) we obtain
\[
\lim_{k \to \infty} F(d(y_{2n(k)}, y_{2m(k)})) \leq \lim_{k \to \infty} F(d(Ax_{2m(k)}, Bx_{2n(k)+1})) \\
\leq \lim_{k \to \infty} \bigg[ \psi \bigg[ a \bigg( F(d(Sx_{2m(k)}, Tx_{2n(k)+1})))^p \\
+ (1 - a) \max \bigg\{ F(d(Ax_{2m(k)}, Sx_{2m(k)})), F(d(Bx_{2n(k)+1}, Tx_{2n(k)+1})) \bigg\}^p \bigg] \bigg] \frac{1}{p}
\]
\[
\leq \lim_{k \to \infty} \bigg[ a \bigg( F(d(y_{2m(k)} - 1, y_{2m(k)})))^p \\
+ (1 - a) \max \bigg\{ F(d(y_{2m(k)}), y_{2m(k)-1})), F(d(y_{2m(k)+1}, y_{2m(k)))) \bigg\}^p \bigg] \frac{1}{p}
\]
\[
\leq \lim_{k \to \infty} \bigg[ \frac{1}{p} \bigg] \frac{1}{p}.
\] (3.9)

Applying (3.5), (3.7), (3.8) and (3.9) we find as \(k \to \infty\)
\[
F(\varepsilon) \leq \bigg[ \psi \bigg( a(F(\varepsilon))^p + (1 - a)(F(\varepsilon))^p \bigg) \bigg] \frac{1}{p} < F(\varepsilon)
\]
which is a contradiction. Hence, \(\{y_n\}\) is a Cauchy sequence in \(X\).  

\[\Box\]
Theorem 3.2. Let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X, d)$ into itself satisfying (3.1) and (3.2). Suppose that one of $S(X)$ or $T(X)$ or $B(X)$ or $A(X)$ is complete and the pairs $(A, S)$ and $(B, T)$ are weakly compatible. Then, $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.

Proof. By Lemma 3.1, the sequence $\{y_{2n+1}\} = \{Sx_{2n+2}\} \subset S(X)$ is a Cauchy sequence in $S(X)$. Since $S(X)$ is complete, it converges to a point $z = Su$ for some $u \in X$. Therefore, the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n}+1\}$ also converge to $z$. If $Au \neq z$, using (3.2) we get

$$
(F(d(Au, Bx_{2n+1})))^p \leq \psi[a(F(d(Su, Tx_{2n+1}))))^p + (1 - a) \max\{F(d(Au, Su)), F(d(Bx_{2n+1}, Tx_{2n+1}))\},
$$

$$(F(d(Au, Su)))^\frac{1}{2} \cdot (F(d(Au, Tx_{2n+1})))^\frac{1}{2},
$$

$$(F(d(Su, Bx_{2n+1})))^\frac{1}{2} \cdot (F(d(Au, Tx_{2n+1})))^\frac{1}{2}]^p].
$$

Letting $n \to \infty$ we obtain

$$
(F(d(Au, z)))^p \leq \psi((1 - a)(F(d(Au, z)))^p) < (F(d(Au, z)))^p
$$

which is a contradiction. Then, $z = Au = Su$. As $A(X) \subset T(X)$, there exists a $v \in X$ such that $z = Tv$. If $z \neq Bv$, applying (3.2) we have

$$
(F(d(z, Bv)))^p = (F(d(Au, Bv)))^p
$$

$$
\leq \psi[a(F(d(Su, Tv)))^p + (1 - a) \max\{F(d(Au, Su)), F(d(Bv, Tv)), (F(d(Au, Su)))^\frac{1}{2} \cdot (F(d(Au, Tv)))^\frac{1}{2},
$$

$$(F(d(Su, Bv)))^\frac{1}{2} \cdot (F(d(Au, Tv)))^\frac{1}{2}]^p]
$$

$$
= \psi((1 - a)(F(d(z, Bv)))^p)
$$

$$
< (F(d(z, Bv)))^p
$$

which is a contradiction. Therefore, $z = Bv = Tv$. As the pair $(A, S)$ is weakly compatible, we have $SAu = ASu$; i.e., $Az = Sz$. If $Az \neq z$, using (3.2) we obtain

$$
(F(d(Az, z)))^p = (F(d(Az, Bv)))^p
$$

$$
\leq \psi[a(F(d(Sz, Tv)))^p + (1 - a) \max\{F(d(Az, Sz)), F(d(Bv, Tv)), (F(d(Az, Sz)))^\frac{1}{2} \cdot (F(d(Az, Tv)))^\frac{1}{2},
$$

$$(F(d(Sz, Bv)))^\frac{1}{2} \cdot (F(d(Az, Tv)))^\frac{1}{2}]^p]
$$

$$
= \psi((F(d(Az, z)))^p)
$$

$$
< (F(d(Az, z)))^p
$$

which is a contradiction. So, $z = Az = Sz$. Similarly, we can prove that $z = Bz = Tz$. The same result of Theorem 3.2 holds if we assume that $T(X)$ or $B(X)$ or $A(X)$ is complete instead of $S(X)$.

Suppose there exists an $n$ such that $y_n = y_{n+1}$. Therefore, $y_n = y_{n+k}$ for $k \geq 1$ and so there exists $u, v \in X$ such that $Au = Su$ and $Bv = Tv$. As in Theorem 3.2, we can prove that $z = Az = Bz = Tz$. The uniqueness of $z$ follows from (3.2). □

If $F(t) = \int_0^t \varphi(s) \, ds$ in Theorem 3.2, where $t \in [0, A)$, $\int_0^\epsilon \varphi(t) \, dt > 0$ for each $\epsilon \in (0, A)$, we get Theorem 3 of [6].

If $F(t) = t$ in Theorem 3.2, where $t \in [0, A)$, we obtain Theorem 7 of [5].

If $B = A$ and $T = S$ in Theorem 3.2, we get a corollary which generalizes Corollary 1 of [6].

If $p = a = 1$, $S = T = I_X$ and $F(t) = \int_0^t \varphi(s) \, ds$ in Theorem 3.2, where $t \in [0, A)$, $\int_0^\epsilon \varphi(t) \, dt > 0$ for each $\epsilon \in (0, A)$, we obtain Lemma 1 of [20].
Theorem 3.3. Let \( A, B, S \) and \( T \) be mappings from a metric space \((X, d)\) into itself satisfying (3.1) and

\[
F(d(Ax, By)) < aF(d(Sx, Ty)) + (1 - a) \max\left\{ F(d(Ax, Sx)) , F(d(By, Ty)) , (F(d(Ax, Sx)))^{1/2} \cdot (F(d(Ax, Ty)))^{1/2} ,
\right. \\
\left. (F(d(Sx, By)))^{1/2} \cdot (F(d(Ax, Ty)))^{1/2} \right\},
\]

(3.10)

for all \( x, y \) in \( X \) for which the right-hand side of (3.10) is positive, where \( 0 < a < 1 \) and \( F \in \mathcal{I} [0, 1) \). Suppose that \((A, S)\) or \((B, T)\) satisfies property (E.A), one of \( A(X) \), \( B(X) \), \( S(X) \), \( T(X) \) is a closed subspace of \( X \) and the pairs \((A, S)\) and \((B, T)\) are weakly compatible. Then, \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

Proof. Suppose that \((B, T)\) satisfies property (E.A). Then, there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = z \) for some \( z \in X \). Therefore, \( \lim_{n \to \infty} d(Bx_n, Tx_n) = 0 \). Since \( B(X) \subset S(X) \), there exists in \( X \) a sequence \( \{y_n\} \) such that \( Bx_n = Sy_n \). Hence, \( \lim_{n \to \infty} Sy_n = z \). Let us show that \( \lim_{n \to \infty} Ay_n = z \).

Suppose that \( \limsup_{n \to \infty} d(Ay_n, z) = \varepsilon > 0 \). Applying (3.10) we get

\[
F(d(Ay_n, Bx_n)) < aF(d(Sy_n, Tx_n))^{1/2} + (1 - a) \max\left\{ F(d(Ay_n, Sy_n)) , F(d(Bx_n, Tx_n)) , (F(d(Ay_n, Sy_n)))^{1/2} \cdot (F(d(Bx_n, Tx_n)))^{1/2} ,
\right. \\
\left. (F(d(Sy_n, Bx_n)))^{1/2} \cdot (F(d(Ay_n, Tx_n)))^{1/2} \right\}.
\]

Taking the limit as \( n \to \infty \), we obtain

\[
F(\varepsilon) \leq (1 - a) F(\varepsilon) < F(\varepsilon)
\]

which is a contradiction. Hence, \( \varepsilon = 0 \); i.e., \( \lim_{n \to \infty} Ay_n = z \). Suppose that \( S(X) \) is a closed subspace of \( X \). Then, \( z = Su \) for some \( u \in X \). If \( Au \neq z \), using (3.10) we get

\[
F(d(Au, Bx_{2n+1})) < aF(d(Su, Tx_{2n+1}))^{1/2} + (1 - a) \max\left\{ F(d(Au, Su)) , F(d(Bx_{2n+1}, Tx_{2n+1})) ,
\right. \\
\left. (F(d(Au, Su)))^{1/2} \cdot (F(d(Au, Bx_{2n+1})))^{1/2} ,
\right. \\
\left. (F(d(Su, Bx_{2n+1})))^{1/2} \cdot (F(d(Au, Tx_{2n+1})))^{1/2} \right\}.
\]

Letting \( n \to \infty \) we obtain

\[
F(d(Au, z)) \leq (1 - a) F(d(Au, z)) < F(d(Au, z))
\]

which is a contradiction. Then, \( z = Au = Su \). Since \( A(X) \subset T(X) \), there exists a \( v \in X \) such that \( z = Tv \). If \( z \neq Bv \), applying (3.10) we have

\[
F(d(z, Bv)) = F(d(Au, Bv)) < aF(d(Su, Tv))^{1/2} + (1 - a) \max\left\{ F(d(Au, Su)) , F(d(Bv, Tv)) , (F(d(Au, Su)))^{1/2} \cdot (F(d(Au, Tv)))^{1/2} ,
\right. \\
\left. (F(d(Su, Bv)))^{1/2} \cdot (F(d(Au, Tv)))^{1/2} \right\}
\]
Let \( \text{Theorem 3.4.} \) common property (E.A) which generalizes Theorem 5 of [6].

The right-hand side of (3.3) holds if we assume that (A,S) follow from (3.12) or (3.13). Hence, \( z = Bv = Tv \). As the pair (A,S) is weakly compatible, we have \( SAu =ASu \); i.e., \( Az =Sz \). If \( Az \neq z \), using (3.10) we obtain

\[
F(d(Az, z)) = F(d(Az, Bv)) < a F(d(Sz, Tv)) + (1 - a) \max \{ F(d(Az, Sz)), F(d(Bv, Tv)), (F(d(Az, Sz)))^{\frac{1}{2}} \cdot (F(d(Az, Tv)))^{\frac{1}{2}}, (F(d(Sz, Bv)))^{\frac{1}{2}} \cdot (F(d(Az, Tv)))^{\frac{1}{2}} \}
\]

which is a contradiction. Hence, \( z = Az = Sz \). In the same manner, we can prove that \( z = Bz = Tz \). The same result of Theorem 3.3 holds if we assume that \( S(X) \) or \( B(X) \) or \( T(X) \) is complete instead of \( A(X) \). The uniqueness of \( z \) follows from (3.10). \( \square \)

Now, we prove a common fixed point theorem of Gregus type using a strict contraction of integral type and a common property (E.A) which generalizes Theorem 5 of [6].

**Theorem 3.4.** Let \( A, B, S \) and \( T \) be mappings from a metric space \( (X, d) \) into itself satisfying (3.10) for which the right-hand side of (3.10) is positive. Suppose that \( (A, S) \) and \( (B, T) \) satisfy a common property (E.A), \( S(X) \) and \( T(X) \) are closed subspaces of \( X \) and the pairs \( (A, S) \) and \( (B, T) \) are weakly compatible. Then, \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** Suppose that \( (A, S) \) and \( (B, T) \) satisfy a common property (E.A). Then, there exists two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Tx_n = z \) for some \( z \in X \). Assume that \( S(X) \) and \( T(X) \) are closed subspaces of \( X \). Therefore, \( z = Su = Tv \) for some \( u, v \in X \). If \( Au \neq z \), Applying (3.10) we get

\[
F(d(Au, By_n)) < a F(d(Su, Ty_n)) + (1 - a) \max \{ F(d(Au, Su)), F(d(By_n, Ty_n)), (F(d(Au, Su)))^{\frac{1}{2}} \cdot (F(d(Au, Ty_n)))^{\frac{1}{2}}, (F(d(Su, By_n)))^{\frac{1}{2}} \cdot (F(d(Au, Ty_n)))^{\frac{1}{2}} \}
\]

Letting \( n \to \infty \) we obtain

\[
F(d(Au, z)) < (1 - a) F(d(Au, z)) < F(d(Au, z))
\]

which is a contradiction. So, \( z = Au = Su \). The rest of the proof follows as in Theorem 3.3. \( \square \)

Similarly, we can prove the following theorem which generalizes Theorem 2.1 of [3] and Theorem 1 of [24].

**Theorem 3.5.** Let \( A, B, S \) and \( T \) be mappings from a metric space \( (X, d) \) into itself satisfying (3.1) and

\[
F(d(Ax, By)) \leq \psi \left( F \left( \max \left\{ d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{d(Ax, Ty) + d(Sx, By)}{2} \right\} \right) \right)
\]

for all \( x, y \) in \( X \), \( F \in F[0, A) \) and \( \psi \in \Psi[0, F(A - 0)) \). Suppose that one of \( A(X) \) or \( B(X) \) or \( S(X) \) or \( T(X) \) is complete and the pairs \( (A, S) \) and \( (B, T) \) are weakly compatible. Then, \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

Now, we are going to generalize Theorem 3.5 of [18].
Lemma 3.6. Let $A, B, S$ and $T$ be self-mappings of a metric space $(X, d)$ satisfying (3.1) and
\[
\left[F(d(Ax, By))\right]^{2p} \leq a \psi_0\left(\left[F(d(Sx, Ty))\right]^{2p}\right) + (1 - a) \max\left\{\psi_1\left(\left[F(d(Sx, Ty))\right]^{2p}\right), \psi_2\left(\left[F(d(Ax, Sx))\right]^{q} \cdot \left[F(d(By, Ty))\right]^{q'}\right), \psi_3\left(\left[F(d(Sx, By))\right]^{r} \cdot \left[F(d(Ax, Ty))\right]^{s}\right), \psi_4\left(\frac{1}{2} \left[F(d(Ax, Sx))\right]^{q} \cdot \left[F(d(Ax, Ty))\right]^{q'}\right), \psi_5\left(\frac{1}{2} \left[F(d(By, Ty))\right]^{r} \cdot \left[F(d(Sx, By))\right]^{s}\right)\right\}
\]
(3.11)
for all $x, y \in X$, where $\psi_i \in \Psi[0, F(A - 0), F \in F[0, A)$ satisfying $F(2t) \leq 2F(t)$ for all $t > 0$, $i = 0, 1, 2, 3, 4, 5$, $0 \leq a, b, c \leq 1$ and $0 < p, q, q', r, r', s, s', l, l' \leq 1$, such that $2p + q + q' = r + r' = s + s' = l + l'$. Then, the sequence \(\{y_n\}\) defined by (3.3) is a Cauchy sequence in $X$.

Proof. First of all, assume that $y_n \neq y_{n+1}$ for all $n$. Using (3.3) and (3.11) we have
\[
\left(F(d(y_{2n}, y_{2n+1}))\right)^p = \left(F(d(Ax_{2n}, Bx_{2n+1}))\right)^{2p} \leq a \psi_0\left(\left[F(d(y_{2n-1}, y_{2n}))\right]^{2p}\right) + (1 - a) \max\left\{\psi_1\left(\left[F(d(y_{2n-1}, y_{2n}))\right]^{2p}\right), \psi_2\left(\left[F(d(y_{2n-1}, y_{2n}))\right]^{q} \cdot \left[F(d(y_{2n}, y_{2n+1}))\right]^{q'}\right), \psi_3\left(\left[F(d(y_{2n}, y_{2n+1}))\right]^{r} \cdot \left[F(d(y_{2n-1}, y_{2n+1}))\right]^{s}\right), \psi_4\left(\frac{1}{2} \left[F(d(y_{2n}, y_{2n+1}))\right]^{q} \cdot \left[F(d(y_{2n-1}, y_{2n+1}))\right]^{q'}\right), \psi_5\left(\frac{1}{2} \left[F(d(y_{2n-1}, y_{2n}))\right]^{s} \cdot \left[F(d(y_{2n-1}, y_{2n+1}))\right]^{s}\right)\right\}.
\]
If $F(d(y_{2n-1}, y_{2n})) \leq F(d(y_{2n}, y_{2n+1}))$ in the above inequality, then
\[
\left(F(d(y_{2n}, y_{2n+1}))\right)^{2p} \leq a \psi_0\left(\left[F(d(y_{2n}, y_{2n+1}))\right]^{2p}\right) + (1 - a) \max\left\{\psi_1\left(\left[F(d(y_{2n}, y_{2n+1}))\right]^{2p}\right), \psi_2\left(\left[F(d(y_{2n}, y_{2n+1}))\right]^{2p}\right), \psi_3\left(\left[F(d(y_{2n}, y_{2n+1}))\right]^{2p}\right), \psi_4\left(\left[F(d(y_{2n}, y_{2n+1}))\right]^{2p}\right), \psi_5\left(\left[F(d(y_{2n}, y_{2n+1}))\right]^{2p}\right)\right\}.
\]
Applying Lemma 2.10, it follows that
\[
\left(F(d(y_{2n}, y_{2n+1}))\right)^{2p} \leq a \psi\left(\left[F(d(y_{2n}, y_{2n+1}))\right]^{2p}\right) + (1 - a) \psi\left(\left[F(d(y_{2n}, y_{2n+1}))\right]^{2p}\right) < \left(F(d(y_{2n}, y_{2n+1}))\right)^{2p}
\]
which is a contradiction. Therefore
\[
\left(F(d(y_{2n}, y_{2n+1}))\right)^{p} \leq \psi\left(\left(F(d(y_{2n-1}, y_{2n}))\right)^{p}\right).
\]
In the same manner, we get
\[
\left(F(d(y_{2n+1}, y_{2n+2}))\right)^{p} \leq \psi\left(\left(F(d(y_{2n}, y_{2n+1}))\right)^{p}\right).
\]
The rest of the proof follows as in Lemma 3.1. Hence, \(\{y_n\}\) is a Cauchy sequence in $X$. \qed
Theorem 3.7. Let $A, B, S$ and $T$ be mappings from a metric space $(X, d)$ into itself satisfying (3.1) and (3.11). Suppose that one of $S(X)$ or $T(X)$ or $A(X)$ or $B(X)$ is complete and the pairs $(A, S)$ and $(B, T)$ are weakly compatible. Then $A, B, S$ and $T$ have a unique common fixed point in $X$.

Proof. By Lemma 3.6, the sequence $\{y_{2n+1}\} = \{Sx_{2n+2}\} \subset S(X)$ is a Cauchy sequence in $S(X)$. Since $S(X)$ is complete, it converges to a point $z = Su$ for some $u \in X$. Therefore, the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\}$ also converge to $z$. If $Au \neq z$, using (3.11) we get

$$
[F(d(Au, Bx_{2n+1}))]^{2p} \leq a\psi_0([F(d(Su, Tx_{2n+1}))]^{2p}) + (1 - a)\max\left\{\psi_1([F(d(Su, Tx_{2n+1}))]^{2p}), \psi_2([F(d(Au, Su))]^q \cdot [F(d(Bx_{2n+1}, Tx_{2n+1}))]^{q'}), \psi_3([F(d(Su, Bx_{2n+1}))]^r \cdot [F(d(Au, Tx_{2n+1}))]^{r'}), \psi_4\left(\frac{1}{2}[F(d(Au, Su))]^s \cdot [F(d(Au, Tx_{2n+1}))]^{s'}\right), \psi_5\left(\frac{1}{2}[F(d(Bx_{2n+1}, Tx_{2n+1}))]^t \cdot [F(d(Su, Bx_{2n+1}))]^{t'}\right)\right\}.
$$

Letting $n \to \infty$ we obtain

$$
[F(d(Au, z))]^{2p} \leq (1 - a)\psi\left(\frac{1}{2}[F(d(Au, z))]^{2p}\right) < [F(d(Au, z))]^{2p}
$$

which is a contradiction. Then $z = Au = Su$. As $A(X) \subset T(X)$, there exists a $v \in X$ such that $z = Tv$. If $z \neq Bv$, applying (3.11) we have

$$
[F(d(z, Bv))]^{2p} = [F(d(Au, Bv))]^{2p} \leq \psi_5\left(\frac{1}{2}[F(d(z, Bv))]^{2p}\right)
$$

which is a contradiction. Therefore, $z = Bv = Tv$. As the pair $(A, S)$ is weakly compatible, we have $SAu = ASu$; i.e., $Az = Sz$. If $Az \neq z$, using (3.11) we obtain

$$
[F(d(Az, Bv))]^{2p} \leq a\psi_0([F(d(Sz, Tv))]^{2p}) + (1 - a)\max\left\{\psi_1([F(d(Sz, Tv))]^{2p}), \psi_3([F(d(Sz, Bv))]^r \cdot [F(d(Az, Tv))]^{r'}), \psi_4\left(\frac{1}{2}[F(d(Az, z))]^s \cdot [F(d(Az, z))]^{s'}\right), \psi_5\left(\frac{1}{2}[F(d(Bx_{2n+1}, Tx_{2n+1}))]^t \cdot [F(d(Su, Bx_{2n+1}))]^{t'}\right)\right\}
$$

which is a contradiction. So, $z = Az = Sz$. Similarly, we can prove that $z = Bz = Tz$. The same result of Theorem 3.2 holds if we assume that $T(X)$ or $B(X)$ or $A(X)$ is complete instead of $S(X)$. The uniqueness of $z$ follows from (3.11). $\Box$

If $F(t) = \int_0^t \psi(s) \, ds$ in Theorem 3.7, where $t \in [0, A), \int_0^t \psi(t) \, dt > 0$ for each $\epsilon \in (0, A)$, we get Theorem 3 of [6] or $F(t) = t$ in Theorem 3.7, where $t \in [0, A)$, we obtain a generalization of a Theorem 3.5 of [18].

The following example shows that Theorem 3.2 is a generalization of Theorem 7 of [5] if $\psi(t) = ht$ for all $t > 0$, $h \in (0, 1)$.

Example 3.8. Let $X = \{\frac{1}{n} : n \in \mathbb{N}^*\} \cup \{0\}$ with the Euclidean metric and $A, B, S$ and $T$ are self mappings of $X$ defined by
Therefore, for all \( x \neq 718 \) we have \( \psi(t) = h t \) for all \( t > 0 \); i.e., there exists \( h \in [0, 1) \) such that \( d^{\rho}(Ax, By) \leq h M_{\rho}(x, y) \) for all \( x, y \in X \), where

\[
M_{\rho}(x, y) = ad^{\rho}(Sx, Ty) + (1 - a) \max \left\{ d^{\rho}(Ax, Sx), d^{\rho}(By, Ty), \right. \\
\min \left\{ d^{\rho}(Ax, Sx) \cdot d^{\rho}(Ax, Ty), d^{\rho}(By, Ty) \cdot d^{\rho}(Sx, By) \right\}, \\
d^{\rho}(Sx, By) \cdot d^{\rho}(Ax, Ty), \left. \frac{d^{\rho}(Ax, Sx) + d^{\rho}(By, Ty)}{2} \right\}.
\]

Therefore, for all \( x \neq y \) we have \( \frac{d^{\rho}(Ax, By)}{M_{\rho}(x, y)} \leq h < 1 \). Using Example 4 of [20] we obtain

\[
\sup_{x \neq y} \frac{d^{\rho}(Ax, By)}{M_{\rho}(x, y)} = \sup_{m \in \mathbb{N}^*} \frac{\frac{1}{m+1}^{(m+1)p}}{\left(\frac{1}{m+1}\right)^{(m+2)p} + (1 - a) \frac{1}{m^{p+1}} (m+1)^p} = \sup_{m \in \mathbb{N}^*} \frac{m^p}{(m+2)^p} = 1.
\]

So, there is no \( h \in [0, 1) \) such that \( d^{\rho}(Ax, By) \leq h M_{\rho}(x, y) \). Hence, Theorem 7 of [5] cannot be used if \( \psi(t) = ht \) for all \( t > 0 \). On the other hand, the inequality (3.2) is satisfied. To see this, let \( F(s) = s^{1/2} \) and \( \psi(t) = \frac{t}{2} \). Then \( F \in F \{0, A\} \) and \( \psi \in \Psi[0, e^{\frac{t}{2}}] \), where \( A = e > D \). Using Example 4 of [20] we have

\[
d^{\rho}(Ax, By) \leq \left| \frac{1}{n+1} - \frac{1}{m+1} \right|.
\]

Therefore

\[
F(d^{\rho}(Ax, By)) \leq F\left( \left| \frac{1}{n+1} - \frac{1}{m+1} \right| \right) = \left| \frac{1}{n+1} - \frac{1}{m+1} \right|^{\frac{1}{\pi+1}}.
\]

Using Example 3.6 of [4], we get

\[
F(d^{\rho}(Ax, By)) \leq \frac{1}{2} \left| \frac{1}{n} - \frac{1}{m} \right|^{\frac{1}{\pi+1}} = \frac{1}{2} F(d^{\rho}(Sx, Ty))
\]

and so

\[
\left[ F(d^{\rho}(Ax, By)) \right]^p \leq \frac{1}{2^p} \left[ F(d^{\rho}(Sx, Ty)) \right]^p
\]

\[
\leq \frac{1}{2} \left[ F(d^{\rho}(Sx, Ty)) \right]^p
\]

\[
= \psi\left[ \left[ F(d^{\rho}(Sx, Ty)) \right]^p \right]
\]

\[
\leq \psi\left( a \left( F(d^{\rho}(Sx, Ty)) \right)^p \right)
\]

\[
+ (1 - a) \max \left[ F(d^{\rho}(Ax, Sx)), F(d^{\rho}(By, Ty)) \right],
\]

\[
\min \left\{ \left( F(d^{\rho}(Ax, Sx)) \right)^{\frac{1}{2}}, \left( F(d^{\rho}(Ax, Ty)) \right)^{\frac{1}{2}}, \left( F(d^{\rho}(By, Ty)) \right)^{\frac{1}{2}}, \left( F(d^{\rho}(Sx, By)) \right)^{\frac{1}{2}} \right\} \leq \psi\left( a \left( F(d^{\rho}(Sx, Ty)) \right)^p \right)
\]

\[
\left( F(d^{\rho}(Sx, By)) \right)^{\frac{1}{2}} \left( F(d^{\rho}(Ax, Ty)) \right)^{\frac{1}{2}} \left( F(d^{\rho}(Ax, Sx)) \right)^{\frac{1}{2}} \left( F(d^{\rho}(By, Ty)) \right)^{\frac{1}{2}} \left( F(d^{\rho}(Sx, By)) \right)^{\frac{1}{2}} \right] \leq \psi\left( a \left( F(d^{\rho}(Sx, Ty)) \right)^p \right).
\]

In the same manner, we can prove that the inequality (3.11) is satisfied, but Theorem 3.5 of [18] cannot be used if \( \psi_i(t) = ht, i = 0, 1, 2, 3, 4, 5 \), for all \( t > 0 \), where \( h \in [0, 1) \).
Acknowledgments

The author would like to thank Dr X. Zhang for his paper [24] and the reviewers for their useful suggestions.

References