A Sturm Type Comparison Theorem for Nonlinear Problems

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1. Introduction

The qualitative study of second order linear equations originated in Sturm's classical paper of 1836. A series of comparison theorems, see [1, 6], have been established for pairs of linear equations:

\[(p_j(t)u')' + q_j(t)u = 0 \quad \text{for} \quad j = 1, 2, \quad (1.1)\]

where \(p_j\) and \(q_j\) are real-valued continuous functions on an interval \(J = [a, b]\), and also satisfy

\[p_1 \geq p_2 > 0 \quad \text{and} \quad q_1 \leq q_2 \quad \text{on} \quad [a, b]. \quad (1.2)\]

If (1.2) holds, (1.1) is called a Sturm majorant for (1.1). The Sturm comparison theorem plays an important role in the study of second order linear equations, so it is natural to consider similar comparison theorems for the study of nonlinear equations. Basic models leading to such nonlinear equations are, e.g.,

(i) the degenerate Laplace equation,

\[\nabla \cdot (|\nabla u|^{m-2} \nabla u) + g(x, u) = 0, \quad m > 1; \quad (1.3)\]

(ii) the steady state porous medium equation,

\[A\rho(u) + g(x, u) = 0; \quad (1.4)\]

and the following generalization of (i) and (ii),

\[\nabla \cdot (u^{\gamma} |\nabla u|^{m-2} \nabla u) + g(x, u) = 0, \quad m > 1. \quad (1.5)\]
It is easy to show that radially symmetric solutions of (1.3)-(1.5) satisfy ordinary differential equations of the form
\[(r^{N-1} \phi(u)|u'|^{m-2} u')' + r^{N-1} g(r, u) = 0, \quad N \geq 1 \quad \text{and} \quad m > 1. \quad (1.6)\]

In standard proofs of the Sturm theorem, the condition (1.2) are essential, especially, the requirement \(p_2(0) > 0\). However in (1.6), we have \(p_1(r) = r^{N-1}\) and \(p_2(0) = 0\), so that the classical approach cannot be applied. Moreover the nonlinear terms \(\phi(u) |u'|^{m-1} u'\) and \(g(r, u)\) present intrinsic difficulties in applying the Sturm theorem to (1.6).

Recently, see for instance \([2, 3, 5]\), several authors have studied Sturm-type comparison theorems for various special cases of (1.6). Gui has obtained a comparison theorem for linear equations of type (1.6) when \(m = 2, \phi \equiv 1\) and \(g(r, u) = k(r) u\). In \([3]\), a comparison between solutions of (1.6) and solutions of the equation
\[(r^{N-1} |u'|^{m-2} u')' + r^{N-1} Mu^{m-1} = 0 \quad (1.7)\]
is given when \(|g(r, u)| \leq Mu^{m-1}\). In \([3, 5]\), the comparison theorems are then applied in the derivation of interesting properties for radially symmetric solutions of certain nonlinear elliptic problems.

The purpose of this paper is to extend the classical Sturm theorem for second order linear equations to nonlinear equations (and inequalities) and to apply the result to existence and uniqueness problems for positive radially symmetric solutions of nonlinear elliptic equations defined on bounded domains or in the whole space \(\mathbb{R}^N\). Our comparison theorem includes the comparison theorems of Gui, Fillipucci, Ghiselli Ricci and Pucci, as well as the classical Sturm separation theorem, see \([7]\).

Let us consider the following problems
\[
\begin{align*}
\{ (p_1(r) \phi(u, u'))' + g_1(r, u) & \geq 0 \quad \text{for} \quad r > 0, \\
u(0) = u_0, \quad u'(0) = u'_0
\end{align*}
\]
and
\[
\begin{align*}
\{ (p_2(r) \phi(v, v'))' + g_2(r, v) & \leq 0 \quad \text{for} \quad r > 0, \\
v(0) = v_0, \quad v'(0) = v'_0
\end{align*}
\]
Here we assume that
\[
\phi(u, w) = \phi(u) |w|^{m-2} w, \quad m > 1, \quad (1.10)
\]
and that there are functions \(q_1(r), q_2(r)\) and \(f(u)\) such that
\[
g_i(r, u) \leq q_i(r) f(u) \quad (1.11)
\]
and
\[
g_2(r, u) \geq q_2(r) f(u)
\]
(1.12)

for all \( r \geq 0 \) and \( u \) in \( \mathbb{R} \). Equations (1.1.1) and (1.1.2) are obvious special cases of (1.8) and (1.9). Also (1.6) is included in both (1.8) and (1.9).

Throughout the paper we assume the following principal conditions on the functions \( p_i \) and \( q_i \),

\begin{enumerate}
  \item[(H\_1)] \( p_i \in C[0, \infty) \) and \( q_i \in C(0, \infty) \) for \( i = 1, 2 \).
  \item[(H\_2)] \( p_i(r) \geq p_2(r) > 0 \) and \( q_i(r) \geq q_2(r) \geq 0 \) for \( r > 0 \).
  \item[(H\_3)] \( \delta(r) = \int_0^r p_i(s) \, ds / p_i(r) \to 0 \) as \( r \to 0 \).
\end{enumerate}

As in the classical case, we then call (1.8) a Sturm majorant for (1.9). It is worth noticing that the smoothness conditions on \( p_i \) and \( q_i \) are essentially minimal and no differentiability is required. When \( p_1(0) > 0 \), of course (H\_1) is automatically satisfied. On the other hand, even if \( p_1(0) = 0 \), (H\_3) continues to hold when \( p_1(r) = r^{N-1} \) or \( p_1(r) \) is monotone increasing. By (H\_1)-(H\_3), it is easy to check that \( \delta \in C[0, \infty) \) by setting \( \delta(0) = 0 \). We should also point out that (H\_3) is only imposed on \( p_1 \) and not on \( p_2 \).

According to [10], radially symmetric solutions of (1.6) always satisfy
\[
u' = 0 \quad \text{at} \quad r = 0.
\]

For general differential inequalities, however we impose the initial relations
\[
u_0 \geq v_0 > 0 \quad \text{and} \quad \nu'_0 \geq v'_0 \quad \text{with} \quad v'_0 \leq 0.
\]

Let
\[
u(r) > 0 \quad \text{and} \quad v(r) > 0 \quad \text{for} \quad r \in [0, R).
\]

Here \( R \) is either a positive number or \( +\infty \). The existence of such a value \( R \) is guaranteed by (1.13). We shall concentrate on the comparison between solutions of (1.8) and (1.9) on \([0, R)\).

Instead of the standard case of \( C^2 \) solutions, here we consider solutions in the following weaker sense. That is, \( u \) and \( v \) are said to be solutions of (1.8) and (1.9) on \([0, R)\) if
\[
u, v \in C^1[0, R), \quad p_1 \phi(u) |u'|^m - 2 u', p_2 \phi(v) |v'|^m - 2 v' \in C^1(0, R)
\]

and (1.8), (1.9) hold.

The nonlinearities in (1.8) and (1.9) essentially determine the behavior of solutions. The following crucial assumptions will be imposed on \( \phi \) and \( f \).
\((A_1)\) \(f \in C^1(0, \infty), f > 0\) and \(f\) is nondecreasing on \((0, \infty)\).

\((A_2)\) \(\phi \in C(0, \infty), \phi > 0\).

\((A_3)\) \(\phi^{-\mu}(u) \cdot d[\phi(u)]/du\) is non-increasing on \((0, \infty)\) for \(\mu = 1/(m-1)\).

Conditions \((A_1)\) and \((A_2)\) are natural, and \((A_3)\) is essential for our main theorem. Note that in the canonical case

\(\phi(u) = u^\alpha, \quad f(u) = u^\beta\)

condition \((A_3)\) takes the simpler form, \(\beta \leq \alpha + m - 1\). Finally, if \(\phi \equiv 1\) and \(m = 2\), \((A_3)\) reduces to the requirement that \(f\) be concave.

**Theorem A (Main Comparison Theorem).** Suppose that \(u\) and \(v\) are positive solutions of (1.8) and (1.9), and that (1.10)–(1.13) hold. If \(u_0' \geq 0\), then

\[u(r) \geq v(r) \quad \text{for} \quad r \in [0, R).\]

If \(u_0' < 0\) and

\[(H_4) \quad p_2(r) |v_0'|^{m-1} \left(\frac{\phi(v_0)}{f(v_0)}\right) \geq p_1(r) |u_0'|^{m-1} \left(\frac{\phi(u_0)}{f(u_0)}\right) \quad \text{for small} \quad r,
\]

then

\[u(r) \geq v(r) \quad \text{for} \quad r \in [0, R).\]

In both cases, also

\[\int_{s(r)}^{s(v)} \left(\frac{\phi(s)}{f(s)}\right)^m ds \geq \int_{s(r)}^{s(v)} \left(\frac{\phi(s)}{f(s)}\right)^{m-1} ds \quad \text{for} \quad r \in [0, R).\]

Roughly speaking, the comparison theorem holds as long as the functions \(g_1(r, u)\) and \(g_2(r, u)\) can be separated by a positive concave function \(f(u)\) for each fixed \(r\). It is easy to see that if

\[p_2(0) |v_0'|^{m-1} \left(\frac{\phi(v_0)}{f(v_0)}\right) > p_1(0) |u_0'|^{m-1} \left(\frac{\phi(u_0)}{f(u_0)}\right),\]

then \((H_4)\) holds automatically. As an easy corollary of Theorem A, see section 4, the following conclusion holds.
Theorem B. Consider the Dirichlet problem
\[
\begin{cases}
\nabla \cdot (|u|^{m-2} \nabla u) + g(|x|, u) = 0, & x \in B_R \\
u = 0, & x \in \partial B_R,
\end{cases}
\]
where \( B_R \) is a ball of radius \( R \) in \( \mathbb{R}^N \) and
\[
0 \leq g(|x|, u) \leq Mu^{m-1} \quad \text{for} \quad u \geq 0.
\]
If
\[
R < \left( \frac{N}{M} \right)^{\frac{1}{m}} \left( \frac{m}{m-1} \right)^{\frac{m}{m-1}},
\]
then there is no radial solution of (1.14) which is positive in \( B_R \).

It is not simple to prove Theorem B by means of difference arguments, because of the nonlinearities. On the other hand, by using Theorem A, the conclusion becomes straightforward.

We arrange the paper as follows. Section 2 contains several technical results, including a Riccati transformation for nonlinear equations.Section 3 is dedicated to the proof of the main comparison Theorem A. In Section 4, we give various applications of the main comparison theorem to existence and uniqueness problem of radial solutions of nonlinear elliptic Dirichlet problems. Theorem 4.1 and Theorem 4.2 are analogous to the uniqueness theorem in [11], and Theorem 4.3 improves the result of the main comparison theorem in [3].

The method that we use here is different from those in [3, 6, 7]. In [3] the authors take advantage of the condition \(|g(r, u)| \leq Mu^{m-1}\), and construct an auxiliary equation. Since we are dealing with more general nonlinear functions \( \phi \) and \( f \), it seems hard to follow that approach.

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2. Preliminaries

We begin by considering the following simpler problems
\[
\begin{cases}
(p_1(r) \phi(u) |u'|^{m-2} u')' + q_1(r) f(u) \geq 0 & \text{for} \quad r > 0 \\
u(0) = u_0, & u'(0) = u'_0,
\end{cases}
\]

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and

\[ \begin{cases} (p_2(r) \phi(v)) v'' + q_2(r) f(v) \leq 0 & \text{for } r > 0 \\ v(0) = v_0, & \phi'(0) = \phi_0. \end{cases} \] (2.2)

The relations between initial conditions are

\[ u_0 \geq v_0 > 0 \] (2.3)

and

\[ u'_0 \geq v'_0 \quad \text{with} \quad v'_0 < 0. \] (2.4)

It is easy to see that (2.1) and (2.2) are special cases of (1.8) and (1.9). When (1.11) and (1.12) hold, it is clear that the comparison theorem for (1.8) and (1.9) can be deduced from that for (2.1) and (2.2).

The functions \( u \) and \( v \) are positive solutions of (2.1) and (2.2) on \([0, R)\), where

\[ R = \max \{ r \in (0, \infty) \mid u(r) > 0 \text{ and } v(r) > 0 \}. \]

By (2.4) and the structural conditions of section 1, we see that \( v'(r) \leq 0 \) for \( r \in [0, R) \). This fact is essential in the course of our discussion.

In the standard proof of Sturm theorems, see [6], the Prüfer transformation plays an important role. Suppose that \( u \) is a solution of the equation

\[ p(r) u'' + q(r) u = 0. \] (2.5)

The Prüfer transformation for (2.5) is

\[ \varphi = \tan^{-1} \left( \frac{u}{pu'} \right) \quad \text{and} \quad \rho = (u^2 + p^2 u'^2)^{1/2}, \]

so that \( \varphi \) and \( \rho \) satisfy the following differential system

\[ \begin{cases} \varphi'(r) = 1/p(r) \cos \varphi + q(r) \sin^2 \varphi \\ \rho'(r) = -[q(r) - 1/p(r)] \rho \sin \varphi \cos \varphi. \end{cases} \]

The above transformation heavily depends on the linearity of equation (2.5) and the assumption \( p(0) > 0 \), and is not easy to carry over to non-linear equations of type (1.6).

Instead, we define Riccati transformations for (2.1) and (2.2) by

\[ \omega(r) = \frac{p_2(r) \phi(u) |u'|^{m-2} u'}{f(u)} \] (2.6)
and

\[ \theta(r) = \frac{p_2(r) \phi(v) |v'|^{m-2} v'}{f(v)}. \]  

(2.7)

The corresponding Riccati transformation for linear equations of type (2.5) appears in [1, 6].\(^1\) By (A1) and since \(u\) and \(v\) are positive on \([0, R)\), it is clear that \(\omega\) and \(\theta\) are well-defined on \([0, R)\). Furthermore the following conclusion holds for \(\omega\) and \(\theta\).

**Proposition 2.1.** Suppose that \(u\) and \(v\) are positive solutions of (2.1) and (2.2).

Then

\[ \omega, \theta \in C([0, R]) \cap C^1([0, R]). \]

Moreover, on the interval \((0, R)\)

\[ \omega'(r) \geq -q_1(r) - \frac{(m-1) |\omega(r)|^{m+1}}{p_1'(r)} \frac{d\omega^m(u)/du}{\phi^m(u)} \]  

and

\[ \theta'(r) \leq -q_2(r) - \frac{(m-1) |\theta(r)|^{m+1}}{p_2'(r)} \frac{d\theta^m(v)/du}{\phi^m(v)} \]  

(2.8)

(2.9)

**Proof.** By definition of solutions we have

\[ u, v \in C^1[0, R), \]

\[ p_1 \phi(u) |u'|^{m-2} u', p_2 \phi(v) |v'|^{m-2} v' \in C^1(0, R). \]

Since \(f \in C^1(0, \infty)\), then

\[ \omega, \theta \in C([0, R]) \cap C^1([0, R]). \]

Differentiating (2.6) and (2.7) with respect to \(r\) yields

\[ \omega'(r) = \frac{(p_1(r) \phi(u) |u'|^{m-2} u')' f(u) - p_1(r) \phi(u) |u'|^{m} df(u)/du}{f^2(u)} \]  

and

\[ \theta'(r) = \frac{(p_2(r) \phi(v) |v'|^{m-2} v')' f(v) - p_2(r) \phi(v) |v'|^{m} df(v)/du}{f^2(v)} \]  

(2.10)

(2.11)

\(^1\text{I.e. } \omega(r) = p_1 u/\phi, \theta(r) = p_2 v/\phi. \)
Since \( u \) and \( v \) satisfy (2.1) and (2.2) for \( r > 0 \), by (2.10) and (2.11) the proof is completed.

**Proposition 2.2.** Suppose that \( u \) and \( v \) are positive solutions of (2.1) and (2.2). If we assume

\[
\omega \geq 0 \quad \text{on} \quad [0, a),
\]

(2.12)

for some \( a \in (0, R] \), then

\[
\int_{v(r)}^{u(r)} \left( \frac{\phi(s)}{f(s)} \right)^\omega \, ds \geq \int_{v_0}^{u_0} \left( \frac{\phi(s)}{f(s)} \right)^\omega \, ds \quad \text{for} \quad r \in [0, a). \tag{2.13}
\]

**Proof.** By (2.2) and the facts that \( f > 0 \) and \( q \geq 0 \) for \( r \in (0, R) \), we have

\[
(p_2(r) \phi(v) |v'|^{m-2} v')' \leq 0 \quad \text{for} \quad r \in (0, R).
\]

From \((H_2),(A_2)\) and (2.4), we see that

\[
p_2(r) \phi(v) |v'|^{m-2} v' \leq 0 \quad \text{at} \quad r = 0.
\]

Then obviously

\[
p_2(r) \phi(v) |v'|^{m-2} v' \leq 0 \quad \text{for} \quad r \in [0, R) \tag{2.14}
\]

and in particular \( v' \leq 0 \) on \([0, R)\). Using (2.12) and \((H_2)\) on \((0, a)\) gives

\[
\frac{\phi(u) |u'|^{m-2} u'}{f(u)} \geq \frac{\phi(v) |v'|^{m-2} v'}{f(v)} \quad \text{for} \quad r \in (0, a).
\]

Since \( v' \leq 0 \) on \([0, a)\), then

\[
u' \left( \frac{\phi(u)}{f(u)} \right)^\omega \geq v' \left( \frac{\phi(v)}{f(v)} \right)^\omega \quad \text{for} \quad r \in (0, a). \tag{2.15}
\]

Integrating (2.15) from 0 to \( r \in [0, a) \) yields,

\[
\int_{v(r)}^{u(r)} \left( \frac{\phi(s)}{f(s)} \right)^\omega \, ds \geq \int_{v_0}^{u_0} \left( \frac{\phi(s)}{f(s)} \right)^\omega \, ds \quad \text{for} \quad r \in [0, a),
\]

which is exactly (2.13).
Corollary 2.3. Suppose that $u$ and $v$ are positive solutions of (2.1) and (2.2). If we assume that
\[ \omega \geq \theta \quad \text{on} \quad [0, a) \]
for some $a \in (0, R]$, then
\[ u \geq v \quad \text{on} \quad [0, a). \]

By (2.13) and (2.3), Corollary 2.3 follows at once. If in addition we assume $\omega > \theta$ on $(0, a)$, then
\[ u > v \quad \text{on} \quad (0, a). \]

The advantage of using the Riccati transformations (2.6) and (2.7) is that a second order nonlinear differential inequality is transformed into a first order differential relation which can be handled by elementary methods.

We close with a simple fact. According to (2.6), (2.7) and (2.4), it is evident that $\omega(0) \geq \theta(0)$ whenever $u'(0) \geq 0$. On the other hand, if $u'(0) < 0$ then $(H_4)$ and the continuity of $p_1$ and $p_2$ again imply that $\omega(0) \geq \theta(0)$.

3. The Main Comparison Theorem

In this section we prove the main comparison theorem, which is applicable to a wide class of nonlinear functions. Throughout this section, except for Corollary 3.5, we assume that condition $(H_4)$ is valid, as well as $(A_1)$–$(A_3)$ and $(H_1)$–$(H_3)$. For simplicity, we divide the proof into several lemmas.

Lemma 3.1. Suppose that $u$ and $v$ are positive solutions of (2.1) and (2.2), and also assume that
\[ \omega > \theta \quad \text{on} \quad (0, \gamma) \]
for some $\gamma \in (0, R)$. Then
\[ \omega > \theta \quad \text{on} \quad (0, R). \]

Proof. As shown in the proof of Proposition 2.2, see (2.14),
\[ p_2 \phi(v) |v'|^{n-2} v' \leq 0. \]
Therefore, from (2.7) we obtain $\theta \leq 0$ on $[0, R)$. Moreover, by (2.9), $(H_2)$ and $(A_1)$ we have $\theta' \leq 0$ on $[0, R)$. There are now three possible cases to consider.
Case 1. \( \theta < 0 \) on \((0, R)\).

We claim the assertion holds. Otherwise, there is a point \( R' \in (0, R) \) such that
\[
\omega(R') = \theta(R') \quad \text{and} \quad \omega(r) > \theta(r) \quad \text{for} \quad r \in [\gamma, R').
\]
By the assumption we have
\[
\omega(R') < 0.
\]
The continuity of \( \omega \) implies that there is \( R'' < R' \) such that \( \omega \leq 0 \) on \([R'', R']\). Consequently
\[
|\omega| \leq |\theta| \quad \text{on} \quad [R', R''].
\]
Combining (2.8) with (3.2) and using \((A_1)\) yields
\[
\omega(r) \geq -q_2(r) \frac{(m-1)|\theta(r)|^{\nu+1}}{p_2^*(r)} \cdot \frac{df^n(u)/du}{\phi^n(u)}
\]
Since \( \omega(r) \geq \theta(r) \) for \( r \in [0, R') \), then applying Corollary 2.3 on \([0, R')\) gives
\[
u \geq v \quad \text{on} \quad [0, R'].
\]
Now (3.3), \((A_1)\) and (2.9) imply that
\[
\omega(r) \geq -q_2(r) \frac{(m-1)|\theta(r)|^{\nu+1}}{p_2^*(r)} \cdot \frac{df^n(u)/du}{\phi^n(u)} \geq \theta'(r) \quad \text{for} \quad r \in [R', R'].
\]
Since \( \omega(R') > \theta(R') \), by (3.1) we get
\[
\omega(R') > \theta(R').
\]
This contradicts the first part of (3.1) and completes the proof of the claim.

Case 2. \( \theta \equiv 0 \) on \([0, R)\).

If \( \theta \equiv 0 \), then by (2.7) \( p_2 \phi(v) \mid v' \mid^{m-2} v' \equiv 0 \). Thus (2.2) and \((H_2)\) imply that \( q_1 \equiv q_2 \equiv 0 \) on \([0, R)\). By (2.1)
\[
(p \phi(u) \mid u' \mid^{m-2} u') \geq 0 \quad \text{on} \quad (0, R).
\]
Since \( \omega(r) > \theta(r) \equiv 0 \) for \( r \in (0, \delta) \), then by (2.6) we get
\[
p_1(r) \phi(u) \mid u' \mid^{m-2} u' > 0 \quad \text{for} \quad r \in (0, \delta).
\]
Consequently \( \omega(r) > 0 \) for \( r \in (0, R) \), that is \( \omega > \theta \) on \((0, R)\) as required.
Case 3. \( \theta(0) = 0 \), but \( \theta \not= 0 \) on \([0, R)\).
Since \( \theta' \leq 0 \), then there exists \( R_0 \) on \([0, R)\) such that
\[
\theta = 0 \quad \text{on} \quad [0, R_0]
\]
and
\[
\theta < 0 \quad \text{on} \quad (R_0, R).
\]
If \( R_0 = 0 \), this is exactly Case 1. Otherwise, by the argument of Case 2 we get \((p_1 \phi(u) |u'|^{m-2} u')' \geq 0\) on \((0, R_0)\), see (3.4). Since \( p_1 \phi(u) |u'|^{m-2} u' > 0 \) on \((0, \gamma)\), then \( p_1 \phi(u) |u'|^{m-2} u' > 0\) on \([0, R_0]\). By (2.6) we get \( \omega > 0 \) on \([0, R_0]\). The continuity of \( \omega \) and \( \theta \) implies that there is a small \( \gamma > 0 \) such that \( \omega > \theta \) in \([R_0, R_0 + \gamma]\). Thus, again applying the argument of Case 1 on \([ R_0, R)\), the proof is completed.

By the remark at the end of section 2 and the conclusion of Lemma 3.1, we also have \( \omega \geq \theta \) on \([0, R)\). Now we are ready to prove a comparison theorem between solutions of (2.2) and the following auxiliary problem

\[
\begin{aligned}
(p_1(r) \phi(u) |u'|^{m-2} u')' + q_1(r) f(u) &\geq \zeta(r) p_1(r) \\
u(0) &= u_0, \quad u'(0) = u'_0.
\end{aligned}
\tag{3.5}
\]

**Lemma 3.2.** Suppose that \( u \) and \( v \) are positive solutions of (3.5) and (2.2), where the function \( \zeta \) is in \( C[0, \infty) \) with
\[
\zeta(r) \geq 0 \quad \text{for} \quad r \in [0, \infty) \quad \text{and} \quad \zeta(0) > 0. \tag{3.6}
\]
Then
\[
\omega(r) \geq \theta(r) \quad \text{for} \quad r \in [0, R).
\]

**Proof.** In order to prove this lemma, we consider the following three cases, \( u'_0 > 0, \ u'_0 = 0 \) and \( u'_0 < 0 \).

**Case 1.** \( u'_0 > 0 \).
Since \( u \in C^1[0, R) \), then by the assumption we get \( u'(r) > 0 \) for \( r \in [0, \gamma) \), provided that \( \gamma \) is small enough. From (2.6) we obtain
\[
\omega(r) > 0 \quad \text{for} \quad r \in (0, \gamma).
\]
Since \( v'(r) \leq 0 \) for \( r \in [0, R) \), then by (2.7) we get \( \theta \leq 0 \) on \((0, \gamma)\). Hence
\[
\omega(r) > \theta(r) \quad \text{for} \quad r \in (0, \gamma).
\]
Therefore, Lemma 3.1 implies that
\[ \omega(r) \geq \theta(r) \quad \text{for} \quad r \in [0, R). \]

**Case 2.** \( u_0 = 0. \)

Repeating the calculation in Proposition 2.1, but using (3.5) instead of (2.1), yields (see (2.10))
\[ \omega'(r) \geq -q_2(r) - \frac{r_1(r) \, \varphi(u) \, |u'|^m \, df(u) / du}{f'(u)} \]

and similarly
\[ \theta'(r) \leq -q_2(r) \, \phi(v) \, |v'|^m \, df(v) / du \]

By (H2) and the above inequalities we obtain
\[
\omega'(r) - \theta'(r) \geq q_2(r) - q_2(r) + \frac{\varphi(u)}{f(u)} \left( p_1(r) \, \phi(u) \, |u'|^m \, df(u) / du + p_2(r) \, \phi(v) \, |v'|^m \, df(v) / du \right)
\]
\[
\geq \frac{p_1(r) \, \varphi(u)}{f(u)} \left( -p_1(r) \, |u'|^m \, \phi^{-\mu_0}(u) \, df'(u) \, \phi_1^{-\mu_0}(u) / f'(u) \right)
\]
\[
+ \mu^{-1} \frac{p_2(r) \, \phi(v) \, |v'|^m \, \phi^{-\mu_0}(v) \, df'(v) \, \phi_1^{-\mu_0}(v) / f'(v)} {f'(v)} \]  
(3.7)

By the continuity assumptions on \( f, \phi \) and \( p_1 \), and the facts that \( p_1(r) > 0 \) for \( r \in (0, \infty) \), \( f(u_0) > 0 \) and \( \varphi(0) > 0 \), we get
\[
P_1(r) \varphi(u) \left( -p_1(r) \, |u'|^m \, \phi^{-\mu_0}(u) \, df'(u) \, \phi_1^{-\mu_0}(u) / f'(u) \right) > 0 \quad \text{for} \quad r \in (0, \gamma),
\]
where \( \gamma \) is a small positive number. Clearly,
\[
\mu^{-1} p_2 \phi(v) \, |v'|^m \, \phi^{-\mu_0}(v) \, df'(v) \, \phi_1^{-\mu_0}(v) / f'(v) \geq 0.
\]

Thus \( (r) > \theta'(r) \) for \( r \in (0, \gamma). \) Since \( u_0 = 0 \) and \( \varphi_0 \leq 0, \) then by (2.6) and (2.7) we obtain \( \omega(0) \geq \theta(0). \) Hence
\[ \omega(r) > \theta(r) \quad \text{for} \quad r \in (0, \gamma). \]

By Lemma 3.1, the proof is completed.
Case 3. \( u_0 < 0 \)

By (2.4) we easily get \( v_0 < 0 \). The condition \((H_4)\) implies that there is a small positive number \( \gamma_1 \) such that

\[
 p_2(r) \geq p_1(r) \frac{\phi(u_0)}{\phi(v_0)} |u_0|^{m-1} f(v_0) \quad \text{for} \quad r \in (0, \gamma_1). 
\]

Therefore we have

\[
 p_2^{\pm}(r) \geq p_1^{\pm}(r) \frac{\phi^{\pm}(u_0)}{\phi^{\pm}(v_0)} |u_0|^{m} f^{\pm}(v_0) \quad \text{for} \quad r \in (0, \gamma_1) 
\]

where \( \mu = 1/(m - 1) \). Using \( p_1(r) \geq p_2(r) \geq 0 \) for \( r \in [0, \infty) \) yields

\[
 p_2(r) \geq p_1(r) \frac{\phi^{\pm}(u_0)}{\phi^{\pm}(v_0)} |u_0|^{m} f^{\pm}(v_0) \quad \text{for} \quad r \in (0, \gamma_1). \quad (3.8)
\]

Since \( u_0 \geq v_0 \), then by \((A_3)\)

\[
 \phi^{-\rho}(v_0) df^\rho(v_0)/du \geq \phi^{-\rho}(u_0) df^\rho(u_0)/du. \quad (3.9)
\]

Now set

\[
 F(r) = \frac{\zeta(r)}{f(u)} \mu^{-1} |u'|^m \phi^{-\rho}(u) \frac{df^\rho(u)}{du} \phi^{\pm}(u) + \mu^{-1} |v'|^m \phi^{-\rho}(v) \frac{df^\rho(v)}{du} \phi^{\pm}(v) |v'|^m f^{\pm}(v_0) 
\]

It is easy to check that \( F \) is a continuous function on \([0, R]\). By (3.6), (3.8) and (3.9) we obtain

\[
 F(0) \geq \frac{\zeta(0)}{f(u_0)} > 0. 
\]

The continuity of \( F(r) \) ensures that there is a positive number \( \gamma \leq \gamma_1 \) such that \( F(r) > 0 \) for \( r \in (0, \gamma) \). Multiplying \( F \) by \( p_1 \) and using (3.8) yield

\[
 p_1(r) \frac{\zeta(r)}{f(u)} - \mu^{-1} p_1(r) |u'|^m \phi^{-\rho}(u) \frac{df^\rho(u)}{du} \phi^{\pm}(u) + \mu^{-1} p_2(r) \phi(v) |v'|^m \phi^{-\rho}(v) \frac{df^\rho(v)}{du} \phi^{\pm}(v) > 0 \quad \text{on} \quad (0, \gamma). 
\]
Therefore, from (3.7) we see that \((r) > \theta'(r)\) for \(r \in (0, \gamma)\). By the remark of the end of section 2, we get \(\omega(0) \geq \theta(0)\). Thus
\[\omega(r) > \theta(r) \quad \text{for} \quad r \in (0, \gamma).\]
Again by Lemma 3.1,
\[\omega(r) \geq \theta(r) \quad \text{for} \quad r \in [0, R).
\]
This completes the proof.

The next lemma is essential for the crucial comparison Theorem 3.4. Since the proof is technical, we shall give it in section 5. Now suppose that \(u\) is a positive \(C^1\) function on \([0, R)\) with
\[p_1 \phi(u) |u'|^{m-2} u' \in C^1(0, R)\]
Then we have\(^2\)

**Lemma 3.3.** For every positive number \(\epsilon\), there is a function \(u_\epsilon \in C^1[0, R)\) such that
\[u_\epsilon(0) = u(0) = u_0, \quad u'_\epsilon(0) = u'(0) = u'_0\]
and
\[\phi(u_\epsilon) |u'_\epsilon|^{m-2} u'_\epsilon = \phi(u) |u'|^{m-2} u' + \epsilon \delta(r) \quad \text{on} \quad [0, R), \quad (3.10)\]
where \(\delta\) is defined in \((H_3)\). Moreover, for \(r \in [0, R)\)
\[u_\epsilon(r) \geq u(r)\]
and
\[\lim_{\epsilon \to 0} u_\epsilon(r) = u(r).\]

In the special case
\[p_1(r) = r^{N-1}, \quad \phi(s) \equiv 1, \quad m = 2,\]
the auxiliary function \(u_\epsilon\) is given by
\[u_\epsilon(r) = u(r) + \frac{\epsilon r^2}{2N}.
\]
\(^2\) Note that \(u\) here need not in fact be a solution of (2.1).
It is easy to check that this function satisfies Lemma 3.3. On the other hand, in general the auxiliary function is not easy to obtain explicitly.

**Theorem 3.4.** Suppose that $u$ and $v$ are positive solutions of the differential inequalities \eqref{2.1} and \eqref{2.2}. Then

\[ u(r) \geq v(r) \quad \text{for} \quad r \in [0, R) \]

and

\[ \int_{v(r)}^{u(r)} \left( \frac{\phi(s)}{f(s)} \right)^n \, ds \geq \int_{v(r)}^{u(r)} \left( \frac{\phi(s)}{f(s)} \right)^m \, ds \quad \text{for} \quad r \in [0, R). \]

**Proof.** According to Lemma 3.3 there is a function $u_\epsilon \in C^1(0, R)$ such that

\[ p_1(r) \phi(u_\epsilon) |u_\epsilon'|^{m-2} u_\epsilon'(r) = p_1(r) \phi(u) |u'|^{m-2} u'(r) + \int_0^r p_1(s) \, ds \]

for $r \in [0, R)$. \hfill \ (3.11)

Since $p_1 \phi(u) |u'|^{m-2} u' \in C^1(0, R)$, then

\[ p_1(r) \phi(u_\epsilon) |u_\epsilon'|^{m-2} u_\epsilon' \in C^1(0, R). \]

Differentiating \eqref{3.11} with respect to $r$ yields

\[ (p_1(r) \phi(u_\epsilon) |u_\epsilon'|^{m-2} u_\epsilon')' + q_1(r) f(u_\epsilon) \geq \varepsilon p_1(r) \quad \text{on} \quad (0, R). \]

Since $u_\epsilon \geq u$, then by \eqref{A.1} we get $f(u_\epsilon) \geq f(u)$. Hence, $u_\epsilon$ is a solution of the problem

\[ \begin{aligned}
(p_1(r) \phi(u_\epsilon) |u_\epsilon'|^{m-2} u_\epsilon')' + q_1(r) f(u_\epsilon) &\geq \varepsilon p_1(r) \\
u_\epsilon(0) &= u_\epsilon_0, \\
u'_\epsilon(0) &= u'_0.
\end{aligned} \]

Applying Lemma 3.2 to \eqref{3.12} and \eqref{2.2} yields

\[ \co(r) \geq \theta(r) \quad \text{for} \quad r \in [0, R). \]

By Proposition 2.2 and Corollary 2.3 we obtain

\[ u_\epsilon(r) \geq v(r) \quad \text{for} \quad r \in [0, R) \]

and

\[ \int_{v(r)}^{u_\epsilon(r)} \left( \frac{\phi(s)}{f(s)} \right)^n \, ds \geq \int_{v(r)}^{u_\epsilon(r)} \left( \frac{\phi(s)}{f(s)} \right)^m \, ds \quad \text{for} \quad r \in [0, R). \]
Let $\varepsilon \to 0$ to get

$$u(r) \geq v(r) \quad \text{for} \quad r \in [0, R)$$

and

$$\int_{\varepsilon(r)}^{r} \left( \frac{\phi(s)}{f(s)} \right)^{m} \, ds \geq \int_{\varepsilon(0)}^{0} \left( \frac{\phi(s)}{f(s)} \right)^{m} \, ds \quad \text{for} \quad r \in [0, R).$$

We can now obtain the main comparison theorem, Theorem A. Indeed, since $u$ and $v$ are positive solutions of (1.8) and (1.9), then by (1.11) and (1.12) $u$ and $v$ satisfy the following inequalities

$$\begin{align*}
&\left( p_{1}(r) \phi(u) \mid u' \mid^{m-2} u' \right)' + q_{1}(r) f(u) \geq 0 \\
&u(0) = u_{0}, \quad u'(0) = u'_{0},
\end{align*}$$

and

$$\begin{align*}
&\left( p_{2}(r) \phi(v) \mid v' \mid^{m-2} v' \right)' + q_{2}(r) f(v) \leq 0 \\
v(0) = v_{0}, \quad v'(0) = v'_{0}.
\end{align*}$$

According to Theorem 3.4, the conclusion of Theorem A then follows at once.

It is worth noticing that if $p_{1} \equiv p_{2}$ and $u_{0} = v_{0}$ in the problems (2.1) and (2.2), then ($H_{4}$) holds automatically. That is, if $u'_{0} \geq 0$, then by ($A_{3}$), $u'_{0} \geq v'_{0}$ and $u_{0} = v_{0}$ we get

$$\frac{\phi^{m+1}(v_{0}) \mid v'_{0} \mid^{m}}{f^{m+1}(v_{0})} \geq \frac{\phi^{m+1}(u_{0}) \mid u'_{0} \mid^{m}}{f^{m+1}(u_{0})}.$$ 

Since $p_{1} \equiv p_{2}$, it follows that

$$p_{2}(r) \frac{\phi^{m+1}(v_{0}) \mid v'_{0} \mid^{m}}{f^{m+1}(v_{0})} \geq p_{1}(r) \frac{\phi^{m+1}(u_{0}) \mid u'_{0} \mid^{m}}{f^{m+1}(u_{0})} \quad \text{for} \quad r \in [0, R),$$

which is exactly ($H_{4}$).

We conclude with a simple application of Theorem A. Further applications of the comparison theorem are given in section 4. Consider the equation

$$( p(r) \phi(u) \mid u' \mid^{m-2} u' \right)' + q(r) f(u) = 0 \quad (3.13)$$

with initial conditions

$$u(0) > 0 \quad \text{and} \quad u'(0) \leq 0.$$
The uniqueness of positive solutions is hard to derive from the standard theory of ordinary differential equations. Using Theorem A, however, the following uniqueness theorem holds.

**Corollary 3.5.** Suppose that $u$ is a positive solution of (3.13), and $(H_1) - (H_3)$ and $(A_1) - (A_3)$ hold for $\phi, f, p$ and $q$, then $u$ is unique.

**Proof.** Suppose that $u$ and $v$ are two solutions of (3.13). Then we have

$$\begin{cases}
(p(r) \phi(u) |u'|^{m-2} u')' + q(r) f(u) = 0 \\
u(0) = u_0, \quad u'(0) = u'_0
\end{cases}$$

and

$$\begin{cases}
(p(r) \phi(v) |v'|^{m-2} v')' + q(r) f(v) = 0 \\
v(0) = u_0, \quad v'(0) = u'_0
\end{cases}$$

with $u'_0 \leq 0$ and $u_0 > 0$. Let

$$R = \max \{ r \geq 0 | u(r) > 0 \text{ and } v(r) > 0 \}.$$ 

Since $p_1 = p_2 = p$ and $q_1 = q_2 = q$, then $(H_4)$ holds automatically as shown above, hence Theorem A can be applied to couples $\{u, v\}$ and $\{v, u\}$ on $[0, R)$. This gives

$$u \geq v \quad \text{and} \quad v \geq u \quad \text{on} \quad [0, R),$$

which implies that

$$u \equiv v \quad \text{on} \quad [0, R).$$

**Remark.** Some of the assumptions can be weakened. Indeed, a similar comparison theorem can be proved when $u'(0) = -\infty$ and $u \in C^1(0, R)$. If $(H_3)$ is replaced by the weaker hypothesis that either

$$q_i(r) \in C^0[0, \infty) \quad \text{and} \quad q_2(0) > q_1(0) \geq 0$$

or

$$r^{1-N}q_i(r) \in C^0[0, \infty) \quad \text{and} \quad \lim_{r \to 0} r^{1-N}q_2(r) > \lim_{r \to 0} r^{1-N}q_1(r) \geq 0,$$

then the same comparison theorem can also be proved. Finally, with more smoothness assumptions on $f$ and $\phi$, a Gronwall type inequality can be proved for (1.6), see [4]. In this case, Corollary 3.5 can be proved in a different way.
4. APPLICATIONS

In this section, we present some applications of our main comparison theorem to nonlinear problems. Sturm type comparison theorems have been successfully used by many authors to obtain existence and uniqueness theorems for nonlinear elliptic equations, see for instance, [2, 3, 5, 9, 11, 12]. Here we establish some further results for existence and uniqueness of positive radial solutions of the $m$-Laplace equation.

Consider the equation

$$(r^{N-1} |u'|^{m-2} u')' + r^{N-1} k(r) f(u) = 0, \quad (4.1)$$

where $k$ is a nonnegative function in $C[0, \infty)$ and $f$ satisfies $(A_1)$. Since $\phi(u) \equiv 1$, then $(A_2)$ holds automatically and $(A_3)$ reduces to

$$(A_3)' \quad df/du \text{ is non-increasing in } (0, \infty).$$

By the Schauder fixed point theorem, see [10], there is a local solution of the initial value problem for (4.1). Moreover, this solution is unique if $u(0) > 0$ and $u'(0) = 0$, as long as it remains positive, see the Appendix [4], or Corollary 3.5.

**Theorem 4.1.** Suppose that $u$ and $v$ are two positive solutions of the initial value problem for (4.1) on $[0, R)$, with initial conditions $u(0) > v(0) > 0$ and $u'(0) = v'(0) = 0$. Then $u$ and $v$ do not intersect on $[0, R)$.

**Proof.** Since $u(0) > v(0) > 0$ and $u'(0) = v'(0) = 0$, then by Theorem A we get

$$u(r) \geq v(r) \quad \text{for} \quad r \in [0, R).$$

Suppose that there is point $a \in (0, R)$ such that

$$u(a) = v(a) \quad \text{and} \quad u(r) > v(r) \quad \text{for} \quad r \in [0, a).$$

Applying Theorem A on $[0, a)$ yields

$$0 = \int_{r(a)}^{r(a)} f^{-\mu}(s) \, ds \geq \int_{r(a)}^{r(a)} f^{-\mu}(s) \, ds > 0,$$

which is a contradiction.
Remark. Consider the equation
\[ u'' + \frac{u'}{r} + u^2 = 0 \] (4.2)
with initial conditions
\[ u(0) > 0 \quad \text{and} \quad u'(0) = 0. \]

For (4.2), it is easy to check that \( N = 2, \ m = 2, \ \mu/(m-1) = 1 \) and \( df/du = 2u \) is increasing (so that \((A_3)\)' is not verified). Using the computer software *Phase Portrait*, one can find two intersecting positive solutions of (4.2). Therefore, Theorem 4.1 and Theorem A fail for this equation, showing that the condition \((A_3)\)' is somehow inevitable.

1. The \( m \)-Laplace equation on bounded domains in \( \mathbb{R}^N \).

Let \( B_R \) denote an open ball in \( \mathbb{R}^N \) with center at the origin and radius \( R \). We consider the Dirichlet problem on \( B_R \) for the \( m \)-Laplace equation
\[
\begin{aligned}
\nabla \cdot (|\nabla u|^{m-2} \nabla u) + k(|x|) f(u) = 0, \quad m > 1. \\
\end{aligned}
\] (4.3)

It is easy to check that radial solutions of this problem satisfy the ordinary differential equation
\[
(r^{N-1} |u'|^{m-2} u')' + r^{N-1} k(r) f(u) = 0
\]
with boundary conditions
\[ u(R) = 0 \quad \text{and} \quad u'(0) = 0. \] (4.5)

We should point out that the following theorem for the case \( \phi \equiv 1 \) and \( m = 2 \), see [11].

**Theorem 4.2.** Suppose that \( f^{\nu} \in L^1[0, \infty) \). Then (4.3) has at most one radial solution which is positive in \( B_R \).

**Proof.** Suppose that there are two positive solutions \( u \) and \( v \) of (4.4). Without loss of generality, assume \( u_0 > v_0 \). Otherwise, if \( u_0 = v_0 \), then \( u \equiv v \) on \([0, R]\) by the uniqueness of the initial value problem. From Theorem A we get
\[
\int_{u(r)}^{v(r)} f^{-\nu}(s) \, ds \geq \int_{u_0}^{v_0} f^{-\nu}(s) \, ds \quad \text{for} \quad r \in [0, R].
\]
Since $f^{-\mu} \in L^1[0, \infty)$ and $v(R) = u(R) = 0$, it follows easily that
\[
\int_0^\infty f^{-\mu}(s) \, ds \geq \int_0^\infty f^{-\mu}(s) \, ds.
\]
This contradicts the assumption $u_0 > v_0$. Thus the proof is completed.

It is well-known that for the linear equation
\[
u'' + \lambda^2 u = 0 \quad \text{with} \quad \lambda \neq 0,
\]
the distance from the origin to the first zero of any solution when $u(0) > 0$ and $u'(0) = 0$ is equal to the positive constant $\pi/2 |\lambda|$. Next we prove an analogous result for nonlinear equations of type (4.1), the result being a lower bound for the distance from the origin to the first zero.

**Theorem 4.3.** Suppose that $u$ is a positive solution of the problem
\[
\begin{cases}
(r^{N-1}|u'|^{m-2}u')' + r^{N-1}g(r, u) = 0 \\
u(R) = 0, \quad u'(0) = 0
\end{cases}
\]
with
\[
0 \leq g(r, u) \leq Mu^{m-1} \quad \text{for} \quad u \geq 0 \quad \text{and} \quad r \in [0, R).
\]
Then
\[
R \geq \left( \frac{N}{M} \right)^{1/m} \left( \frac{m}{m-1} \right)^{(m-1)/m}.
\]

**Proof.** If $u$ is a positive solution of (4.6), then obviously on the interval $[0, R)$ we have $u' \leq 0$. By (4.6) and (4.7) also we see that $u$ solves the problem
\[
\begin{cases}
(r^{N-1}|u'|^{m-2}u')' + r^{N-1}Mu^{m-1} \geq 0 \\
u(0) = u_0, \quad u'(0) = 0.
\end{cases}
\]
Let
\[
v(r) = c(b - r^{m(m-1)})
\]
with
\[
b = \left( \frac{N}{M} \right)^{1/m} \left( \frac{m}{m-1} \right) \quad \text{and} \quad c = u_0/b.
\]
It is easy to see that \( v(0) = u_0 \) and \( v'(0) = 0 \). From direct calculation we obtain

\[
\begin{align*}
(r^{N-1} |v'|^{m-2} v')' + r^{N-1} M v^{m-1} &\leq 0 \\
v(0) = u_0, \quad v'(0) = 0.
\end{align*}
\]  
(4.9)

Applying Theorem A to (4.8) and (4.9) yields

\[ u(r) \geq v(r) \quad \text{for} \quad r \in [0, R). \]

Since \( v(r) > 0 \) for \( 0 \leq r < (N/M)^{1/m} \cdot [m/(m-1)]^{(m-1)/m} \), it follows that

\[ R \geq \left( \frac{N}{M} \right)^{1/m} \cdot \left( \frac{m}{m-1} \right)^{(m-1)/m}. \]

We can now prove Theorem B. It is clear that positive radially symmetric solutions of (1.14) solve the problem (4.6), and that \( g(|x|, u) \) satisfies (4.7). If there is a positive radial solution of (1.14) when \( R < (N/M)^{1/m} \cdot [m/(m-1)]^{(m-1)/m} \), we get a contradiction. Thus the proof is completed.

If \( N = 1, m = 2 \) and \( g(r, u) = u \), Theorem 4.3 implies that the distance from the origin to the first zero of any solution of (4.4) is greater than \( \sqrt{2} \), the exact distance is \( \pi/2 \), greater than \( \sqrt{2} \). For general equation (4.4), of course, it seems impossible to find an exact value for the distance from the origin to the first zero of solutions of (4.4). Comparing Theorem B with Theorem 2.1 in [3], we see that the bound (1.15) is better then that in [3], namely \( R \geq (1/M)^{1/m} \cdot [m/(m-1)]^{(m-1)/m} \). It is worth noticing that (1.15) depends not only on \( m \) but also the dimension \( N \).

2. The \( m \)-Laplace equation in \( \mathbb{R}^N \).

Many results have been obtained for the existence and uniqueness of positive entire solutions of the problem (4.3) when \( m = 2 \), see for instance [5, 8, 10, 13]. We now consider the \( m \)-Laplace equation (4.3) in \( \mathbb{R}^N \). It is easy to see that radial solutions of (4.3) satisfy the ordinary differential equation (4.4). The following theorem for the \( m \)-Laplace equation is analogous to existence theorems given in [8] and [13].

**Theorem 4.4.** If there exists a positive number \( \xi \) such that

\[
\frac{\xi}{f'(|\xi|)} \geq \int_0^\infty \left( \int_0^r \left( \frac{s}{r} \right)^{N-1} k(s) \, ds \right)^m \, dr,
\]

then there are infinitely many radial solutions of (4.3) which are positive in \( \mathbb{R}^N \).
Proof. By a standard local existence theorem, see [10], there is a positive function \( u \) on \([0, R)\) such that

\[
\begin{cases}
(r^{N-1} |u'|^{m-2} u')' + r^{N-1} k(r) f(u) = 0 \\
u(0) = \xi, \quad u'(0) = 0,
\end{cases}
\]  

(4.11)

with

\[ R = \max \left\{ r > 0 \mid u(r) > 0 \right\}. \]

Now set

\[ v(r) = \xi - f''(\xi) \int_0^r \left( \int_0^s \left( \frac{s}{r} \right)^{N-1} k(s) \, ds \right)^{\frac{1}{N-1}} \, dr. \]

Then \( v \) is a \( C^1 \) function on \([0, \infty)\) and also a solution of the problem

\[
\begin{cases}
(r^{N-1} |v'|^{m-2} v')' + r^{N-1} k(r) f(v) = 0 \\
v(0) = \xi, \quad v'(0) = 0.
\end{cases}
\]  

(4.12)

By (4.11) we get \( u' \leq 0 \), so that \( u \leq \xi \). Condition \((A_1)\) implies that \( f(u)/f(\xi) \leq 1 \). Therefore, \( u \) solves the problem

\[
\begin{cases}
(r^{N-1} |u'|^{m-2} u')' + r^{N-1} k(r) f(u) \geq 0 \\
u(0) = \xi, \quad u'(0) = 0.
\end{cases}
\]  

(4.13)

Since \( v(r) \geq 0 \) for \( r \in [0, \infty) \) by the hypothesis (4.10), then using Theorem A, and taking \( p_1 = p_2 = r^{N-1} \), \( q_1 = q_2 = r^{N-1} k(r) f(\xi) \), we get \( u \geq v > 0 \) for \( r > 0 \). Therefore \( R = \infty \), and \( u \) is a positive entire solution of (4.11). Now consider the problem

\[
\begin{cases}
(r^{N-1} |w'|^{m-2} w')' + r^{N-1} k(r) f(w) = 0 \\
w(0) > \xi, \quad w'(0) = 0.
\end{cases}
\]  

(4.14)

Again applying Theorem A to (4.11) and (4.14), we obtain \( w(r) \geq u(r) \) for \( r \in [0, \infty) \). Thus problem (4.3) admits infinitely many radial solutions which are positive in \( \mathbb{R}^n \).

Remark. Consider the quasilinear equation

\[ \nabla \cdot (\phi(u) |\nabla u|^{m-2} \nabla u) + k(\epsilon |x|) f(u) = 0, \quad m > 1. \]
If condition (4.10) is replaced by
\[
\int_0^1 \left( \frac{\phi(s)}{f(s)} \right)^\mu (s) \, ds > \int_0^\infty \left( \frac{s}{r} \right)^{N-1} k(s) \, ds, \]
then the conclusion of Theorem 4.4 still holds.

We end the paper with a uniqueness theorem for positive radial solutions of (4.3) in \( \mathbb{R}^N \).

**Theorem 4.5.** Suppose that both \( u \) and \( v \) are positive radial solutions of (4.3) in \( \mathbb{R}^N \). If
\[
\lim_{r \to \infty} u(r) = \lim_{r \to \infty} v(r) = x \geq 0 \quad (4.15)
\]
and
\[
f^{-\mu} \in L^1[0, \infty) \quad \text{if} \quad x = 0, \quad (4.16)
\]
then \( u \equiv v \) on \( [0, \infty) \).

**Proof.** By (4.4) we see that \( u' \leq 0 \) and \( v' \leq 0 \). Moreover,
\[
u_0 \geq u > 0 \quad \text{and} \quad v_0 \geq v > 0 \quad \text{on} \quad [0, \infty).
\]
Without loss of generality, assume that \( u_0 > v_0 \). Otherwise, if \( u_0 = v_0 \), then \( u \equiv v \) by the uniqueness of the initial value problem. Applying Theorem A yields
\[
\int_{r(\xi)}^{r(\eta)} f^{-\mu}(s) \, ds \geq \int_{r(\xi)}^{r(\eta)} f^{-\mu}(s) \, ds,
\]
Now let \( r \to \infty \) to get
\[
\int_{\alpha}^{\beta} f^{-\mu}(s) \, ds \geq \int_{\alpha}^{\beta} f^{-\mu}(s) \, ds. \quad (4.17)
\]
By (A₁) and (4.16), it is easy to see that (4.17) contradicts the assumption \( u_0 > v_0 \). Thus the proof is completed.

5. Appendix

**Proof of Lemma 3.3.** For \( \varepsilon > 0 \), define
\[
\psi_\varepsilon = A_\varepsilon |A_\varepsilon|^{\mu-1} \quad (5.1)
\]
where

\[ A_{2} = \phi(u) |u'|^{m-2} u' + \varepsilon \delta. \]  

(5.2)

Since \( u \in C^1([0, R]) \) and \( \delta \in C([0, R]) \), one checks easily that \( \psi_s \) is continuous on \([0, R]\), with \( \psi_s(0) = [\phi(u_0)]^m u_0' \). We extend \( \psi_s \) to be continuous on \((-R, R)\) by setting \( \psi_s(r) = \psi_s(0) \) for \( r < 0 \).

\[ h(r, u) = \int_{s_0}^{s} [\phi(s)]^{n} ds - \int_{0}^{r} \psi_s(s) ds \quad \text{for} \quad r \in (-R, R) \quad \text{and} \quad u \geq 0. \]  

(5.3)

It is easy to see that

\[ h(r, u) \in C^1((-R, R) \times (0, \infty)) \quad \text{and} \quad h(0, u_0) = 0. \]

We now divide the proof into three steps.

**Step 1.** Differentiate (5.3) with respect to \( u \) and set \( r = 0, u = u_0 \) to get

\[ \frac{\partial h}{\partial u}(0, 0) = [\phi(u_0)]^n > 0. \]

According to the implicit function theorem, there exists an open interval \( J \) containing 0 and a positive function \( u = u_0 \in C^1(J) \) such that

\[ u_s(0) = u_0 \quad \text{and} \quad h(r, u_s(r)) = 0 \quad \text{for} \quad r \in J. \]

Since \( \phi(s) > 0 \) for \( s > 0 \), we can extend \( u_s \) onto \([0, R_s]\), where

\[ R_s = \max \{ r \in (0, \infty) | u_s(r) > 0 \}. \]

Therefore,

\[ \int_{s_0}^{u_s(r)} [\phi(s)]^{n} ds = \int_{0}^{r} \psi_s(s) ds \quad \text{for} \quad r \in [0, R_s]. \]  

(5.4)

**Step 2.** Differentiating (5.4) with respect to \( r \) yields

\[ [\phi(u_s)]^m u_s' = \psi_s \quad \text{on} \quad [0, R_s]. \]

By (5.1) and (5.2) we get

\[ \phi(u_s) |u_s'|^{m-2} u_s' = \phi(u) |u'|^{m-2} u' + \varepsilon \delta \quad \text{on} \quad [0, R_s]. \]  

(5.5)
From \((H_3)\) and \((5.5)\), it is obvious that \(u_t(0) = u'(0)\). Since \(\delta \geq 0\) on \([0, \infty)\), it follows that

\[
\phi(u_t)|u_t'|^{m-2} u_t' \geq \phi(u) |u'|^{m-2} u' \quad \text{on} \quad [0, R_\epsilon). \tag{5.6}
\]

If \(u_t' \geq 0\) and \(u' \geq 0\), then by raising each side to the power \(\mu\) we get

\[
[\phi(u_t)]^\mu u_t' \geq [\phi(u)]^\mu u' \quad \text{on} \quad [0, R_\epsilon). \tag{5.7}
\]

If \(u_t' \geq 0\) and \(u' < 0\), then \((5.7)\) is obvious. For the case \(u_t' < 0\) and \(u' < 0\), taking absolute values for both sides of \((5.6)\), then by raising each side to the power \(\mu\), \((5.7)\) follows at once. Integrating both sides of \((5.7)\) with respect to \(r\) gives

\[
\int_{m_0}^{\mu(r)} [\phi(s)]^\mu ds \geq \int_{m_0}^{\mu(r)} [\phi(s)]^\mu ds.
\]

Hence \(u_t \geq u\) on \([0, R_\epsilon)\).

**Step 3.** It remains to prove the convergence \(u_t \to u\) as \(\epsilon \to 0\). Put

\[
\psi_0 = [\phi(u)]^\mu u'. \tag{5.8}
\]

We shall show that

\[
\psi_t \to \psi_0 \quad \text{as} \quad \epsilon \to 0 \quad \tag{5.9}
\]

uniformly on \([0, \bar{R}]\) for any fixed \(\bar{R} \in [0, R]\). There are several cases to consider.

First, suppose that \(A_t \geq 0\). If also \(u_t' \geq 0\) and \(\mu \geq 1\), then by the mean value theorem there is a constant \(C_t\) depending on supremum of the continuous functions \(\phi(u), u'\) and \(\delta\) on \([0, \bar{R}]\), such that

\[
|\psi_t - \psi_0| = |[\phi(u) |u'|^{m-2} u' + \epsilon \delta]^\mu - [\phi(u) |u'|^{m-2} u']^\mu| \leq C_t \epsilon.
\]

Next, if \(u_t' \geq 0\) and \(0 < \mu < 1\), then by the inequality

\[
x^{y} \geq (x + y)^{y} - y^{y} \quad \text{for} \quad x \geq 0 \quad \text{and} \quad y \geq 0,
\]

we get \(|\psi_t - \psi_0| \leq (\epsilon \delta)^{\mu} \). Therefore, putting \(C_2 = \sup \{\delta^{\mu}(r) | r \in [0, \bar{R}]\}\),

\[|\psi_t - \psi_0| \leq C_2 \epsilon^{\mu}.
\]

Finally, if \(u_t' < 0\), then by \((5.1), (5.2)\) and \((5.8)\) we obtain for all \(\mu > 0\)

\[
|\psi_t - \psi_0| \leq (\epsilon \delta)^{\mu} + [\phi(u)]^\mu |u'|.
\]
Since $A_+ \geq 0$ and $u' < 0$,

\[ [\phi(u)]^n |u'| \leq (a\delta)^n, \]

so

\[ |\psi_r - \psi_0| \leq 2C_+ v^n. \]

Combining the previous estimates now gives

\[ |\psi_r - \psi_0| = O(v + v^n) \tag{5.10} \]

for $r \in [0, \tilde{R}]$ and $A_+ \geq 0$.

Last, we consider $A_+ < 0$. Hence necessarily $u' < 0$. By (5.1), (5.2) and (5.8) we get

\[ |\psi_r - \psi_0| = \left| \left[\phi(u) \ |u'|^{m-1} - \varepsilon \delta \right]^n - \left[\phi(u) \ |u'|^{m-1} \right]^n \right|. \]

Then exactly as in the case $u' \geq 0$ above, we obtain

\[ |\psi_r - \psi_0| = O(v + v^n) \tag{5.11} \]

for $r \in [0, \tilde{R}]$ and $A_+ < 0$. Combining (5.10) and (5.11) gives (5.9).

Therefore, by (5.9) and the uniform convergence theorem we get

\[ \lim_{r \to 0} \int_0^r \left| \psi_r(s) - \psi_0(s) \right| ds = 0 \quad \text{for} \quad r \in [0, \tilde{R}). \tag{5.12} \]

Integrating (5.8) form 0 to $r$ yields

\[ \int_0^r \psi_0(s) ds = \int_{s_0}^{s(r)} \left[ \phi(u) \right]^n \quad \text{for} \quad r \in [0, \tilde{R}). \tag{5.13} \]

Hence, from (5.4), (5.12) and (5.13)

\[ \lim_{r \to 0} \int_{s_0}^{s(r)} \phi(u) \right]^{r} ds = \int_{s_0}^{s(r)} \phi(u) ds. \]

This implies

\[ \lim_{r \to 0} u_s(r) = u(r). \]

Thus the proof is completed.
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