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Doubling the Cube: A New Interpretation of Its Significance for Early Greek Geometry

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It is widely known that Hippocrates of Chios reduced the problem of doubling the cube to the problem of finding two mean proportionals between two given lines. Nothing, however, is known about how this reduction was justified. To answer this question, propositions and patterns of arguments in Books VI, XI, and XII of the *Elements* are examined. A reconstruction modelled after Archimedes' *On Sphere and Cylinder*, Proposition II-1, is proposed, and its plausibility is discussed. © 1995 Academic Press, Inc.

Il est admis qu'Hippocrate de Chio a réduit la duplication du cube au problème de la recherche de deux moyennes proportionnelles entre deux segments donnés, mais rien n'est certain quant à la justification de cette réduction. Pour répondre à cette question, sont examinées les propositions comme les argumentations caractéristiques des livres VI, XI, et XII des *Eléments*. Une reconstruction sera proposée d'après *De la sphère et du cylindre*, proposition II-1 d'Archimède, et son caractère plausible sera discuté. © 1995 Academic Press, Inc.

Es ist wohlbekannt, daβ Hippocrates von Chios das Problem der Verdoppelung des Würfels auf das Problem der Auffindung von zwei mittleren Proportionalen zwischen zwei gegebenen Geraden zurückführte. Jedoch ist nichts bekannt über die Rechtfertigung dieser Reduktion. Um diese Frage zu beantworten, werden Sätze und Strukturen von Argumenten aus den Büchern VI, XI, und XII der *Elemente* untersucht. Eine Rekonstruktion wird dann vorgeschlagen, die sich auf Satz II-1 in Archimedes *Über Kugel und Zylinder* stützt, und ihre Plausibilität wird diskutiert. © 1995 Academic Press, Inc.

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1. INTRODUCTION

The problem of doubling the cube is one of the central problems in Greek mathematics. Since it attracted many mathematicians of the period, we possess exceptionally rich ancient materials on this problem. As is well known, Hippocrates of Chios (fl. ca. 440 B.C.) is said to have originated the tradition of investigating cube duplication. Several sources credit him with reducing this problem to another, that of finding two mean proportionals between two given lines. For example, in his commentary to Archimedes' *On Sphere and Cylinder* (hereafter SC), Eutocius cites Eratosthenes:

> And it was sought among the geometers in what way one could double the given solid, keeping it in the same shape, and they called this sort of problem the duplication of the cube. And when they all puzzled for a long time, Hippocrates of Chios first conceived that if, for two

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> given lines, two mean proportionals were found in continued proportion, the cube will be doubled. Whence he turned his puzzle into another no less puzzling.[1]

Hereafter we call this transformation of cube duplication into the problem of finding two mean proportionals "Hippocrates' reduction." It was in this reduced form that the solution of cube duplication was sought during antiquity. The traditions of these ancient studies have recently been so thoroughly studied by Knorr [9, 11–153] that it seems as if nothing further could be teased out of the extant materials.

In this article, however, we concentrate on a problem about which almost all the ancient sources are strangely taciturn: how cube duplication was reduced to finding two mean proportionals between two given lines. Strangely enough, in contrast to the abundance of solutions for inserting two mean proportionals, we have little testimony on how Hippocrates' reduction was proved in antiquity. No extant text explains how Hippocrates arrived at this reduction. It would therefore seem worthwhile to try to understand the kind of arguments he was likely to have made.[2] Throughout this paper, we will be less concerned with the heuristic context that motivated Hippocrates' work than with the justification of the discovery, the proof of the equivalence of cube duplication to finding two mean proportionals. Our concern will lead us to propose a new interpretation of the status of the theory of proportion in the early stage of Greek mathematics, where we will highlight a group of propositions which may have served as a tool for problem-solving (including that of cube duplication) for the mathematicians of the period.

2. THE PROBLEM AND ITS REDUCTION

First, let us briefly analyze the problem of doubling the cube, taking into account ancient testimonies and modern studies on its reduction. Cube duplication is a problem that appears to have been originally stated as follows:

[CD] To find a cube which is twice as large (in size, not side) as a given one.

Fairly early on, however, this problem was studied in the following generalized form:

[CDx] To find a cube whose ratio to a given cube equals the ratio of two given lines (not necessarily in the ratio 2 : 1) (see also [8, 23]).

This generalized problem was then reduced to the following problem:

[1] [5, 3:88]. We cite Knorr's translation in [9, 147]. Knorr persuasively asserts the genuineness of Eratosthenes' accounts. See [8, 17–24] and [9, 131–153].

[2] It might be doubted if Hippocrates really proved the truth of his reduction, since Eratosthenes simply says that Hippocrates "first conceived" the truth of the reduction. Therefore, the reader with maximum reserve should read the phrase "Hippocrates' proof (or justification) of his reduction" in this paper as "the first rigorous proof of Hippocrates' reduction." This proof was surely found very soon, if not by Hippocrates himself, since it was concerned with such an important problem as cube duplication. Archytas already contrived a method to insert two mean proportionals [5, 3:84–88; 9, 100–110], so that we may assume that the proof was already available to Archytas. It seems to us that the attribution of both the discovery and the proof of the "reduction" to Hippocrates himself fits better with other testimonies which rank him as one of the great mathematicians of the period.

[CDxr] To find two mean proportionals between two given lines.

Thus given two lines a, b, one must find x, y, such that $a:x = x:y = y:b$. The last problem, [CDxr], was usually investigated by the ancients under the name of cube duplication, although strictly speaking, this only covers the case $b = 2a$.

We have a few sources that demonstrate the truth of Hippocrates' reduction, and they are based on the use of the concept of triplicate ratio (as used in a proposition like *Elements* XI-33), or the compounding of ratios. We can confirm this in Diocles [15, 102], Pappus [6, 1:66–68] (see also [9, 90–91]) and Philoponus [18, 104–105] (for translation, see [9, 20]). We cannot, however, regard these arguments as reflecting the first proof of Hippocrates' reduction for reasons that will be explained below.

Some of the modern reconstructions follow the same lines as these authors of late antiquity. For example, Knorr explains the equivalence of cube duplication [CDx] and its reduction [CDxr] as follows:

> If for any two given lines, A and B, we can insert the two mean proportionals, X and Y, then $A:X = X:Y = Y:B$. Thus, by compounding the ratios, one has $(A:X)^3 = (A:X)(X:Y)(Y:B)$, that is, $A^3:X^3 = A:B$. Thus, X will be the side of a cube in the given ratio (B : A) to the given cube (A^3). [8, 23]

In this passage, Knorr is less interested in restoring Hippocrates' line of thought than in convincing modern readers of the truth of Hippocrates' reduction. Indeed, he does not pretend to restore any ancient argument. Using modern notations, he assumes the equivalence of triplicate ratio with the operation of compounding the same ratio three times,[3] and he further assumes that the triplicate ratio of two lines is equal to the ratio of the cubes on them. The truth of the former, though evident to us, is not explicitly proved in the extant *Elements*, while the latter is a special case of *Elements* XI-33.[4]

In earlier studies [12; 13], we have argued that the concepts of duplicate and triplicate ratio, as well as that of compound ratio, appear to have been relatively late arrivals in the corpus of Greek mathematics, and that the same holds true for Proposition XI-33. If this thesis is correct, then these notions were certainly not available to Hippocrates when he formulated his reduction of cube duplication. It would, therefore, be worthwhile to search for a proof without these notions.[5]

In this context, we should note another way to formulate cube duplication which must have been clear to Greek geometers from the time of Hippocrates. The problem [CDx] can easily be represented in the following more geometric form:

[CDxg] Given a rectangular prism on a square base, to find a cube equal to it.

[3] Knorr is far from the first to make this assumption. For example Heath, in his account of Hippocrates' reduction, states that "Hippocrates could work with compound ratios" [3, 1: 200].

[4] Another modern reconstruction in [1, 57ff.] is based on the corollary of XI-33.

[5] In this respect, two modern reconstructions [16, 119; 14, 128] are to be noted. These authors do not resort to multiplicate and compound ratios, but use instead more fundamental theorems. However, as they did not intend to offer historical reconstructions, they offered no textual evidence for these arguments.

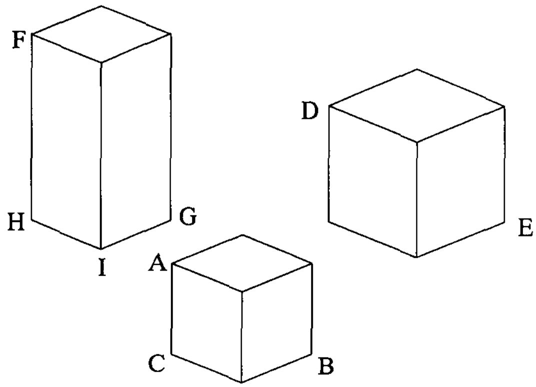

FIGURE 1

Obviously, the solution of [CDxg] implies that of [CDx]. Let there be given a cube AB and two lines a, b (see Fig. 1). To find a cube whose ratio to AB is the given ratio $a:b$, one needs only to construct a rectangular prism FG with square base HG congruent to BC and whose height FH is such that FH : AC = $a:b$. It follows immediately that if cube DE is equal to prism FG, then DE : AB = $a:b$.

Thus, the problem [CDx] is reduced to [CDxg]. In this respect, cube duplication is a natural analogue of the problem of squaring a rectangle in plane geometry. We therefore begin by examining the problem of squaring, and its relationship to cube duplication.

3. THE PROBLEM OF SQUARING

This section examines the problem of squaring a rectangle, the counterpart of cube duplication in plane geometry. The first problem is solved by finding one mean proportional, whereas the second requires finding two mean proportionals (see [3, 1:201; 8, 22]).

In the *Elements*, the problem of squaring a rectangle is treated in II-14. In Book VI, the same construction is presented as a method for finding one mean proportional between two given lines. In VI-17, the problem of squaring a given rectangle is reduced to another problem, that of finding a mean proportional between two given lines. This step corresponds to Hippocrates' reduction of cube duplication. Then, the problem of finding one mean proportional is solved by means of the construction in VI-13, which is identical with that given in II-14. The counterpart of this step in cube duplication is finding two mean proportionals. It is important to note that proposition VI-17, the counterpart to Hippocrates' reduction, does not depend on propositions concerning duplicate or compound ratios. In other words, a reduction of cube duplication using triplicate ratio or compound ratio, such as that given by Diocles or Pappus, is a different sort of solution from Euclid's solution of squaring.

So what means did Hippocrates use in his reduction? To answer this question, it is necessary to know which propositions he had at his disposal. Let us therefore examine the propositions concerning ratios between figures both in Book VI (plane figures) and in Books XI and XII (solid figures).

Book VI of the *Elements* contains several propositions concerning ratios between plane figures. Here we choose those relevant to the problem of squaring, and classify these propositions according to the scheme introduced by Mueller [10]:

D	base-area proportionality	VI-1
F	equal-area propositions	VI-14,15,16,17
G	duplicate-ratio between similar figures	VI-19,20
J	compound-ratio proposition	VI-23

The letters in the first column indicate the symbols used by Mueller[6] to denote each type of proposition. The second column gives a short explanation of each type of proposition. Thus, type D refers to the proportionality relationship between base and area in two parallelograms or in two triangles of equal height. Only Proposition VI-1 belongs to this type and it constitutes the basis for all other propositions. Although Proposition VI-1 is concerned with the relationship between base and area, not with that involving height and area, the corresponding height–area proportionality can be established through VI-1, with the aid of VI-4. In the solid geometry of Books XI and XII, however, D (base–volume proportionality) and E (height–volume proportionality) are treated as distinct propositions.

Type F, which we call "equal-area propositions," deserves a closer look. The first of these is:

> In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal. [*Elements* V1-14]

For the proof, Euclid begins with two equiangular parallelograms, AB and BC (see Fig. 2). By using VI-1, he proves that if AB = BC then DB : BE = GB : BF, and *vice versa*. Proposition VI-15 proves the corresponding theorem for triangles that have one angle equal. Proposition VI-16 treats the special case of VI-14, where the parallelograms are rectangles. VI-17 then handles the particular case of VI-16 in which one of the rectangles is a square. The proof of VI-17 depends on propositions VI-14 through VI-16.

Type G contains two propositions (VI-19, VI-20) proving that similar plane figures are in the duplicate ratio of their corresponding sides.. Finally, type J contains the only theorem in Book VI concerning compound ratios (VI-23: equiangular parallelograms are in the ratio compounded of the ratios of their sides). This result has the greatest affinity to the modern formula for the numerical area of a parallelogram.

[6] See [10, 217; Appendix 1]. Some types of propositions in Mueller's classification are omitted since they are of little importance here.

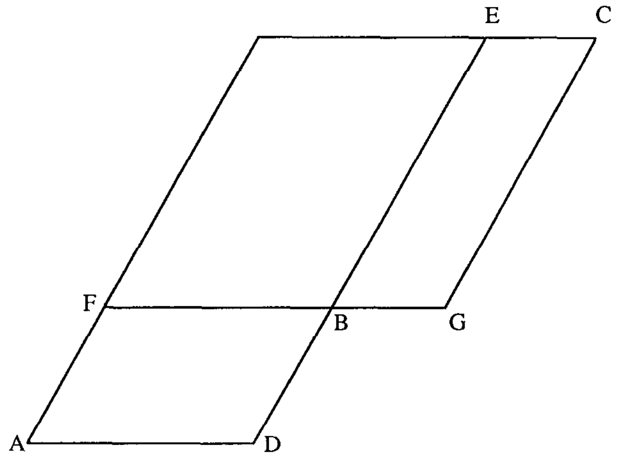

FIGURE 2

De Morgan noted [4, 2:217, 233–234] that the two propositions of types D and J are sufficient to establish all the other propositions concerning ratios between parallelograms. In fact, VI-23 can be proved directly from VI-1, and it thus depends on none of the intermediate results of types F and G. VI-19 is merely a special case of VI-23, whereas the propositions in F could even seem to be trivial corollaries that follow directly from VI-23 (see [10, 161–162]).

It is important to recognize, however, that the logical relations of these propositions, as presented in Book VI, are quite different. First, J (VI-23) is completely isolated in the *Elements*: not only is it never used later but it also depends on the notion of compound ratios, a concept whose definition in Book VI is spurious.

We believe that J was interpolated into an earlier version of Book VI which existed before Eudoxus and Euclid [12, 33–35].[7] Furthermore, the propositions of type G (VI-19, VI-20) also seem to be out of place in Book VI. While VI-19 is used in the proof of VI-20, neither VI-19 nor VI-20 are used anywhere else in Book VI. Instead, the corollary (porism) to VI-19, which avoids the concept of "duplicate ratio," is used to prove VI-22, VI-25, and VI-31. This odd avoidance of type G suggests that the term "duplicate ratio" was also a later insertion into the original version of Book VI.

The very proof of VI-19 supports the plausibility of this interpolation. If one were to demonstrate VI-19 using the full power of the idea of duplicate ratio and following the kind of argument presented in VIII-18 for numbers, it would be natural to proceed as follows:

[7] This interpolation may have been inserted by Euclid himself or by somebody after him. We can only speak of relative chronology here.

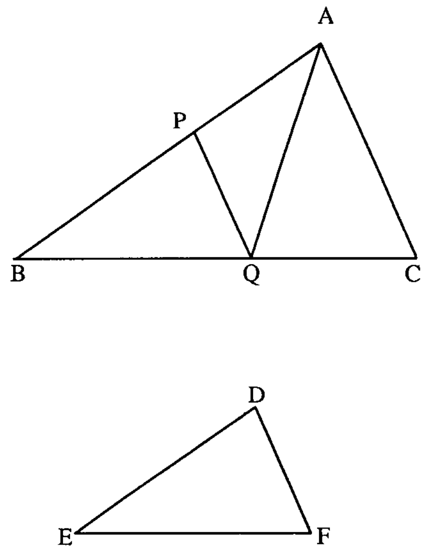

FIGURE 3

Let ABC and DEF be similar triangles (Fig. 3). Construct triangle PBQ congruent with the triangle DEF, and join AQ. Then

$$ABC:ABQ = BC:BQ = BA:BP = ABQ:PBQ = ABQ:DEF,$$

so

$$ABC:ABQ = ABQ:DEF.$$

Therefore, it is evident that the ratio of similar triangles ABC:DEF is the duplicate ratio of their corresponding sides BC:EF (=BC:BQ).[8]

The proof of VI-19 in the *Elements*, however, proceeds quite differently.

Using VI-11, one finds a third proportional BG to BC and EF (Fig. 4):

$$BC:EF = EF:BG.$$

Since

$$AB:DE = BC:EF,$$

one has

[8] This argument tacitly invokes *ex aequali* (V-22). For a more detailed discussion, see [13, 120].

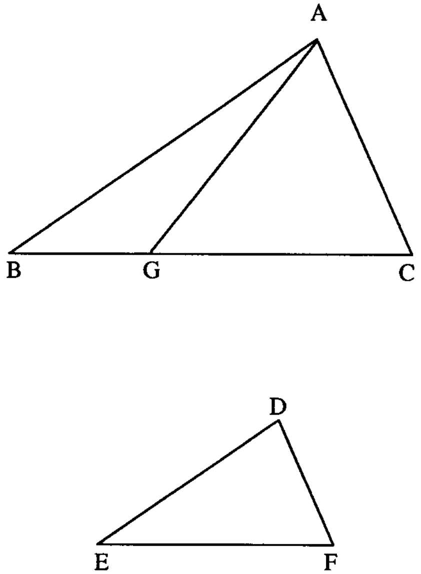

FIGURE 4

$$AB:DE = EF:BG.$$

By F (VI-14), this leads to the equality of two triangles:

$$ABG = DEF.$$

Since, by VI-1, ABC:ABG = BC:BG,

$$ABC:DEF = BC:BG = 2 * (BC:EF).^{9}$$

Thus, Euclid evidently preferred an argument depending on F (VI-14) to a direct proof which he could have formulated after the model of VIII-18. His proof seems to be more appropriate for the corollary to VI-19 rather than for VI-19 itself, because he actually constructs the third proportional in the given figure.[10]

The idiosyncracies we have noted above in Book VI are difficult to account for on mathematical grounds alone. However, they become understandable if we as-

[9] We use this notation for the duplicate ratio, and the triplicate ratio of BC:EF will be written 3 * (BC:EF).

[10] VI-19 and its corollary are as follows:

VI-19. Similar triangles are to one another in the duplicate ratio of the corresponding sides.

VI-19 Corollary. From this it is manifest that, if three straight lines be proportional, then, as the first is to the third, so is the figure described on the first to that which is similar and similarly described on the second.

The corollary refers explicitly to the third proportional, while in the proposition, it is hidden under the term "duplicate ratio."

sume that the application of the theory of proportion to plane geometry began with the problem of squaring simple figures, and that propositions of the types G and J represent the fruits of later developments (for more discussion see [13]).

Although this interpretation may appear speculative, the contrary assumption that the techniques of multiplicate ratios and compound ratios were available at the time of Hippocrates poses a major difficulty: if these notions were familiar over 100 years before Euclid, why are compound ratios so isolated and why are duplicate ratios not exploited more often in the extant text of Book VI of the *Elements*? If we assume, on the other hand, that these concepts were not available to Hippocrates, the question arises: what was Hippocrates' reduction like? To this question, the solid geometry in the *Elements* provides valuable suggestions.

4. PROPOSITIONS CONCERNING THE RATIOS OF SOLID FIGURES IN THE *ELEMENTS*

We now consider Books XI and XII of the *Elements* and analyze the propositions presented therein concerning the ratios between solid figures. The figures treated in these propositions are parallelpipeds, prisms, pyramids, cylinders, cones, and spheres. Here, as in the preceding section, we again make use of Mueller's classification:

		p	tpy	py	c	s
D	base-volume proportionality	XI-32	XII-5	XII-6	XII-11	
E	height-volume proportionality	XI-32			XII-14	
F	equal-volume proposition	XI-34	XII-9		XII-15	
G	triplicate-ratio proposition	XI-33	XII-8		XII-12	XII-18

In this table, the symbols p, tpy, py, c and s in the first line stand for **p**arallelpiped, **t**riangular **py**ramid, **py**ramid (in general), **c**one (and cylinder) and **s**phere, respectively. Propositions like VI-23 (type J, using compound ratios) are absent in the solid geometry of the *Elements*, while those of type E are often independent of type D.

The most striking feature of solid geometry in the *Elements* is that the selections and demonstrations of propositions are not as systematic as one might expect. There is no use of compound ratios although this could greatly simplify the proof of other propositions, as Ian Mueller has noted. In pointing out that Euclid could have proven the theorem that pyramids are to one another in the same ratio as the compound ratio of their bases and heights, Mueller states: "Euclid's failure to prove this extension of XII,9 is perhaps some further confirmation of the view that the connections among compounding, multiplying, and volumes were not so immediately clear to him as they are to us [10, 229]." We should add that the propositions of type J are not the only ones that fail to appear in Book XII. As is seen in the table, most of the propositions concerning pyramids in general are lacking, though these are easily provable.

We now examine the propositions and the logical dependency among them. First, let us analyze the propositions concerning parallelpipeds: Dp and Ep (XI-32), Fp (XI-34), and Gp (XI-33). The basic results Dp and Ep are covered by XI-32, which is the counterpart of VI-1 in solid geometry.[11] Next, Gp (XI-33) and Fp (XI-34) are proved from XI-32. What is striking in these propositions is that Gp appears before Fp, unlike the propositions of types G and F for parallelograms. This reversed order of propositions entails a difference in their proof, of course. Both Gp (XI-33) and Fp (XI-34) are proved directly from Dp and Ep (XI-32). Gp therefore does not depend on Fp. Neither Gtpy nor Gc depends on its corresponding theorem of type F, since they are both proved from Gp.

In examining the propositions on parallelograms, we emphasized the logical dependence of propositions of type G (VI-19, 20) on those of type F (VI-14 to VI-17). We thence claimed that the reduction of squaring to finding one mean proportional (VI-17, a special case of VI-14) was not proved through duplicate ratio.

In solid geometry, this dependence no longer exists. Should we then reject F as irrelevant to cube duplication and assume that the reduction of cube duplication was carried out by means of G (XI-33), taking advantage of triplicate ratio? We do not think so. The independence of Gp from Fp may reflect the period of the draft of these books on solids, which are no doubt later than Book VI. Indeed, the fact that propositions of type F appear in Books XI and XII is significant, because they are no longer necessary. Propositions XI-34, XII-9, and XII-15, which are never used in any substantial way in the *Elements*,[12] suggest again that the theorems of type F reflect the mathematics of an earlier period, as their role in the extant text of the *Elements* is otherwise difficult to explain. We believe this earlier context was intimately connected with the problem of doubling the cube.

To support this thesis, let us now examine Book XII more closely. The propositions of Book XII deal with curved solids, and therefore require the method of exhaustion, which is employed in the proofs of the propositions concerned with cylinders and cones. The method of exhaustion is also used in the last proposition XII-18 (spheres are to one another in the triplicate ratio of their respective diameters), the most sophisticated result on spheres before Archimedes. The propositions on pyramids, which did not require the method of exhaustion, are only useful lemmata for the proof of the propositions on cylinders, cones, and spheres. Clearly, the aim of Book XII was not to provide a compendium of propositions concerning the volumes of pyramids.

[11] Although XI-32 does not explicitly state E, this property is easily derived from XI-32 and VI-1, as Euclid argues in XI-34.

[12] The only use of F, namely of XI-34 in XII-9, does not explain the significance of F, since both XI-34 and XII-9 belong to F.

The method of exhaustion is used in Propositions 2, 5, 11, 12, and 18, and most of the other propositions in Book XII are used as lemmata to establish these five theorems. Only four propositions, namely 9, 13, 14, and 15, have no connection with the method of exhaustion, and they appear to be dead-end propositions with no apparent purpose.[13] We should not, however, simply disregard these propositions as useless. We must first make every effort to discover their significance. What then was the significance of these four propositions? Since they all establish F, the equal-volume propositions (XII-9 for triangular pyramids, and XII-15 for cylinders and cones),[14] it may be assumed that the author had a special interest in establishing propositions of type F.

Our examination of solid geometry in the *Elements* thus again reveals an emphasis on F, the equal-volume propositions. In the next section, we will examine a document that suggests that the equal-volume propositions underpinned Hippocrates' reduction of cube duplication.

5. ARCHIMEDES' *ON SPHERE AND CYLINDER*, PROPOSITION II-1

In this section we examine Archimedes' SC II-1. Its solution is reduced to the same problem as cube duplication: finding two mean proportionals between two given lines. This coincidence inspired Eutocius to add a long commentary on cube duplication. In fact, without Eutocius's commentary, we would have far poorer documentation on the history of cube duplication. We concentrate on this problem in SC because we believe that the coincidence of the solutions of this problem in SC II-1 and of cube duplication reveals a more fundamental affinity.

The first proposition of the second book of Archimedes' *On Sphere and Cylinder* solves the following problem:

SC II-1. Given a cone or cylinder, to find a sphere equal to it.

This proposition consists of two parts: the analysis and the synthesis (though Archimedes does not use these terms here). We examine the analysis, where Archimedes first reduces the problem as follows:

SC II-1′. Given a cylinder, to find an equal cylinder whose height is equal to its diameter.

Archimedes' analysis of II-1′ proceeds as follows.

Let E be the given cylinder (Fig. 5), with height EF, and base diameter CD, and let K be the cylinder to be constructed, with height KL equal to its base diameter GH. Then

$$\text{(circle E)} : \text{(circle K), that is, } \mathrm{sq(CD)} : \mathrm{sq(HG)} = \mathrm{KL} : \mathrm{EF} \qquad (1)$$

and $KL = HG$. Therefore,

[13] See Neuenschwander's diagram of the logical relationships between the propositions in Book XII in [11, 116].

[14] The other propositions, namely XII-13 and XII-14, serve as lemmata for XII-15.

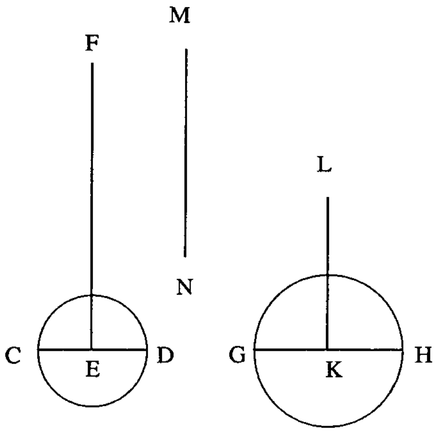

FIGURE 5

$$\mathrm{sq(CD):sq(HG) = HG:EF.} \tag{2}$$

Let [MN be a line such that]

$$\mathrm{sq(HG) = r(CD, MN)} \tag{3}$$

Then

$$\mathrm{CD:MN = sq(CD):sq(HG), \text{that is,} = HG:EF.} \tag{4}$$

Then, alternately,

$$\mathrm{CD:HG = HG:MN = MN:EF.} \tag{5}$$

We now examine each step in Archimedes' argument, filling up small gaps in his exposition. The first step (1) is based on F (the equal-volume proposition) for cylinders (SC Lemma 4 after I-16;[15] *Elements*, XII-15), and the proportionality of the squares and circles (*Elements*, XII-2). The next relation (2) is derived simply from the replacement of KL in (1) by an equal length, HG.

The introduction of the line MN in (3) deserves attention. It is constructed so that the rectangle contained by MN and CD is equal to the square on HG. On the basis of this relation (3), Archimedes then jumps to (4). His small skip would adequately be filled as follows (our reconstruction):[16]

[15] SC Lemma 4 after I-16. In equal cones the bases are reciprocally proportional to the heights; and those cones in which the bases are reciprocally proportional to the heights are equal.

[16] We have to note that Heiberg's reference to Definition 9 of the *Elements* in [5, 1:173, Note 2] is misleading. First of all, Heiberg confuses two notions that are clearly different in Greek mathematics: duplicate ratio (the ratio of the first term to the third when three magnitudes are proportional) and the ratio of squares (see also the text to Note 4, in Section 2). His confusion seems to show that our modern arithmetical approach to geometry using real numbers makes it difficult for us to distinguish them. For further discussion, see [13].

Among modern translations, [2, 182] follows Heiberg in introducing duplicate ratio. We follow Ver Eecke's interpretation in [17, 1:91, Note 2].

Since

$$\mathrm{sq(HG)} = \mathrm{r(CD, MN)} \tag{3}$$

$$\mathrm{sq(CD)} : \mathrm{sq(HG)} = \mathrm{sq(CD)} : \mathrm{r(CD, MN)} = \mathrm{CD} : \mathrm{MN}.$$

Therefore,

$$\mathrm{CD} : \mathrm{MN} = \mathrm{sq(CD)} : \mathrm{sq(HG)}.$$

This is the first half of (4). The rest of (4), sq(CD) : sq(HG) = HG : EF, is the very content of (2).

The deduction of (5) also contains a leap which we supplement in the following manner:

Applying Archimedes' indication "alternately" to the previous relation:

$$\mathrm{CD} : \mathrm{MN} = \mathrm{HG} : \mathrm{EF}, \tag{4}$$

we obtain

$$\mathrm{CD} : \mathrm{HG} = \mathrm{MN} : \mathrm{EF}. \tag{5}$$

The rest of the relation (5), i.e., CD : HG = HG : MN, would be obtained by applying *Elements* VI-16 (another proposition of type F) to the equality (3): sq(HG) = r(CD, MN).

We have now examined every step of Archimedes' analysis in SC II-1. It is worth noting that no recourse is made to either multiplicate ratio (i.e., duplicate and triplicate ratio) or compound ratio.

Archimedes' argument is thus completely clear, in the sense that every proposition he used in each step has been identified. However, we cannot yet be satisfied because we have not yet found an explanation which would make his argument understandable as a whole. Since his argument seems at first sight tortuous, we need to consider the context that may have motivated it.

The affinity of Archimedes' argument with *Elements* VI-19 offers a satisfactory explanation of Archimedes' intention in his analysis of SC II-1. Let us examine this point more closely. In his analysis, Archimedes does not use the propositions concerning multiplicate ratio (type G); their role is replaced by SC Lemma 4 after I-16 which is identical to the *Elements*, XII-15 (type F). In this respect, his argument has much in common with the treatment of squaring in Book VI.

This affinity is far stronger than one might at first imagine. Let us further confirm this point. Both Archimedes and Euclid make use of propositions of type F. Both introduce an auxiliary line which performs an analogous role. This parallelism deserves further explanation. In Archimedes' analysis, the introduction of the line MN has a twofold significance: on the one hand, a proportion

$$\mathrm{CD} : \mathrm{HG} = \mathrm{HG} : \mathrm{MN}$$

is derived from the equality of areas through VI-16 (type F) of the *Elements*. On the other hand, since it supposes an equality of areas (3): sq(HG) = r(CD, MN), it enables one to reduce the ratio between squares sq(CD) : sq(HG) to that between lines:

$$sq(CD) : sq(HG) = sq(CD) : r(CD, MN) = CD : MN.$$

Looking at VI-19 of the *Elements* (see Fig. 4), one notes that the auxiliary line BG thereby has exactly the same twofold role in the proof: to introduce (i) a proportionality (BC : EF = EF : BG) and (ii) an equality of areas which leads to the reduction of a ratio between areas to that between lines (ABC : DEF = BC : BG). There is a perfect parallelism between these arguments. The only difference is that, in VI-19, the line is introduced not through an equality of areas, as in step (3) of Archimedes' analysis, but as the third proportional to two given lines, just as in step (6) of the same analysis. This difference, however, does not signify a substantial difference between these two arguments. Since VI-19 of the *Elements* is a theorem, while Archimedes' passage is an analysis, the arguments are necessarily in reverse order. Thus, SC II-1 and *Elements* VI-19 employ the same technique of introducing an auxiliary line segment.

In a previous article, we have pointed out the use of this same technique in Euclid's *Data*, Proposition 68, and in Apollonius's *Conics*, I-43, calling this a "reduction to linear ratio" [12, 35–48].[17] The function of this technique can be precisely described: it applies a proposition of type F, thereby avoiding an argument by duplicate ratios or compound ratios.

Now we can better understand Archimedes' argument as a whole. It is a due result of the systematic application of an established and widely diffused method, "reduction to linear ratio."

Archimedes' choice of this technique is significant, because another solution by triplicate ratio was certainly available to him. In a lemma in the first book of SC Archimedes states that similar cones are in triplicate ratio of the base diameters.[18] This is also part of XII-12 of the *Elements*, where the same property is also established for cylinders. Though Archimedes does not seem to have directly consulted Euclid's *Elements*, we may certainly assume that Archimedes was also aware that his lemma applied to cylinders as well as to cones.

With this lemma, problem II-1 can be solved as follows:

A Possible Reconstruction of the Solution of II-1′. Take a point O on EF such that CD = EO (Fig. 6). Then from the lemma above we have

$$\text{cylinder CDO} : \text{cylinder GHL} = 3 * (CD : GH) \tag{1}$$

Since cylinders which are on equal bases are as their axes (SC Lemma 2 after I-16; cf. *Elements*, XII-14),

[17] Of course it should be distinguished from "Hippocrates' reduction" which we investigate in the present paper.

[18] SC I, Lemma 5 after Proposition 16: Cones whose diameters of the bases have the same ratio as their axes are in the triplicate ratio of the diameters of their bases.

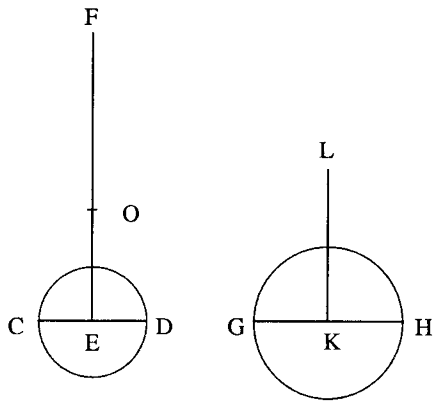

FIGURE 6

$$\text{cylinder CDO} : \text{cylinder CDF} = EO : EF = CD : EF. \quad (2)$$

Also it is supposed that

$$\text{cylinder GHL} = \text{cylinder CDF}. \quad (3)$$

From (1), (2), and (3), we have

$$CD : EF = 3 * (CD : HG). \quad (4)$$

Then, from the definition of triplicate ratio, HG (the diameter of the base of the cylinder to be found) is the first of the two mean proportionals between CD and EF.

This argument, though our reconstruction, depends only upon propositions available to Archimedes and conforms completely to the style of Greek geometry at that time. In short, we see no reason why Archimedes could not have carried out this argument.

Archimedes, however, did not use the lemma concerning triplicate ratio, although it was clearly within his reach, preferring the technique of "reduction to linear ratio." His choice confirms the prevalence of this technique, which we have found in both Euclid and Apollonius. Archimedes' case is particularly significant for two reasons: first, he chose not to use compound and multiplicate ratios in this problem in spite of his perfect mastery of these concepts;[19]second, Archimedes could not have failed to notice that his problem (SC II-1) was essentially equivalent to cube duplication. In fact, it is easy to see that, if one replaces the base circles of the cylinders in SC II-1′ with squares circumscribed on these circles, the problem turns

[19] For compound ratio, see SC II-4. As for multiplicate ratio, Archimedes uses it in SC I-32, 33, and 34, he also refers to the results of his predecessors (cf. *Elements* XII-2 and 18) in terms of multiplicate ratio in his preface to *Quadrature of Parabolas*. Moreover, he even goes on to introduce a "sesquialteral" ratio in SC II-8 (if $a:b = b:c = c:d$, then $a:d$ is the sesquilateral ratio of $a:c$).

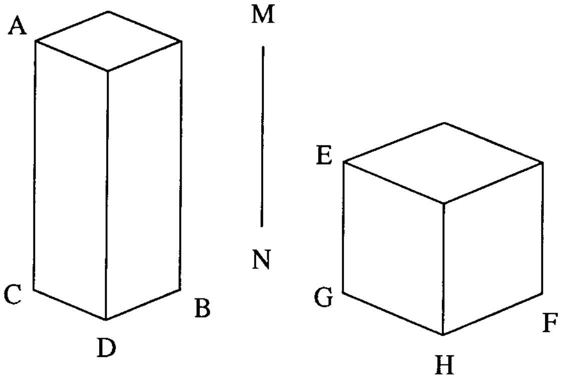

FIGURE 7

into the geometrically represented form of cube duplication [CDxg]; Archimedes indeed performs this very substitution in his argument.

Under these circumstances, Archimedes' argument in SC II-1 seems to suggest something more than his preference for the technique of "reduction to linear ratio." To make our point clear, let us pose a question. The equivalence of the problem of doubling the cube and finding two mean proportionals was no doubt a commonplace knowledge for mathematicians at Archimedes' time. What then was its proof at that time? The assumption that it was based on triplicate ratio (this seems to be the opinion of the majority of modern scholars) makes it difficult to explain Archimedes' avoidance of triplicate ratio in SC II-1. It would seem plausible, on the other hand, that the proof of Hippocrates' reduction of cube duplication was similar to Archimedes' analysis in SC II-1 and that the latter was in fact an adaptation of the former.

This latter assumption leads to a natural reconstruction of the reduction of cube duplication modelled on Archimedes' analysis in SC II-1. This reconstruction goes as follows:

Let there be given a rectangular prism AB, with square base BC (Fig. 7). It is required to find a cube EF which is equal to prism AB. Let it be assumed that the cube EF is constructed. Then, by Fp (XI-34),

$$\mathrm{sq}(CD):\mathrm{sq}(GH) = EG:AC. \qquad (1)$$

Since EF is a cube, GH = EG. Therefore,

$$\mathrm{sq}(CD):\mathrm{sq}(GH) = GH:AC. \qquad (2)$$

Let MN be a line such that

$$\mathrm{sq}(GH) = \mathrm{r}(CD, MN). \qquad (3)$$

Then

$$\mathrm{sq(CD):sq(GH) = sq(CD):r(CD, MN) = CD:MN.}$$

Therefore,

$$\mathrm{CD:MN = sq(CD):sq(GH), \text{ that is, } GH:AC.} \tag{4}$$

Then, alternately,

$$\mathrm{CD:GH = MN:AC.}$$

From (3), by F(VI-16), CD:GH = GH:MN. Therefore,

$$\mathrm{CD:GH = GH:MN = MN:AC.} \tag{5}$$

Therefore, GH, the side of the cube to be constructed, is the first of the two mean proportionals between given two lines CD and AB.

In the reconstruction above, we have given the same numbers to the corresponding relations so as to make the parallelism between the two arguments apparent. To conclude, let us examine what we have established, and discuss the plausibility of our analysis and the above reconstruction.

6. CONCLUDING OBSERVATIONS

First, we have established that Archimedes' argument in SC II-1, which seems at first sight roundabout and tortuous, is in fact a systematic application of a technique which we call "reduction to linear ratio." In fact, this proposition provides further evidence that this technique enjoyed wide diffusion and long persistence. At the same time, we have discovered the significance of equal-volume propositions (those of type F) in Books XI and XII of the *Elements*. These results are indispensable for the technique of "reduction to linear ratio" (just as are their counterparts in plane geometry), and this method enabled Greek mathematicians to avoid using multiplicate and compound ratios. How old is this technique of "reduction" then? Although the lack of extant documents prevents us from giving a definitive answer to this question, the technique would seem fairly ancient. For Archimedes, both the technique of "reduction" and the method of multiplicate and compound ratios were clearly available, and certain types of problems could be solved by either of them. In the *Data* and the *Elements*, however, we see a preference for the "reduction" method. There seems to be no reason to believe that this technique was first introduced by Euclid, since it requires only basic theorems of proportion theory. At any rate, one should be wary of invoking the use of the multiplicate and compound ratios in reconstructing early Greek geometry, since these methods appear to have been developed later and are not directly supported by pre-Euclidean documentary evidence.

Second, the proof of Hippocrates' reduction which we have reconstructed from Archimedes' argument, or something quite close to this, very probably served as a standard justification of Hippocrates' reduction. For although our reconstruction has no direct historical evidence, the assumption that this argument was unfamiliar would at once entail the following assertions which seem very difficult to sustain:

• The technique of "reduction to linear ratio," which was widely used in a variety of contexts, was not applied to cube duplication;

• In spite of this, Archimedes adopted the technique in SC II-1, a problem similar to cube duplication, when he could have just as easily utilized the method of multiplicate and compound ratios.

• Books XI and XII of Euclid's *Elements* contain some equal-volume propositions (type F) with no apparent purpose.

Taking these points into consideration, it would seem safe to assume that the justification of Hippocrates' reduction of cube duplication existed in the form we have reconstructed it, by no later than Archimedes' time.

Finally, let us briefly discuss the possibility of attributing our reconstruction to Hippocrates himself. The technique of "reduction to linear ratio" may safely be dated before Eudoxus (fl. ca. 368 B.C.), since Eudoxus's theory of proportion could justify the use of multiplicate ratios and these were indeed used in XII-1 (see [13, 130–135]). If so, our reconstructed proof would have originated by no later than the first half of the fourth century B.C., which would be much less than a century after Hippocrates.

If we may be permitted to make some conjectures here, we would like to propose that Hippocrates himself gave a rigorous proof similar to our reconstruction, within the standards of his time, for his reduction of cube duplication, and that he developed the technique of "reduction to linear ratio" based on the propositions of type F contained in his *Elements*. In this connection, we would emphasize the importance of the propositions of type F as a tool for the technique of "reduction to linear ratio" in early Greek geometry. The greatest advantage of this approach is that it is in accordance with the idiosyncrasies in the *Elements* and other Greek mathematical works.

Thus our study of how Hippocrates or his contemporaries may have justified Hippocrates' reduction has resulted in a proposal for a new interpretation of the development of the theory of proportion in early Greek geometry. We believe that, despite the disturbing lack of documentary evidence, we can still hope to improve our understanding of pre-Euclidean geometry through careful analysis of the theorems and techniques used in the extant texts. We hope that our approach will inspire further researches in this direction.

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