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Existence and uniqueness for a problem involving Hilfer fractional derivative

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ABSTRACT

We consider an initial value problem for a class of nonlinear fractional differential equations involving Hilfer fractional derivative. We prove existence and uniqueness of global solutions in the space of weighted continuous functions. The stability of the solution for a weighted Cauchy-type problem is also analyzed.

1. Introduction

We consider the initial value problem

$$D_{a+}^{\alpha,\beta} y(x) = f(x, y), \quad x > a,\ 0 < \alpha < 1,\ 0 \le \beta \le 1, \tag{1}$$

$$I_{a+}^{1-\gamma} y(a^+) = y_a, \quad \gamma = \alpha + \beta - \alpha\beta, \tag{2}$$

where $D_{a+}^{\alpha,\beta}$ is the generalized Riemann–Liouville fractional derivative operator introduced by Hilfer in [1–3].

In recent years there has been a considerable interest in the theory and applications of fractional differential equations. See for example [4–14] and references therein.

Fractional calculus approach has been introduced in many models. Fractional models provide a tool for capturing and understanding complex phenomena in many areas; see for example [15–18]. Indeed, some of these models are supported by experimental evidence and yield results that agree with the observed behavior [19].

The two parameter family of fractional derivatives $D_{a+}^{\alpha,\beta}$ of order α and type β allows one to interpolate between the Riemann–Liouville and the Caputo derivatives described in [20–22]. The type-parameter produces more types of stationary states and provide an extra degree of freedom on the initial condition. Models based on this derivatives are considered in [2,3,7,15].

In this paper, we prove the existence and uniqueness for the nonlinear initial value problem (1)–(2) in a weighted space of continuous functions. We start with some preliminaries in Section 2. In Section 3, we set up the Cauchy type problem and

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define the generalized derivative and the spaces of solutions. In Section 4, we establish the equivalence with the Volterra integral equation. In Section 5, we prove the existence and uniqueness of the solution. Finally, in Section 6 we present a stability result.

2. Preliminaries

In this section we present some definitions, lemmas, properties and notation which we use later. For more details please see [20].

Let $-\infty < a < b < \infty$. Let $C[a,b]$, $AC[a,b]$ and $C^n[a,b]$ denote the spaces of continuous, absolutely continuous and n times continuously differentiable functions on $[a,b]$, respectively. We denote by $L^p(a,b)$, $p \geq 1$, the spaces of Lebesgue integrable functions on (a,b).

Definition 1. We consider the weighted spaces of continuous functions

$$C_\gamma[a,b] = \{f : (a,b] \to \mathbb{R} : (x-a)^\gamma f(x) \in C[a,b]\}, \quad 0 \leq \gamma < 1,$$

and

$$C_\gamma^n[a,b] = \left\{f \in C^{n-1}[a,b] : f^{(n)} \in C_\gamma[a,b]\right\}, \quad n \in \mathbb{N},$$
$$C_\gamma^0[a,b] = C_\gamma[a,b],$$

with the norms

$$\|f\|_{C_\gamma} = \|(x-a)^\gamma f(x)\|_C\,,$$

and

$$\|f\|_{C_\gamma^n} = \sum_{k=0}^{n-1} \left\|f^{(k)}\right\|_C + \left\|f^{(n)}\right\|_{C_\gamma}.$$

These spaces satisfy the following properties.

- $C_0[a,b] = C[a,b]$.
- $C_\gamma^n(a,b) \subset AC^n[a,b]$.
- $C_{\gamma_1}[a,b] \subset C_{\gamma_2}[a,b]$, $0 \leq \gamma_1 < \gamma_2 < 1$.

Lemma 2. *Let $0 \leq \gamma < 1$, $a < c < b$, $g \in C_\gamma[a,c]$, $g \in C[c,b]$ and g is continuous at c. Then $g \in C_\gamma[a,b]$.*

Lemma 3 ([23]). *Let $\lambda,\ \nu,\ \omega > 0$, then*

$$\int_0^t (t-s)^{\nu-1} s^{\lambda-1} e^{-\omega s} ds \leq C t^{\nu-1}, \quad t > 0,$$

where

$$C = \max\left\{1, 2^{1-\nu}\right\} \Gamma(\lambda)(1 + \lambda(\lambda+1)/\nu)\omega^{-\lambda} > 0.$$

The following is a special case of Jensen's Inequality.

Lemma 4. *For nonnegative a_i, $i = 1, \ldots, k$,*

$$\left(\sum_{i=1}^k a_i\right)^p \leq k^{p-1} \sum_{i=1}^k a_i^p, \quad p \geq 1. \tag{3}$$

Lemma 5 ([24]). *Let $a(t)$ and $b(t)$ be continuous positive functions defined on $[t_0, \infty)$, $t_0 \geq 0$. Let $w : [0,\infty) \to [0,\infty)$ be a continuous monotonic nondecreasing function such that $w(0) = 0$ and $w(x) > 0$ for $x > 0$. If u is a positive differentiable function on $[t_0,\infty)$ that satisfies*

$$u'(t) \leq a(t)w(u(t)) + b(t), \quad t \in [t_0, \infty),$$

then we have

$$u(t) \leq G^{-1}\left[G\left(u(t_0) + \int_{t_0}^t b(s)ds\right) + \int_{t_0}^t a(s)ds\right],$$

for the values of t for which the right-hand side is well-defined, where

$$G(r)=\int_{r_0}^{r}\frac{ds}{w(s)},\quad r>r_0>0.$$

Theorem 6 *([20], Banach Fixed Point Theorem)*. *Let (U,d) be a nonempty complete metric space. Let $T:U\to U$ be a map such that for every $u,v\in U$, the relation*

$$d(Tu,Tv)\le w\,d(u,v),\quad 0\le w<1$$

holds. Then the operator T has a unique fixed point $u^*\in U$.

Furthermore, if T^k, $k\in\mathbb{N}$, is the sequence of operators defined by

$$T^1=T,\qquad T^k=TT^{k-1},\quad k\in\mathbb{N}\setminus\{1\},$$

then for any $u_0\in U$ the sequence $\left\{T^k u_0\right\}_{k=1}^{\infty}$ converges to the above fixed point, u^*.

The left-sided Riemann–Liouville fractional integrals and derivatives are defined as follows.

Definition 7. Let $f\in L^1(a,b)$. The integral

$$I_{a^+}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(s)}{(x-s)^{1-\alpha}}ds,\quad x>a,\ \alpha>0,$$

is called the left-sided Riemann–Liouville fractional integral of order α of the function f.

Definition 8. The expression

$$D_{a^+}^{\alpha}f(x):=DI_{a^+}^{\alpha-1}f(x),\quad x>a,\ 0<\alpha<1, D=\frac{d}{dx},$$

is called the left-sided Riemann–Liouville fractional derivative of order α of f provided the right-hand side exists.

For power functions we have the following properties.

Lemma 9. *For $x>a$ we have*

$$\left[I_{a^+}^{\alpha}(t-a)^{\beta-1}\right](x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1},\quad \alpha\ge 0,\ \beta>0.$$
$$\left[D_{a^+}^{\alpha}(t-a)^{\alpha-1}\right](x)=0,\quad 0<\alpha<1.$$

The following lemmas provide some mapping properties of $I_{a^+}^{\alpha}$. Proofs can be found in [25].

Lemma 10. *For $\alpha>0$, $I_{a^+}^{\alpha}$ maps $C[a,b]$ into $C[a,b]$.*

Lemma 11. *Let $\alpha>0$ and $0\le\gamma<1$. Then $I_{a^+}^{\alpha}$ is bounded from $C_\gamma[a,b]$ into $C_\gamma[a,b]$.*

Lemma 12. *Let $\alpha>0$ and $0\le\gamma<1$. If $\gamma\le\alpha$, then $I_{a^+}^{\alpha}$ is bounded from $C_\gamma[a,b]$ into $C[a,b]$.*

Lemma 13. *Let $0\le\gamma<1$ and $f\in C_\gamma[a,b]$. Then*

$$I_{a^+}^{\alpha}f(a):=\lim_{x\to a^+}I_{a^+}^{\alpha}f(x)=0,\quad 0\le\gamma<\alpha.$$

Proof. Note that by Lemma 12, $I_{a^+}^{\alpha}f\in C[a,b]$. Since $f\in C_\gamma[a,b]$ then $(x-a)^\gamma f(x)$ is continuous on $[a,b]$ and thus

$$|(x-a)^\gamma f(x)|<M,\quad x\in[a,b],$$

for some positive constant M. Therefore

$$\left|I_{a^+}^{\alpha}f(x)\right|<M\left[I_{a^+}^{\alpha}(t-a)^{-\gamma}\right](x),$$

and by Lemma 9

$$\left|I_{a^+}^{\alpha}f(x)\right|\le M\frac{\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)}(x-a)^{\alpha-\gamma}.$$

Since $\alpha>\gamma$, the right-hand side $\to 0$ as $x\to a^+$. This completes the proof. $\square$

Lemma 14. *Let $f \in L^1(a,c)$. Then*

$$\lim_{x\to c^+}\int_a^c (x-t)^{\alpha-1} f(t)\,dt = \int_a^c (c-t)^{\alpha-1} f(t)\,dt = \Gamma(\alpha) I_{a^+}^{\alpha} f(c), \quad \alpha > 0.$$

Proof. We have

$$\begin{aligned}\left|\int_a^c (c-t)^{\alpha-1} f(t)dt - \int_a^c (x-t)^{\alpha-1} f(t)\,dt\right| &\le \int_a^c \left|(c-t)^{\alpha-1} - (x-t)^{\alpha-1}\right| |f(t)|\,dt \\ &= \int_a^c k(x,t)|f(t)|\,dt.\end{aligned}$$

This proves the result since $\lim_{x\to c^+} k(x,t) = 0$ for all $a \le t < c$. □

The following lemma follows by direct calculations using Dirichlet formula.

Lemma 15. *Let $\alpha \ge 0$, $\beta \ge 0$, and $f \in L^1(a,b)$. Then*

$$I_a^{\alpha} I_a^{\beta} f(x) \overset{a.e.}{=} I_a^{\alpha+\beta} f(x), \quad t \in [a,b].$$

In particular, if $f \in C_\gamma[a,b]$ or $f \in C[a,b]$ then equality holds at every $x \in (a,b]$ or $x \in [a,b]$, respectively.

This lemma leads to the left inverse operator as follows.

Lemma 16. *Let $\alpha > 0$, $1 \le \gamma < 1$, and $f \in C_\gamma[a,b]$. Then*

$$D_{a^+}^{\alpha} I_{a^+}^{\alpha} f(x) = f(x)$$

for all $x \in (a,b]$.

The following composition is proved in [22].

Lemma 17. *Let $0 < \alpha < 1$, $0 \le \gamma < 1$. If $f \in C_\gamma[a,b]$ and $I_{a^+}^{1-\alpha} f \in C_\gamma^1[a,b]$, then*

$$I_{a^+}^{\alpha} D_{a^+}^{\alpha} f(x) = f(x) - \frac{I_{a^+}^{1-\alpha} f(a)}{\Gamma(\alpha)} (x-a)^{\alpha-1},$$

for all $x \in (a,b]$.

3. Generalized Cauchy problem

In this section we set up the Cauchy type problem (1)–(2) and define the generalized derivative and the spaces of solutions.

Definition 18. The right-sided fractional derivative operator of order $0 < \alpha < 1$ and type $0 \le \beta \le 1$ is defined by

$$D_{a^+}^{\alpha,\beta} = I_{a^+}^{\beta(1-\alpha)} D I_{a^+}^{(1-\beta)(1-\alpha)}.$$

Remark 19. The following follows from the definitions.

R1. The operator $D_{a^+}^{\alpha,\beta}$ can be written as

$$D_{a^+}^{\alpha,\beta} = I_{a^+}^{\beta(1-\alpha)} D I_{a^+}^{1-\gamma} = I_{a^+}^{\beta(1-\alpha)} D_{a^+}^{\gamma}, \quad \gamma = \alpha + \beta - \alpha\beta.$$

R2. The $D_{a^+}^{\alpha,\beta}$ derivative is considered as an interpolator between the Riemann–Liouville and Caputo derivative since

$$D_{a^+}^{\alpha,\beta} = \begin{cases} D_{a^+}^{\alpha}, & \beta = 0, \\ I_{a^+}^{1-\alpha} D, & \beta = 1. \end{cases}$$

R3. The parameter γ satisfies

$$0 < \gamma \le 1, \qquad \gamma \ge \alpha, \qquad \gamma > \beta, \qquad 1-\gamma < 1-\beta(1-\alpha).$$

We introduce the spaces

$$C_{1-\gamma}^{\alpha,\beta}[a,b] = \left\{ f \in C_{1-\gamma}[a,b], D_{a^+}^{\alpha,\beta} f \in C_{1-\gamma}[a,b] \right\}, \tag{4}$$

and

$$C_{1-\gamma}^{\gamma}[a,b] = \left\{ f \in C_{1-\gamma}[a,b], D_{a+}^{\gamma} f \in C_{1-\gamma}[a,b] \right\}.$$

Since $D_{a^+}^{\alpha,\beta} f = I_{a^+}^{\beta(1-\alpha)} D_{a+}^{\gamma} f$, it follows from Lemma 11 that

$$C_{1-\gamma}^{\gamma}[a,b] \subset C_{1-\gamma}^{\alpha,\beta}[a,b].$$

The following lemma follows directly from the semigroup property in Lemma 15.

Lemma 20. *Let* $0 < \alpha < 1$, $0 \le \beta \le 1$, *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $f \in C_{1-\gamma}^{\gamma}[a,b]$ *then*

$$I_{a+}^{\gamma} D_{a+}^{\gamma} f = I_{a+}^{\alpha} D_{a+}^{\alpha,\beta} f,$$

and

$$D_{a+}^{\gamma} I_{a+}^{\alpha} f = D_{a^+}^{\beta(1-\alpha)} f.$$

Lemma 21. *Let* $f \in L^1(a,b)$. *If* $D_{a^+}^{\beta(1-\alpha)} f$ *exists and in* $L^1(a,b)$ *then*

$$D_{a^+}^{\alpha,\beta} I_{a+}^{\alpha} f = I_{a^+}^{\beta(1-\alpha)} D_{a^+}^{\beta(1-\alpha)} f.$$

Proof.

$$D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} = I_{a^+}^{\beta(1-\alpha)} D I_{a^+}^{(1-\beta)(1-\alpha)} I_{a^+}^{\alpha} = I_{a^+}^{\beta(1-\alpha)} D I_{a^+}^{1-\beta(1-\alpha)} = I_{a^+}^{\beta(1-\alpha)} D_{a^+}^{\beta(1-\alpha)}. \quad \square$$

Lemma 22. *Let* $0 < \alpha < 1$, $0 \le \beta \le 1$, *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $f \in C_{1-\gamma}[a,b]$ *and* $I_{a^+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma}^{1}[a,b]$ *then* $D_{a^+}^{\alpha,\beta} I_{a+}^{\alpha} f$ *exists in* $(a,b]$ *and*

$$D_{a^+}^{\alpha,\beta} I_{a+}^{\alpha} f(x) = f(x), \quad x \in (a,b].$$

Proof. From Lemmas 13, 17 and 21 we have

$$I_{a^+}^{\beta(1-\alpha)} D_{a^+}^{\beta(1-\alpha)} f(x) = f(x) + \frac{I_{a^+}^{1-\beta(1-\alpha)} f(a)}{\Gamma(\beta(1-\alpha))} (x-a)^{\beta(1-\alpha)-1} = f(x), \quad x \in (a,b]. \quad \square$$

Next we investigate the solutions of (1)–(2) by reducing the problem to a Volterra integral equation and then applying the Banach fixed point theorem.

4. Equivalent Volterra integral equation

The following theorem yields the equivalence between the Cauchy type problem (1)–(2) and the Volterra integral equation of the second kind

$$y(x) = \frac{y_a}{\Gamma(\alpha+\beta-\alpha\beta)} (x-a)^{(\alpha-1)(1-\beta)} + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t, y(t))\, dt, \quad x > a. \tag{5}$$

Theorem 23. *Let* $\gamma = \alpha + \beta - \alpha\beta$ *where* $0 < \alpha < 1$ *and* $0 \le \beta \le 1$. *Let* $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $f(.,y(.)) \in C_{1-\gamma}[a,b]$ *for any* $y \in C_{1-\gamma}[a,b]$. *If* $y \in C_{1-\gamma}^{\gamma}[a,b]$, *then* y *satisfies* (1)–(2) *if and only if* y *satisfies* (5).

Proof. First we prove the necessity. Let $y \in C_{1-\gamma}^{\gamma}[a,b]$ be a solution of (1)–(2). We want to prove that y is also a solution of the integral equation (5). By the definition of $C_{1-\gamma}^{\gamma}[a,b]$, Lemma 12, and Definition 8, we have

$$I_{a^+}^{1-\gamma} y \in C[a,b] \quad \text{and} \quad D_{a^+}^{\gamma} y = D(I_{a^+}^{1-\gamma} y) \in C_{1-\gamma}[a,b].$$

Thus by Definition 1 we have

$$I_{a^+}^{1-\gamma} y \in C_{1-\gamma}^{1}[a,b].$$

Now we apply Lemma 17 to obtain

$$I_{a+}^{\gamma} D_{a+}^{\gamma} y(x) = y(x) - \frac{y_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}, \quad x \in (a,b]. \tag{6}$$

Since by our hypothesis $D_{a+}^{\gamma} y \in C_{1-\gamma}[a,b]$, Lemma 20 yields

$$I_{a+}^{\gamma} D_{a+}^{\gamma} y = I_{a+}^{\alpha} D_{a+}^{\alpha,\beta} y = I_{a+}^{\alpha} f, \quad \text{in } (a,b]. \tag{7}$$

From (6) and (7) we obtain

$$y(x) = \frac{y_a}{\Gamma(\gamma)}(x-a)^{\gamma-1} + \left[I_{a+}^{\alpha} f(t,y(t))\right](x), \quad x \in (a,b] \tag{8}$$

which is the Eq. (5).

Now we prove the sufficiency. Let $y \in C_{1-\gamma}^{\gamma}[a,b]$ satisfy Eq. (5) which can be written as (8). Applying the operator D_{a+}^{γ} to both sides of (8), it follows from Lemma 9, Lemma 20, and Definition 8 that

$$D_{a+}^{\gamma} y = D_{a+}^{\beta(1-\alpha)} f. \tag{9}$$

From (9) and the hypothesis $D_{a+}^{\gamma} y \in C_{1-\gamma}[a,b]$, we have

$$D I_{a+}^{1-\beta(1-\alpha)} f = D_{a+}^{\beta(1-\alpha)} f \in C_{1-\gamma}[a,b]. \tag{10}$$

Also, since $f \in C_{1-\gamma}[a,b]$, by Lemma 11,

$$I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma}[a,b]. \tag{11}$$

It follows from (10) and (11) and the Definition 1 that

$$I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma}^{1}[a,b].$$

Thus f and $I_{a+}^{1-\beta(1-\alpha)} f$ satisfy the conditions of Lemma 17.

Now by applying $I^{\beta(1-\alpha)}$ to both sides of (9) and using Definition 18 and Lemma 17 we can write

$$D_{a+}^{\alpha,\beta} y(x) = f(x,y(x)) - \frac{\left[I_{a+}^{1-\beta(1-\alpha)} f(t,y(t))\right](a)}{\Gamma(\beta(1-\alpha))}(x-a)^{\beta(1-\alpha)-1}. \tag{12}$$

Since $1-\gamma < 1-\beta(1-\alpha)$, Lemma 13 implies that

$$\left[I_{a+}^{1-\beta(1-\alpha)} f(t,y(t))\right](a) = 0.$$

Hence the relation (12) reduces to

$$D_{a+}^{\alpha,\beta} y(x) = f(x,y(x)), \quad x \in (a,b].$$

Now we show that the initial condition (2) also holds. We apply $I_{a+}^{1-\gamma}$ to both sides of (8), then Lemmas 9 and 15 imply that

$$I_{a+}^{1-\gamma} y(x) = y_a + \left[I_{a+}^{1-\beta(1-\alpha)} f(t,y(t))\right](x). \tag{13}$$

In (13), taking the limit as $x \to a$, we obtain

$$I_{a+}^{1-\gamma} y(a) = y_a + \left[I_{a+}^{1-\beta(1-\alpha)} f(t,y(t))\right](a) = y_a$$

This completes the proof. □

Remark 24. Note that under the hypotheses of Theorem 23, the solution satisfies the relation

$$D_{a+}^{\alpha,\beta} y(x) = D_{a+}^{\alpha} y(x) - \frac{y_a}{\Gamma(\gamma-\alpha)}(x-a)^{\gamma-\alpha-1}.$$

Thus, from Lemma 13, the solution satisfies the Cauchy-type problem

$$D_{a+}^{\alpha} y = \frac{y_a}{\Gamma(\gamma-\alpha)}(x-a)^{\gamma-\alpha-1} + f(x,y),$$

$$I_{a+}^{1-\alpha} y(a^+) = 0$$

with $D_{a+}^{\alpha} y \in C_{1-\gamma+\alpha}[a,b]$ in general. This problem is a weaker form of (1)–(2).

5. Existence and uniqueness of the solution

In this section we establish the existence of a unique solution to the Cauchy-type problem (1)–(2) in the space $C^{\alpha,\beta}_{1-\gamma}[a,b]$ defined in (4). The result is obtained under the conditions of Theorem 23 and the Lipschitz condition on $f(.,y)$ with respect to the second variable,

$$|f(x,y_1)-f(x,y_2)| \le A\,|y_1-y_2|\,, \tag{14}$$

for all $x\in(a,b]$ and for all $y_1,y_2\in G\subset\mathbb{R}$, where $A>0$ is constant.

Theorem 25. *Let $0<\alpha<1, 0\le\beta\le 1$, and $\gamma=\alpha+\beta-\alpha\beta$. Let $f:(a,b]\times\mathbb{R}\to\mathbb{R}$ be a function such that $f(.,y(.))\in C^{\beta(1-\alpha)}_{1-\gamma}[a,b]$ for any $y\in C_{1-\gamma}[a,b]$ and satisfies the Lipschitz condition (14) with respect to the second argument.*

Then there exists a unique solution y for the Cauchy type problem (1)–(2) in the space $C^{\gamma}_{1-\gamma}[a,b]$.

Proof. According to Theorem 23, it suffices to prove the result for the equivalent Volterra integral equation (5) which can be written in the operator form

$$y(x)=Ty(x), \tag{15}$$

where

$$Ty(x)=y_0(x)+\left[I^{\alpha}_{a^+}f(t,y(t))\right](x), \tag{16}$$

with

$$y_0(x)=\frac{y_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}. \tag{17}$$

First we prove the existence of a unique solution y in the space $C_{1-\gamma}[a,b]$. Our proof is based on partitioning the interval $(a,b]$ into subintervals on which the operator T is a contraction, then we use the Banach fixed point theorem. Note that $C_{1-\gamma}[c_1,c_2]$, $a\le c_1<c_2\le b$ is a complete metric space with the metric d defined by

$$d(y_1,y_2)=\|y_1-y_2\|_{C_{1-\gamma}[c_1,c_2]}:=\max_{x\in[c_1,c_2]}\left|(x-a)^{1-\gamma}[y_1(x)-y_2(x)]\right|.$$

Select $x_1\in(a,b]$ such that

$$w_1=\frac{A\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(x_1-a)^{\alpha}<1, \tag{18}$$

where $A>0$ is the Lipschitz constant in (14). Clearly $y_0\in C_{1-\gamma}[a,x_1]$. Also, by Lemma 11, $Ty\in C_{1-\gamma}[a,x_1]$. Therefore T maps $C_{1-\gamma}[a,x_1]$ into itself. Moreover, from (14), (16), and Lemma 11, and for any $y_1,y_2\in C_{1-\gamma}[a,x_1]$ we have

$$\begin{aligned}
\|Ty_1-Ty_2\|_{C_{1-\gamma}[a,x_1]} &= \left\|I^{\alpha}_{a^+}f(t,y_1(t))-I^{\alpha}_{a^+}f(t,y_2(t))\right\|_{C_{1-\gamma}[a,x_1]}\\
&\le \left\|I^{\alpha}_{a^+}\left[|f(t,y_1(t))-f(t,y_2(t))|\right]\right\|_{C_{1-\gamma}[a,x_1]}\\
&\le (x_1-a)^{\alpha}\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\,\|f(t,y_1(t))-f(t,y_2(t))\|_{C_{1-\gamma}[a,x_1]}\\
&\le A(x_1-a)^{\alpha}\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\,\|y_1(t)-y_2(t)\|_{C_{1-\gamma}[a,x_1]}\\
&\le w_1\,\|y_1(t)-y_2(t)\|_{C_{1-\gamma}[a,x_1]}\,.
\end{aligned}$$

Our assumption (18) allows us to apply the Banach fixed point theorem to obtain a unique solution $y_0^*\in C_{1-\gamma}[a,x_1]$ to Eq. (5) on the interval $(a,x_1]$.

If $x_1\neq b$ then we consider the interval $[x_1,b]$. On this interval we consider solutions $y\in C[x_1,b]$ for the equation

$$y(x)=Ty(x):=y_{01}(x)+\left[I^{\alpha}_{x_1^+}f(t,y(t))\right](x),\quad x\in[x_1,b], \tag{19}$$

where

$$y_{01}(x)=\frac{y_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_a^{x_1}(x-t)^{\alpha-1}f(t,y(t))\,dt. \tag{20}$$

Now we select $x_2\in(x_1,b]$ such that

$$w_2=\frac{A}{\alpha\Gamma(\alpha)}(x_2-x_1)^{\alpha}<1. \tag{21}$$

Since the solution is uniquely defined on the interval $(a, x_1]$, we can consider y_{01} to be a known function. For $y_1, y_2 \in C[x_1, x_2]$, it follows from Lipschitz condition (14) and Lemma 10 that

$$\begin{aligned}\|Ty_1 - Ty_2\|_{C[x_1,x_2]} &= \left\| I^{\alpha}_{x_1^+} f(t, y_1(t)) - I^{\alpha}_{x_1^+} f(t, y_2(t)) \right\|_{C[x_1,x_2]} \\ &\le \left\| I^{\alpha}_{x_1^+} \left[|f(t, y_1(t)) - f(t, y_2(t))| \right] \right\|_{C[x_1,x_2]} \\ &\le \frac{1}{\alpha\Gamma(\alpha)} (x_2 - x_1)^{\alpha} \|f(t, y_1(t)) - f(t, y_2(t))\|_{C[x_1,x_2]} \\ &\le \frac{A}{\alpha\Gamma(\alpha)} (x_2 - x_1)^{\alpha} \|y_1(t) - y_2(t)\|_{C[x_1,x_2]} \\ &= w_2 \|y_1(t) - y_2(t)\|_{C[x_1,x_2]} .\end{aligned}$$

Since $0 < w_2 < 1$, T is a contraction. Since $f(x, y(x)) \in C[x_1, x_2]$ for any $y \in C[x_1, x_2]$, Lemma 10 implies that $I^{\alpha}_{x_1^+} f \in C[x_1, x_2]$. Thus the right-hand side of (19) is in $C[x_1, x_2]$. Therefore T maps $C[x_1, x_2]$ into itself. By Theorem 6, there exists a unique solution $y_1^*(x) \in C[x_1, x_2]$ to Eq. (5) on the interval $[x_1, x_2]$. Moreover, it follows from Lemma 14 that, $y_1^*(x_1) = y_0^*(x_1)$. Therefore, if

$$y^*(x) = \begin{cases} y_0^*(x), & a < x \le x_1, \\ y_1^*(x), & x_1 < x \le x_2, \end{cases}$$

then by Lemma 2, $y^* \in C_{1-\gamma}[a, x_1]$. So y^* is the unique solution of (5) in $C_{1-\gamma}[a, x_2]$ on the interval $(a, x_2]$.

If $x_2 \ne b$, we repeat the process as necessary, say $M - 2$ times, to obtain the unique solutions $y_k^* \in C[x_k, x_{k+1}]$, $k = 2, 3, \ldots, M$, where $a = x_0 < x_1 < \cdots < x_M = b$, such that

$$w_{k+1} = \frac{A}{\alpha\Gamma(\alpha)} (x_{k+1} - x_k)^{\alpha} < 1.$$

As a result we have the unique solution $y^* \in C_{1-\gamma}[a, b]$ of (5) given by

$$y^*(x) = y_k^*(x), \quad x \in (x_k, x_{k+1}], k = 0, 1, \ldots, M - 1. \tag{22}$$

It remains to show that such a unique solution $y^* \in C_{1-\gamma}[a, b]$ is actually in $C^{\gamma}_{1-\gamma}[a, b]$. From Eq. (5) we have

$$y^*(x) = y_0(x) + \left[I^{\alpha}_{a+} f(t, y^*(t)) \right](x).$$

Applying D^{γ}_{a+} to both sides yields

$$\begin{aligned}D^{\gamma}_{a+} y^*(x) &= D^{\gamma}_{a+} \left[I^{\alpha}_{a+} f(t, y^*(t)) \right](x) = \left[D^{\gamma-\alpha}_{a+} f(t, y^*(t)) \right](x) \\ &= \left[D^{\beta(1-\alpha)}_{a+} f(t, y^*(t)) \right](x),\end{aligned}$$

since $\gamma \ge \alpha$. By hypothesis, the right hand side is in $C_{1-\gamma}[a, b]$ and thus $D^{\gamma}_{a+} y^*(x) \in C^{1-\gamma}[a, b]$.

Therefore, by Theorem 23, y^* is the unique solution of (1)–(2). $\square$

6. Stability

In this section we exploit the initial decay of the solution of the following problem with weighted initial data

$$\begin{aligned}&D^{\alpha,\beta}_{a+} u(t) = f(t, u(t)), \\ &t^{(1-\beta)(1-\alpha)} u(t)\Big|_{t=0} = b.\end{aligned} \tag{23}$$

We determine sufficient conditions for maintaining this behavior for all times in case of global existence.

Theorem 26. *Let $0 < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha + \beta - \alpha\beta$, and $b \ne 0$. Let $f(., u(.)) \in C^{\beta(1-\alpha)}_{1-\gamma}[0, \infty]$ for any $u \in C_{1-\gamma}[0, \infty]$ and satisfies*

$$|f(t, u(t))| \le t^{\mu} e^{-\sigma t} \varphi(t) |u|^m, \quad \mu \ge 0, \ m \in \mathbb{N}, \ \sigma > 0, \tag{24}$$

where φ is a continuous nonnegative function on $(0, \infty)$.

Suppose $\mu - (m - 1)(1 - \gamma) > 0$, $\varphi(t)\, t^{-m\beta(1-\alpha)} \in L^q(0, \infty)$ for some $q > 1/\alpha$, and

$$\|\varphi\|_q^{(m-1)q} \left(\int_0^{\infty} s^{-qm\beta(1-\alpha)} \varphi^q(s) ds \right) < K,$$

where

$$K = \frac{1}{(m-1)2^{m(mq-1)}\,|b|^{mq(m-1)}}\left[\frac{\Gamma^p(\alpha)(\sigma p)^{\lambda_1}}{2^{p(1-\alpha)}\Gamma(\lambda_1)(1+\lambda_1/\lambda_2)}\right]^{mq/p},$$
$$\lambda_1 = 1+p[\mu-(1-\gamma)m], \qquad \lambda_2 = 1+p(\alpha-1),$$

and p is the conjugate exponent of q, i.e. $pq = p+q$.

Then there exists a positive constant C such that the solution of (23) satisfies

$$|u(t)| \le Ct^{\gamma-1}, \quad t>0.$$

Proof. Since the hypothesis of Theorem 23 are satisfied, the problem (23) is equivalent to the associated Volterra integral equation

$$u(t) = b\,t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds, \quad t>0. \tag{25}$$

Multiplying both sides of (25) by $t^{1-\gamma}$ and using the inequality (24), we obtain

$$t^{1-\gamma}\,|u(t)| \le |b| + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\mu}e^{-\sigma s}\varphi(s)\,|u(s)|^m\,ds, \quad t>0. \tag{26}$$

Let v denote the left-hand side of (26). Inserting the term $s^{(1-\gamma)m}s^{-(1-\gamma)m}$ inside the integral gives

$$v(t) \le |b| + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\mu-(1-\gamma)m}e^{-\sigma s}\varphi(s)v^m(s)ds, \quad t>0. \tag{27}$$

From Hölder inequality we have

$$\int_0^t (t-s)^{\alpha-1}s^{\mu-(1-\gamma)m}e^{-\sigma s}\varphi(s)v^m(s)\,ds$$
$$\le \left(\int_0^t (t-s)^{p(\alpha-1)}s^{p[\mu-(1-\gamma)m]}e^{-p\sigma s}ds\right)^{1/p}\left(\int_0^t \varphi^q(s)v^{qm}(s)ds\right)^{1/q}, \quad t>0.$$

Note that the second integral in the right-hand side is finite for each fixed t since φ is continuous and u is in $C_{1-\gamma}[0,\infty]$.

It follows from the hypothesis that $\lambda_1 > 0$ and $\lambda_2 > 0$ and thus

$$p[\mu-(1-\gamma)m] = \lambda_1 - 1 > 0, \qquad p(\alpha-1) = \lambda_2 - 1 > 0.$$

By Lemma 3 (with ν replaced by λ_2, λ replaced by λ_1 and ω replaced by $p\sigma$) we have

$$\int_0^t (t-s)^{\alpha-1}s^{\mu-(1-\gamma)m}e^{-\sigma s}\varphi(s)v^m(s)ds \le C_1 t^{\alpha-1}\left(\int_0^t \varphi^q(s)v^{qm}(s)ds\right)^{1/q}, \tag{28}$$

with

$$C_1 = \left[2^{p(1-\alpha)}\Gamma(\lambda_1)(1+\lambda_1/\lambda_2)(p\sigma)^{-\lambda_1}\right]^{1/p}.$$

By combining (27) and (28) we obtain

$$v(t) \le |b| + t^{-\beta(1-\alpha)}\,\hat{C}_1\left(\int_0^t \varphi^q(s)v^{qm}(s)ds\right)^{1/q}, \quad t>0, \tag{29}$$

where $\hat{C}_1 = C_1/\Gamma(\alpha)$. Multiplying both sides of (29) by $t^{\beta(1-\alpha)}$ yields

$$t^{\beta(1-\alpha)}v(t) \le |b|\,t^{\beta(1-\alpha)} + \hat{C}_1\left(\int_0^t \varphi^q(s)v^{qm}(s)ds\right)^{1/q}, \quad t>0. \tag{30}$$

Let $z(t)$ denote the left-hand side of (30). Inserting the term $s^{-qm\beta(1-\alpha)}s^{qm\beta(1-\alpha)}$ inside the integral gives

$$z(t) \le |b|\,t^{\beta(1-\alpha)} + \hat{C}_1\left(\int_0^t \varphi^q(s)s^{-qm\beta(1-\alpha)}z^{qm}(s)ds\right)^{1/q}, \quad t>0. \tag{31}$$

Raising both sides of (31) to the power q, we get (using Lemma 4)

$$z^q(t) \le 2^{q-1}\left(|b|^q\, t^{q\beta(1-\alpha)} + \hat{C}_1^q \int_0^t \varphi^q(s) s^{-qm\beta(1-\alpha)} z^{qm}(s)ds\right), \quad t > 0. \tag{32}$$

Let

$$w(t) = \hat{C}_1^q \int_0^t \varphi^q(s) s^{-qm\beta(-\alpha)} z^{qm}(s)ds, \quad t > 0. \tag{33}$$

Then, by the continuity of z and the assumption $\varphi(t)t^{-m\beta(1-\alpha)} \in L^q(0,\infty)$ the integrand is summable, $w(0) = 0$, and by differentiation

$$w'(t) = \hat{C}_1^q \varphi^q(t) t^{-qm\beta(1-\alpha)} z^{qm}(t). \tag{34}$$

Since φ, z, and the right hand side of (34) are nonnegative, w is a continuous, nonnegative and nondecreasing function in $(0, \infty)$.

Now, we estimate the right hand side of (34) in terms of w. From (32) and (33) we obtain

$$z^q(t) \le 2^{q-1}(|b|^q t^{q\beta(1-\alpha)} + w(t)), \quad t > 0.$$

Raising both sides to the power m and using Lemma 4, we get

$$z^{qm}(t) \le 2^{mq-1}\left(|b|^{mq}\, t^{mq\beta(1-\alpha)} + w^m(t)\right). \tag{35}$$

Next, a substitution of (35) into (34) yields

$$\begin{aligned} w'(t) &\le 2^{mq-1}\hat{C}_1^q \varphi^q(t) t^{-qm\beta(1-\alpha)} \left(|b|^{mq}\, t^{mq\beta(1-\alpha)} + w^m(t)\right) \\ &\le 2^{mq-1}\, |b|^{mq}\, \hat{C}_1^q \varphi^q(t) + 2^{mq-1}\hat{C}_1^q t^{-qm\beta(1-\alpha)} \varphi^q(t) w^m(t), \quad t > 0. \end{aligned} \tag{36}$$

Applying Lemma 5 (with $w(u) = u^m$) we infer that

$$w(t) \le G^{-1}[G(w(0) + l(t)) + k(t)],$$

where $l(t) = 2^{mq-1}\, |b|^{mq}\, \hat{C}_1^q \int_0^t \varphi^q(s)ds$ and $k(t) = 2^{mq-1}\hat{C}_1^q \int_0^t s^{-qm\beta(1-\alpha)}\varphi^q(s)ds$. Since $G(r) = \int_{r_0}^r \frac{ds}{s^m}$, $r > 0, r_0 > 0$, then

$$G(r) = \frac{r^{1-m}}{1-m} - \frac{r_0^{1-m}}{1-m},$$

and

$$G^{-1}(y) = [r_0^{1-m} - (m-1)y]^{-\frac{1}{m-1}}.$$

That is

$$\begin{aligned} w(t) &\le G^{-1}\left[\frac{l(t)^{1-m}}{1-m} - \frac{l(t_0)^{1-m}}{1-m} + k(t)\right] \\ &\le \left[l(t_0)^{1-m} - (m-1)\left(\frac{l(t)^{1-m}}{1-m} - \frac{l(t_0)^{1-m}}{1-m} + k(t)\right)\right]^{-\frac{1}{m-1}} \\ &\le [l(t)^{1-m} - (m-1)k(t)]^{-\frac{1}{m-1}}, \end{aligned}$$

as long as

$$l(t)^{m-1}k(t) < \frac{1}{m-1}.$$

In particular, if

$$\left(\int_0^t \varphi^q(s)ds\right)^{m-1} \left(\int_0^t s^{-qm\beta(1-\alpha)}\varphi^q(s)ds\right) < K$$

then $w(t) \le K_1$ for some positive constant K_1, and thus from (31) we find that

$$z(t) \le |b|\, t^{\beta(1-\alpha)} + K_1^{1/q},$$

or

$$t^{\beta(1-\alpha)} v(t) \le |b|\, t^{\beta(1-\alpha)} + K_1^{1/q},$$

then

$$v(t) \leq |b| + K_1^{1/q} t^{-\beta(1-\alpha)} \leq C, \quad t \geq t_0 > 0,$$

for some positive constant C. This yields that $|u(t)| \leq Ct^{\gamma-1}$ for $t \geq t_0 > 0$ and the proof is complete. $\square$

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