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On Krasner's Constant*

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Let v be a henselian valuation of any rank of a field K and $\bar{v}$ be the extension of v to a fixed algebraic closure $\overline{K}$ of K. Let $\alpha \in \overline{K} \setminus K$ be separable over K. In this paper the author investigates the condition under which Krasner's constant $\omega_K(\alpha)$ given by max$\{\bar{v}(\alpha - \alpha') | \alpha' \neq \alpha$ runs over K-conjugates of $\alpha\}$, is equal to min$\{\bar{v}(\alpha - \alpha')$: α' runs over K-conjugates of $\alpha\}$; this is the condition for α to be equidistant from all of its K-conjugates. © 1999 Academic Press

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1. INTRODUCTION

Throughout v is a henselian valuation of any rank of a field K with residue field k_v and $\bar{v}$ is a (unique) prolongation of v to a fixed algebraic closure $\overline{K}$ of K. A finite extension (K', v') of (K, v) will be called tame if (a) it is defectless, i.e., $[K' : K] = ef$, where e, f are, respectively, the index of ramification and the residual degree of v'/v; (b) the residue field of v' is a separable extension of k_v; (c) the ramification index of v'/v is not divisible by the characteristic of k_v.

In 1970, Ax [1, Sect. 2, Proposition 2′] pointed out that if K is a perfect field of nonzero characteristic and v is of rank one, then to each $\alpha \in \overline{K} \setminus K$ there corresponds $a \in K$ for which $\bar{v}(\alpha - a) \geq \Delta_K(\alpha)$, where

$$\Delta_K(\alpha) = \min\{\bar{v}(\alpha - \alpha') \colon \alpha' \text{ runs over } K\text{-conjugates of } \alpha\}. \quad (1)$$

In 1991, a counterexample was given to show that this result is false (see [3]). In 1997, while giving a necessary and sufficient condition for Ax's

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result to be true without any assumption on the characteristic of K, we proved the following result (see [4, Theorems 1.2, 2.3]).

THEOREM A. *Let (K', v') be a finite tame extension of a henselian valued field (K, v). Then for any $\alpha \in K' \setminus K$ there exists $a \in K$ such that $v'(\alpha - a) \geq \Delta_K(\alpha)$.*

We showed that the converse of the foregoing theorem is not true [4, Example 2.5], and it was proved to be true [4, Theorem 1.2] with an extra hypothesis, viz., $\Delta_K(\beta) = \omega_K(\beta)$, where Krasner's constant $\omega_K(\beta)$ is given by

$$\omega_K(\beta) = \max\{\bar{v}(\beta - \beta') \mid \beta' \neq \beta \text{ runs over } K\text{-conjugates of } \beta\}.$$

This leads to the following problem: Let (K, v) and $(\bar{K}, \bar{v})$ be as above. If $\alpha \in \bar{K} \setminus K$ is separable over K, are there any necessary and sufficient conditions for $\Delta_K(\alpha)$ to be equal to $\omega_K(\alpha)$?

If α satisfies a polynomial of the type $x^n - a \in K[x]$ and if the characteristic of k_v does not divide n, then any K-conjugate of α is $\alpha\varepsilon$, with $\varepsilon^n = 1$ and hence

$$\omega_K(\alpha) = \bar{v}(\alpha) = \Delta_K(\alpha).$$

There is another situation when the above equality is easily seen to hold, viz.., when $\bar{v}(\alpha) = 0$ and the minimum polynomial $P(x)$ of α over K is such that the corresponding polynomial $P^*(x) \in k_v[x]$ (obtained on replacing the coefficients of $P(x)$ by their images in k_v) has no repeated roots, for in this case $\bar{v}(\alpha - \alpha') = 0$ for all K-conjugates $\alpha' \neq \alpha$ of α.

Observe that if $\alpha \in \bar{K} \setminus K$ is separable over K and if $a, c \neq 0$ are in K, then

$$\Delta_K((\alpha - a)/c) = \Delta_K(\alpha) - v(c),$$
$$\omega_K((\alpha - a)/c) = \omega_K(\alpha) - v(c).$$

It follows that if instead of α, the element $(\alpha - a)/c$ satisfies the requirements of either of the two cases discussed in the preceding paragraph, then $\Delta_K(\alpha) = \omega_K(\alpha)$. In fact these two cases and the foregoing observation are our motivation behind the formulation of Theorem 1.1 stated below.

For any β in the valuation ring of a valuation v' extending the given valuation v, β^* will stand for its v'-residue, i.e., the image of β under the canonical homomorphism from the valuation ring of v' onto the residue field of v'. We shall prove

THEOREM 1.1. *Let v be a henselian valuation of a field K with value group G_v and residue field k_v and $\overline{K}, \bar{v}$ be as above. Suppose that $(K(\alpha), w)$ is a tame extension of (K, v) of degree $n > 1$. Then the following statements are equivalent*:

(i) $\Delta_K(\alpha) = \omega_K(\alpha)$.

(ii) *There exists $a \in K$ such that the value group G_w and the residue field k_w of w are given by*

$$G_w = G_v + \mathbb{Z}\bar{v}(\alpha - a), \quad k_w = k_v\big(((\alpha - a)^r/b)^*\big),$$

where r is the smallest positive integer such that $r\bar{v}(\alpha - a) = v(b)$ is in G_v.

THEOREM 1.2. *Let v be a henselian valuation of a field K. Suppose that α is separable over K of degree a prime number. Then $\Delta_K(\alpha) = \omega_K(\alpha)$.*

2. PROOF OF THEOREM 1.1

LEMMA 2.1. *Let x, y be elements of field K having a valuation v, and r be a natural number. If $v(x - y) > v(x)$, then $v(x^r - y^r) > v(x^r)$.*

Proof. Observe that $v(x) = v(y)$. The desired inequality can be easily verified on writing $x^r - y^r$ as $(x - y)(x^{r-1} + x^{r-2}y + \cdots + y^{r-1})$.

LEMMA 2.2. *Let v be a henselian valuation of field K and $K(\alpha)$ be a tame extension of K (with respect to the unique prolongation of v to $K(\alpha)$ of degree $n > 1$. Then there exists $\delta \in K(\alpha)$ such that $[K(\delta): K] < n$ and $\bar{v}(\alpha - \delta) \geq \omega_K(\alpha)$.*

Proof. As $K(\alpha)/K$ is a tame extension, it must be separable. Let K' be the smallest Galois extension of K containing α and v' be the unique prolongation of v to K'. If T denotes the maximal tame extension of (K, v) contained in (K', v'), then as is well known, T is a Galois extension of K, which contains $K(\alpha)$ by hypothesis (see [2, 21.2]). Hence $K' = T$ is a tame extension of K.

Let H denote the subgroup of $\mathrm{Gal}(K'/K)$ defined by

$$H = \{\sigma \in \mathrm{Gal}(K'/K) | v'(\alpha - \sigma\alpha) \geq \omega_K(\alpha)\}.$$

Let F be the fixed field of H. It is clear from the definition of $\omega_K(\alpha)$ that $\mathrm{Gal}(K'/K(\alpha))$ is properly contained in H. Therefore F is properly contained in $K(\alpha)$. If $\Delta_F(\alpha)$ is defined on replacing K by F in (1), then it can be easily visualized that

$$\Delta_F(\alpha) = \omega_K(\alpha). \tag{2}$$

By virtue of Theorem A stated in the first section and (2), there exists $\delta \in F$ satisfying

$$v'(\alpha - \delta) \geq \Delta_F(\alpha) = \omega_K(\alpha).$$

Since F is properly contained in $K(\alpha)$, the lemma is proved.

Proof of Theorem 1.1. The valuation v being henselian, for any K-conjugate α' of α and $d \in K$, we have

$$\bar{v}(\alpha - d) = \bar{v}(\alpha' - d);$$

hence

$$\bar{v}(\alpha - \alpha') \geq \min\{\bar{v}(\alpha - d), \quad \bar{v}(\alpha' - d)\} = \bar{v}(\alpha - d).$$

Consequently,

$$\Delta_K(\alpha) \geq \bar{v}(\alpha - d), \qquad d \in K. \tag{3}$$

Suppose first that $\Delta_K(\alpha) = \omega_K(\alpha)$. Since $K(\alpha)/K$ is a tame extension, there exists $a \in K$ such that $\bar{v}(\alpha - a) \geq \Delta_K(\alpha)$ by Theorem A. Therefore by virtue of (3) and the supposition, we have

$$\bar{v}(\alpha - a) = \Delta_K(\alpha) = \omega_K(\alpha). \tag{4}$$

We shall denote $\alpha - a$ by β. Let $\bar{u}$ be the prolongation of the valuation $\bar{v}$ of $\overline{K}$ to a simple transcendental extension $\overline{K}(x)$ defined on $\overline{K}[x]$ by

$$\bar{u}\left(\sum c_i x^i\right) = \min\{\bar{v}(c_i) + i\bar{v}(\beta)\}, \qquad c_i \in \overline{K}.$$

We denote by u the valuation of $K(x)$, obtained by restricting $\bar{u}$ to $K(x)$. Observe that the value group of u is $G_v + \mathbb{Z}\bar{v}(\beta)$. So the first assertion of (ii) is proved once we show that $G_u = G_w$; for this it is enough to prove that if $g(x)$ is any polynomial over K of degree less than n, then

$$\bar{v}(g(\beta)) = u(g(x)). \tag{5}$$

To verify (5), let $\beta_1, \beta_2, \beta_3, \ldots$ be the roots of $g(x)$ and c be its leading coefficient. It follows from Krasner's lemma and (4) that

$$\bar{v}(\beta - \beta_i) = \bar{v}(\alpha - a - \beta_i) \leq \omega_K(\alpha) = \bar{v}(\beta).$$

The above inequality together with the triangle law gives

$$\bar{v}(\beta - \beta_i) = \min\{\bar{v}(\beta), \bar{v}(\beta_i)\}. \tag{6}$$

The definition of $\bar{u}$ implies that

$$\bar{u}(x-\beta_i)=\min\{\bar{v}(\beta),\bar{v}(\beta_i)\}. \tag{7}$$

Combining (6) and (7), we see that

$$\bar{v}(\beta-\beta_i)=\bar{u}(x-\beta_i);$$

consequently,

$$\bar{v}(g(\beta))=v(c)+\sum\bar{v}(\beta-\beta_i)=v(c)+\sum\bar{u}(x-\beta_i)=u(g(x)),$$

which proves (5) and hence the first assertion of (ii).

To prove the assertion regarding the residue field, let $(h(\beta))^*$ be any nonzero element of k_w where $h(x)\in K[x]$ is a polynomial of degree less than n. In view of (5),

$$u(h(x))=\bar{v}(h(\beta))=0.$$

So if we write

$$h(x)=a_0+a_1x+a_2x^2+\cdots+a_tx^t,\qquad a_i\in K,$$

then

$$u(h(x))=\min_i\{v(a_i)+i\bar{v}(\beta)\}=0.$$

Hence $v(a_i)+i\bar{v}(\beta)\geq 0$ for every i. Since r is the smallest positive integer such that $r\bar{v}(\beta)=v(b)\in G_v$, it follows that

$$v(a_i)+i\bar{v}(\beta)>0\qquad\text{if } r \text{ does not divide } i.$$

Therefore on taking the $\bar{v}$-residue of $h(\beta)=\sum a_i\beta^i$, we see that

$$(h(\beta))^*=\sum\left(a_{ir}\beta^{ir}\right)^*=\sum\left(b^ia_{ir}\right)^*\left(\beta^{ir}/b^i\right)^*$$

is in $k_v((\beta^r/b)^*)$, as desired.

Conversely, suppose that (ii) holds and suppose, to the contrary, (i) does not hold. By Lemma 2.2, there exists $\delta\in K(\alpha)$ such that $[K(\delta)\colon K]<n$ and $\bar{v}(\alpha-\delta)\geq\omega_K(\alpha)$. Therefore, by virtue of the supposition $\omega_K(\alpha)>\Delta_K(\alpha)$ and (3), we have

$$\bar{v}(\alpha-\delta)\geq\omega_K(\alpha)>\Delta_K(\alpha)\geq\bar{v}(\alpha-a).$$

So we are led to the inequality $\bar{v}(\alpha-\delta)>\bar{v}(\alpha-a)$, which shows that

$$\bar{v}(\alpha-a)=\bar{v}(\delta-a), \tag{8}$$

and

$$\bar{v}\big((\alpha - a)^r - (\delta - a)^r\big) > \bar{v}\big((\alpha - a)^r\big) = v(b),$$

in view of Lemma 2.1. The foregoing inequality implies that

$$\big((\alpha - a)^r/b\big)^* = \big((\delta - a)^r/b\big)^*. \tag{9}$$

If G_o and k_o denote the value group and the residue field of the valuation w_o obtained by restricting w to $K(\delta)$, then it follows from (8), (9), and assumption (ii) of Theorem 1.1 that

$$G_w \subseteq G_o,\ k_w \subseteq k_o. \tag{10}$$

Since $(K(\alpha), w)/(K, v)$ is defectless, we have

$$n = [K(\alpha) \colon K] = [G_w \colon G_v][k_w \colon k_v]. \tag{11}$$

It follows from (10), (11), and the fundamental inequality [2, 13.10] that

$$n \le [G_o \colon G_v][k_o \colon k_v] \le [K(\delta) \colon K] < n.$$

This contradiction proves the result.

3. PROOF OF THEOREM 1.2

Let N be the smallest normal extension of K containing α. Let H denote the subgroup of the Gal(N/K) defined by

$$H = \{\sigma \in \mathrm{Gal}(N/K) | \bar{v}(\alpha - \sigma\alpha) \ge \omega_K(\alpha)\},$$

and let L be its fixed field. Clearly, Gal$(N/(K(\alpha))$ is properly contained in H; consequently, L is properly contained in $K(\alpha)$. As $[K(\alpha) \colon K]$ is a prime number, it follows that $L = K$, and hence $H =$ Gal(N/K), which proves that for all K-conjugates α' of α, $\bar{v}(\alpha - \alpha') \ge \omega_K(\alpha)$.

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