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On improvement of a Hilbert-type inequality

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ABSTRACT

In this paper, by using the Euler–Maclaurin expansion for the zeta function and estimating the weight function effectively, we derive an improvement of a Hilbert-type inequality proved by B.C. Yang.

1. Introduction

If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1$, $n \in \mathbb{N}$ and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m, n\}} < pq \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1.1) and inequality (1.2) led to many papers dealing with alternative proofs, generalizations, numerous variants, and applications in analysis [1].

Recently, by introducing a parameter Yang [2] gave a generalization of inequality (1.2) with the best constant factor as follows:

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, $a_n, b_n > 0$ such that $0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \kappa_\lambda(p) \left\{\sum_{n=1}^{\infty} n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}, \tag{1.3}$$

where $\kappa_\lambda(p) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$ is the best possible.

In this paper, by introducing a parameter and estimating the weight coefficient, we obtain an improvement of (1.3).

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2. Some preliminary results

First, we need the following formula of the Riemann-ζ function (see [3–5]):

$$\zeta(\sigma) = \sum_{k=1}^{n} \frac{1}{k^\sigma} - \frac{n^{1-\sigma}}{1-\sigma} - \frac{1}{2n^\sigma} - \sum_{k=1}^{l-1} \frac{B_{2k}}{2k} \binom{-\sigma}{2k-1} \frac{1}{n^{\sigma+2k-1}} - \frac{B_{2l}}{2l} \binom{-\sigma}{2l-1} \frac{\varepsilon}{n^{\sigma+2l-1}}, \tag{2.1}$$

where $\sigma > 0, \sigma \neq 1, n, l \geq 1, n, l \in \mathbb{N}, 0 < \varepsilon = \varepsilon(\sigma, l, n) < 1$. The numbers $B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, \ldots$ are Bernoulli numbers. In particular, $\zeta(\sigma) = \sum_{k=1}^{\infty} \frac{1}{k^\sigma}$ $(\sigma > 1)$.

Since $\zeta(0) = -1/2$, then the formula of the Riemann-ζ function (2.1) is also true for $\sigma = 0$.

Lemma 2.1. *If $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, 2 - min\{p, q\} < \lambda \leq 2, n \geq 1$ and $n \in \mathbb{N}$, define the weight coefficients as*

$$\omega(n, \lambda, p) = \sum_{k=1}^{\infty} \frac{1}{\max\left\{k^\lambda, n^\lambda\right\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}},$$

and

$$\omega(n, \lambda, q) = \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}},$$

then we have

$$\omega(n, \lambda, p) < n^{1-\lambda} \left[\kappa(\lambda) - \frac{p}{3(p+\lambda-2) n^{\frac{p+\lambda-2}{p}}}\right], \tag{2.2}$$

and

$$\omega(n, \lambda, q) < n^{1-\lambda} \left[\kappa(\lambda) - \frac{q}{3(q+\lambda-2) n^{\frac{q+\lambda-2}{q}}}\right], \tag{2.3}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$.

Proof. When $2 - min\{p, q\} < \lambda \leq 2$, taking $\sigma = \frac{2-\lambda}{p} \geq 0, l = 1$ in (2.1), we obtain

$$\zeta\left(\frac{2-\lambda}{p}\right) = \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} - \frac{1}{2n^{\frac{2-\lambda}{p}}} + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}} \varepsilon_1, \tag{2.4}$$

where $0 < \varepsilon_1 < 1$.

Taking $\sigma = \frac{2}{p} + \frac{\lambda}{q}, l = 1$, we obtain

$$\zeta\left(\frac{2}{p} + \frac{\lambda}{q}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda+2q}{12pqn^{1+\frac{2}{p}+\frac{\lambda}{q}}} \varepsilon_2, \tag{2.5}$$

where $0 < \varepsilon_2 < 1$.

Thus we have,

$$\begin{aligned}
\omega(n, \lambda, p) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&= \sum_{k=1}^{n} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&= \sum_{k=1}^{n} \frac{1}{n^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{p}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}}.
\end{aligned}$$

By (2.4) and (2.5), we get

$$\begin{aligned}\omega(n,\lambda,p) &< \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\left[\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2}+\frac{1}{2n^{\frac{2-\lambda}{p}}}\right]-\frac{1}{n^{\lambda}}\\ &\quad+n^{\frac{2-\lambda}{p}}\left[\frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2}+\frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}}+\frac{p\lambda+2q}{12p\,qn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\right]\\ &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pn^{1-\lambda}}{p+\lambda-2}+\frac{1}{2n^{\lambda}}-\frac{1}{n^{\lambda}}+\frac{qn^{1-\lambda}}{q+\lambda-2}+\frac{1}{2n^{\lambda}}+\frac{p\lambda+2q}{12p\,qn^{1+\lambda}}\\ &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pq\lambda n^{1-\lambda}}{(p+\lambda-2)(q+\lambda-2)}+\frac{p\lambda+2q}{12pqn^{1+\lambda}}\\ &= n^{1-\lambda}\left\{\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left[-\zeta\left(\frac{2-\lambda}{p}\right)-\frac{p\lambda+2q}{12p\,qn^{\frac{p-\lambda+2}{p}}}\right]\right\}.\end{aligned}$$

In (2.4), taking $n=1$, by $2-\min\{p,q\}<\lambda\le 2$, we obtain

$$\begin{aligned}\zeta\left(\frac{2-\lambda}{p}\right) &= 1-\frac{p}{p+\lambda-2}-\frac{1}{2}+\frac{(2-\lambda)\varepsilon_1}{12p}\\ &< \frac{1}{2}-\frac{p}{p+\lambda-2}+\frac{2-\lambda}{12p}\\ &= -\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)}\\ &< 0.\end{aligned}$$

So for $n\ge 1$, $n\in\mathbb{N}$, $2-\min\{p,q\}<\lambda\le 2$, we have

$$\begin{aligned}-\zeta\left(\frac{2-\lambda}{p}\right)-\frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}} &> \frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)}-\frac{p\lambda+2q}{12pq}\\ &= \frac{q(\lambda-2-3p)(\lambda-2-2p)-(p\lambda+2q)(p+\lambda-2)}{12pq(p+\lambda-2)}\\ &= \frac{q(\lambda-2)^2+(p\lambda+5pq+2q)(2-\lambda)-p(p\lambda+2q)+6p^2q}{12pq(p+\lambda-2)}\\ &> \frac{-p(p\lambda+2q)+6p^2q}{12pq(p+\lambda-2)}\\ &\ge \frac{-(2p+2q)+6pq}{12q(p+\lambda-2)}=\frac{p}{3(p+\lambda-2)}.\end{aligned}$$

Using the last result and the inequality for $\omega(n,\lambda,p)$ above, we obtain (2.2).

In a similar way, we can prove (2.3). □

3. Main results

Theorem 3.1. *If $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-\min\{p,q\}<\lambda\le 2$, $a_n\ge 0$, $b_n\ge 0$, for $n\ge 1$, $n\in\mathbb{N}$ and $0<\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p<\infty$, $0<\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q<\infty$, then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max(m^{\lambda},n^{\lambda})} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{q}{3(q+\lambda-2)n^{\frac{q+\lambda-2}{q}}}\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{p}{3(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{3.1}$$

and

$$\sum_{m=1}^{\infty} m^{(p-1)(\lambda-1)} \left(\sum_{n=1}^{\infty} \frac{a_n}{\max\{m^\lambda, n^\lambda\}}\right)^p < \kappa(\lambda)^{p-1} \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{q}{3(q+\lambda-2)n^{\frac{q+\lambda-2}{q}}}\right] n^{1-\lambda} a_n^p, \tag{3.2}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$.

Proof. By Hölder inequality, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max(m^\lambda, n^\lambda)} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left[\frac{a_m}{\max\{m^\lambda, n^\lambda\}^{\frac{1}{p}}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right] \left[\frac{b_n}{\max\{m^\lambda, n^\lambda\}^{\frac{1}{q}}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}}\right] \\
&\le \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left[\frac{a_m^p}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}}\right]\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left[\frac{b_n^q}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{p}}\right]\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{m=1}^{\infty} \omega(m,\lambda,q) a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} \omega(n,\lambda,p) b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Hence, By (2.2), (2.3), inequality (3.1) holds.

By Hölder inequality and Lemma 2.1, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a_n}{\max\{m^\lambda, n^\lambda\}} &= \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\}^{\frac{1}{p}}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}} a_n \frac{1}{\max\{m^\lambda, n^\lambda\}^{\frac{1}{q}}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right] \\
&\le \left\{\sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}} a_n^p\right]\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{p}}\right]\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}} a_n^p\right]\right\}^{\frac{1}{p}} [\omega(m,\lambda,p)]^{\frac{1}{q}} \\
&< \left\{\sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}} a_n^p\right]\right\}^{\frac{1}{p}} [m^{1-\lambda}\kappa(\lambda)]^{\frac{1}{q}}.
\end{aligned}$$

So

$$\begin{aligned}
\sum_{m=1}^{\infty} m^{(p-1)(\lambda-1)} \left(\sum_{n=1}^{\infty} \frac{a_n}{\max\{m^\lambda, n^\lambda\}}\right)^p &< \kappa(\lambda)^{p-1} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}} a_n^p\right] \\
&< \kappa(\lambda)^{p-1} \sum_{n=1}^{\infty} \omega(n,\lambda,q) a_n^p.
\end{aligned}$$

By Lemma 2.1, the proof of Theorem 3.1 is completed. □

Since $p, q > 1$, by Theorem 3.1, we have

Corollary 3.1. *If* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \le 2$, $a_n \ge 0$, $b_n \ge 0$, *for* $n \ge 1$, $n \in \mathbb{N}$ *and* $0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty$, *then*

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max(m^\lambda, n^\lambda)} &< \left\{\sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{3(q+\lambda-2)n^{\frac{q+\lambda-2}{q}}}\right] n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{3(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right] n^{1-\lambda} b_n^q\right\}^{\frac{1}{q}},
\end{aligned} \tag{3.3}$$

and

$$\sum_{m=1}^{\infty} m^{(p-1)(\lambda-1)} \left(\sum_{n=1}^{\infty} \frac{a_n}{\max\{m^{\lambda}, n^{\lambda}\}} \right)^p < \kappa(\lambda)^{p-1} \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{3(q+\lambda-2)n^{\frac{q+\lambda-2}{q}}} \right] n^{1-\lambda} a_n^p, \tag{3.4}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$.

Taking $\lambda = 1$, $p = q = 2$ in (3.1), we have:

Corollary 3.2. *If* $a_n \geq 0$, $b_n \geq 0$, *and* $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$, $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, *then*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max(m, n)} < 4 \left[\sum_{n=1}^{\infty} \left(1 - \frac{1}{6\sqrt{n}} \right) a_n^2 \right]^{\frac{1}{2}} \left[\sum_{n=1}^{\infty} \left(1 - \frac{1}{6\sqrt{n}} \right) b_n^2 \right]^{\frac{1}{2}}. \tag{3.5}$$

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